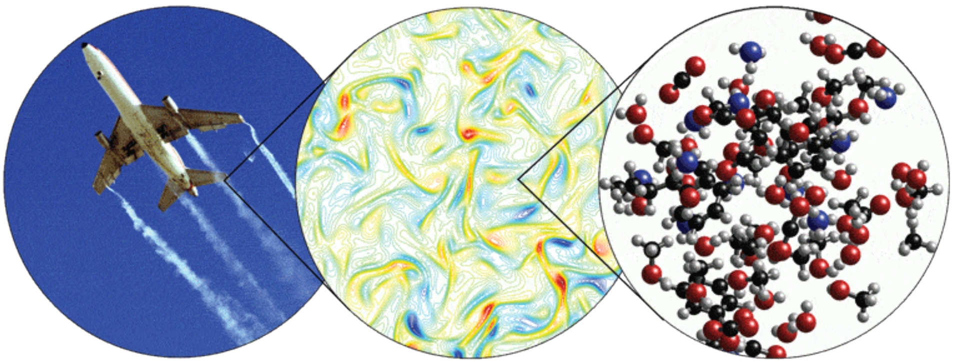
KTH Engineering Sciences

# Multiscale Methods for Highly Oscillatory Ordinary Differential Equations

**Background.** To describe a physical situation, different models must be used depending on the type of information that is of interest and the accuracy required. One can order the models in a hierarchical way, from very detailed microscopic models (e.g. an atomistic description of fluids) to coarser macroscopic models describing averaged quanties (e.g. density, mean velocity). In general, macroscopic models contain fewer degrees of freedom and are less expensive to simulate in a computer than microscopic models.

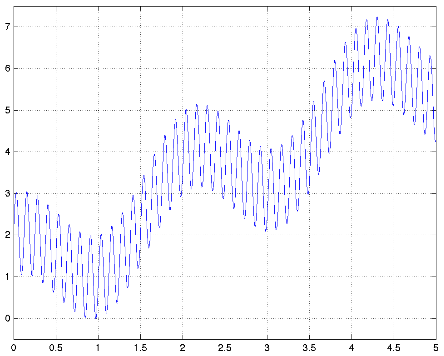


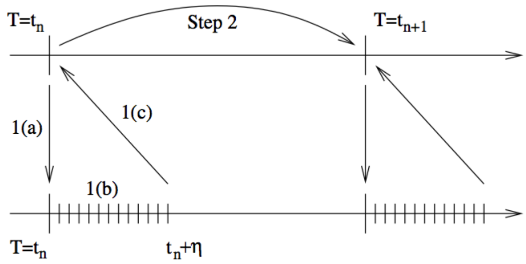
Multiscale problems refer to problems where microscopic and macroscopic models need to be coupled. Such problems are common in areas like physics, chemistry and biology. The problems have multiple time and length scales, such that fine scales cannot be ignored since they have a significant impact on the macro scale dynamics. They can, however, not be solved by direct simulation of the full microscopic model, since this would be too computationally expensive. The numerical treatment of such problems is therefore very difficult.

In this project we will consider multiscale problems described by ordinary differential equations with multiple time scales. Mathematically, we consider the ODE

$$\frac{du_\varepsilon}{dt} = f_\varepsilon(t, u_\varepsilon), \qquad u_\varepsilon(0) = u_0,$$

where the vector valued solution $u_\varepsilon(t) \in \mathbb{R}^n$ consists of fast oscillations with period $\varepsilon \ll 1$, superimposed on a slowly varying function; see left figure for example. There are many examples of such problems and in the project a few of them will be considered. The numerical difficulty with this multiscale structure is here manifested in the following way: the solution is sought for a time interval of size $O(1)$ but the time step $\Delta t$ in the ODE solvers must be taken smaller than $\varepsilon$ to get an accurate result. The computational cost is therefore $O(1/\varepsilon)$ which is very high, since $\varepsilon$ is small.




To solve these problems we will use a *multiscale method*. The computational cost of the method can be made almost independent of $\varepsilon$, and it is hence much faster than standard methods for small $\varepsilon$. The method is based on the framework of *heterogeneous multiscale methods* (HMM). The main idea of such methods is to couple a macroscopic solver, using long time steps ($\gg \varepsilon$), with a microscopic solver, which uses short time steps ($< \varepsilon$), but which only compute the solution in small time-windows. The solver structure is outlined in the right figure.

**Tasks.** (Preliminary list. May change somewhat depending on the interests of the students.)

- Learn about the background of the mathematical problem and some alternative numerical methods for solving it.
- Implement HMM for a model problem. Compare with a direct solver that fully resolves the ODE and verify the method's accuracy and computational cost empirically.
- Study the basic theory of numerical methods for ODEs. and how it can be used to analyse and understand the properties of HMM.
- Implement HMM for a more difficult problem:
  - (Alt 1) The 3-body problem describing the sun, the earth and the moon.
  - (Alt 2) The inverted pendulum problem.
  - (Alt 3) The Fermi–Pasta–Ulam problem.

  Study the accuracy and costs both empirically and theoretically.

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## References


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[3] R. Sharp, Y.-H. Tsai, and B. Engquist. Multiple time scale numerical methods for the inverted pendulum problem. In *Multiscale methods in science and engineering*, volume 44 of Lect. Notes Comput. Sci. Eng., pages 241–261. Springer, Berlin, 2005.