COMPUTING THE STABILITY REGION IN DELAY-SPACE OF A TDS USING POLYNOMIAL EIGENPROBLEMS

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Abstract: In this work we analyze stability properties of retarded linear time invariant multi-dimensional, multi-delay, time delay systems with respect to perturbations in the delay parameters. We analyze two methods which allow the computation of the critical delays, i.e., the points in delay-space which causes the system to have a purely imaginary eigenvalue. The critical delays are potential stability boundaries as the boundaries of the stability region is necessarily a subset of the critical delays.
The two methods originates from a Lyapunov-type condition, which is completely self-contained in this work. The first method corresponds to the case of commensurate delays, for which the the Lyapunov-type condition reduces to a polynomial eigenvalue problem for which the first companion form is exactly the eigenvalue problem occurring in Chen *et al.* (1995). The second method is the result of a simple substitution which allows the computation of the critical delays of an incommensurate system by solving a quadratic eigenvalue problem. For the scalar multi-delay case we find a closed expression for the critical curves using this method. We confirm the methods by comparing it to previous work and published examples.

Keywords: Time-delay systems, critical delays, quadratic eigenproblems, Kronecker products, stability, stability chart

1. INTRODUCTION

In this work we consider linear multi-dimensional retarded *time delay systems* of incommensurate and commensurate type. The boundary of the stability region of the TDS in the delay-parameter space is of particular interest. We address the problem of computing this boundary.

The analysis of the stability region in delay space and delay-dependent stability conditions has, for special cases, received a lot of attention the last decades, for instance the treatment of two delay scalar equation (see Hale and Huang (1993) and references therein). A geometric intepretation of the two-delay problem is considered in Gu and Niculescu (2005). The multi-dimensional one-delay case is treated in Louisell (2001). In Chen *et al.* (1995) the stability region for the commensurate case, i.e., the case where the delays are integer multiples of each other, is computed. It is extended in Niculescu *et al.* (2005) by providing a way to determine the stability switching direction. The large amount of literature about conservative bounds of the stability region, often using *linear matrix inequalities*, is well described in Niculescu (2001) and Gu *et al.* (2003) which also contain more thorough descriptions of methods to analyze stability. For more special cases, see the survey article Sipahi and Olgac (2005*a*). The only existing method finding the boundaries of the stability region for multi-dimensional, incommensurate case is presented in a series of recent papers by Sipahi and Olgac Sipahi and Olgac (2005*b*), Sipahi and Olgac (2003) and Olgac and Sipahi (2002),

In this work we focus on finding the conditions on the delays such that the system is critical, i.e. has an imaginary eigenvalue. It is clear from continuity that this generates the potential boundaries of the stability region in delay space. For many applications the main stable region contains the origin, i.e. the corresponding delay-free system, and is hence easy to identify. We note, however, that if there are more than one stable region, such as the case with *stabilizing delays*, it is not enough to know the critical delays to do a complete stability assessment. For those cases there is a need for a more systematic identification approach, for instance exploiting some *root invariance properties* (see Sipahi and Olgac (2005*b*)).

Here, similar to Sipahi and Olgac (2005*b*), we treat the general time-delay system for the multi-dimensional multi-delay case. But the approach here is completely different. In Sipahi and Olgac (2005*b*) the critical delays (called kernel and offspring curves) are found by making a Rekasius substitution. In the method presented here we define a Lyapunov type matrix operator which, under simple conditions, share roots with the characteristic equation. Moreover, the operator turns out to have a particularly simple structure on the boundary of the stability region, which makes it possible to vectorize it and rewrite it into a polynomial eigenvalue problem. This allows us to apply the rich theories on eigenvalue problems, and in particular computationally efficient iterative methods for eigenvalue problems.

The introduction of the Lyapunov-operator and the corresponding theorems are justified by the fact that when the system is commensurate, the vectorized version of the operator turns into the eigenvalue problem occurring in Chen *et al.* (1995).

We put the method into context by comparing it to the method of Chen *et al.* (1995) and the method of Louisell (2001). We also see that the method is consistent with the theory for the one delay scalar case, for which the (known) explicit expression is found. As a byproduct we also find an explicit parameterization of the boundary of the stability region for the scalar case. We note that an approach, similar to the one here, is taken in Ergenc *et al.* (2006).

This paper is organized as follows. Section 2 defines the problem and some the concepts characteristic eigenproblem and critical curves. Section 3 contains the definition of the Lyapunov type operator as well as the main theorems allowing parameterization of the boundary of the stability region. In Section 4 we apply the method to some examples. We stress that the main contribution of this work is the method presented in Section 3.2 and Section 3.3.

2. DEFINITIONS

The retarded linear m-delay TDS is described by

$$\Sigma = \begin{cases} \dot{x}(t) = \sum_{k=0}^{m} A_k x(t-h_k),\ t > 0 \\ x(t) = \varphi(t),\ t \in [-h_m, 0] \end{cases}$$

with $h_0 = 0 < h_1 < \ldots < h_m$, $x : [-h_m, \infty) \mapsto \mathbb{R}^n$ and $A_k \in \mathbb{R}^{n \times n}$. We will sometimes denote the system Σ with $\Sigma(h_1, \ldots, h_m)$ in order to stress the dependence on the delays.

Definition 1. The *characteristic eigenvalue problem* of Σ is

$$\mathbb{M}(s)v := \left(-sI_n + \sum_{k=0}^{m} A_k e^{-h_k s} \right) v = 0,\ ||v|| = 1, \tag{1}$$

where $v \in \mathbb{C}^n$ is called *eigenvector* and $s \in \mathbb{C}$ an *eigenvalue*. The set of all eigenvalues $\sigma(\Sigma)$ is called the *spectrum*.

This is equivalent to the more common eigenvalue definition $\det(\mathbb{M}(s)) = 0$. We select this definition because we can then save a lot (computationally) in (8) by exploiting the eigenvector structure.

Similar to the delay-free case, a system is exponentially stable if and only if all eigenvalues lie in the open left complex half-plane, i.e. $\sigma(\Sigma) \subset \mathbb{C}_-$. An essential difference is that, unlike the (delay-free) dynamical systems, the spectrum contains a countably infinite number of eigenvalues. Fortunately, it can be proven (see for instance Hale (1977)) that there are only a finite number of eigenvalues in $\mathbb{C}_+$.

From continuity it is clear that the TDS at the boundary of any stability region in the delay-parameter space $h_1, h_2, \ldots, h_m$, has at least one purely imaginary eigenvalue. This justifies the following definitions, inspired by the use of the word *critical* in for instance Plischke (2005) and Gu and Niculescu (2000).

Definition 2.

(1) Σ is called *critical*[1] if and only if $\sigma(\Sigma) \cap i\mathbb{R} \neq \emptyset$.
(2) The set of all points in delay-parameter space $(h_1, h_2, \ldots, h_m)$ for which $\Sigma(h_1, \ldots, h_m)$ is

[1] In the context of nonlinear differential equations a critical system is normally called *non-hyperbolic* (at a fixed point).

critical are called the *critical curves* ($m = 2$) and *critical surfaces* ($m > 2$).[2]

The stability region in the delay-parameter space is bounded by critical surfaces. The rest of this paper will, for that reason, deal with the computation of critical surfaces.

3. RESULTS

First we introduce an operator of Lyapunov-type.

Definition 3. Let

$$\mathbb{L}(X, s) := [\mathbb{M}(s)]\, X + X\, [\mathbb{M}(s)^*] = \\ = \sum_{k=0}^{m} \left(A_k X e^{-h_k s} + X A_k^T e^{-h_k \bar{s}} \right) - 2X \mathrm{Re}\ s,$$

where * denotes complex conjugate transpose.

Note that for the critical case Re $s = 0$, the linear term disappears and the $\mathbb{L}$ operator reduces to a sum of exponential functions.

The following lemma characterizes eigenpairs using the Lyapunov operator.

Theorem 4. Given $s \in \mathbb{C}$ and $v \in \mathbb{C}^n$, $v^* v = 1$ the following are equivalent.

$$\mathbb{M}(s)v = 0 \tag{2}$$

$$\mathbb{L}(vv^*, s) = 0 \wedge v^*\mathbb{M}(s)v = 0 \tag{3}$$

Proof: The forward implication is trivial from definitions, i.e., (1) and (3). The backward implication is clear from the following equalities.

$$\mathbb{M}(s)v = \mathbb{M}(s)vv^*v = \\ = \left(\mathbb{L}(vv^*, s)vv^* - vv^*\mathbb{M}(s)^*vv^*\right) v = 0$$

□

We now characterize the difference of this theorem and the one-delay matrix pencil approach in Louisell (2001). An important property of the operator $\mathbb{L}$ is that, for the critical case, it contains only exponential terms and no linear terms. This is different from for instance the equation considered in Louisell (2001) and (Plischke, 2005, Chapter 6) where the exponential terms are eliminated to form an expression containing only polynomial terms. In the context of the operators here, and in slightly different form, Theorem 3.1 in Louisell (2001) (assuming the system is retarded, i.e., $B = 0$) is

Theorem 5. Given $s \in \mathbb{C}$, $v \in \mathbb{C}^n$, $v \neq 0$ and $\mathbb{M}(s)v = 0$ then

$$\left(\mathbb{M}(s) - e^{-sh_m} A_m\right) vv^* \left(\mathbb{M}(s)^* - e^{-\bar{s}h_m} A_m^T\right) \\ - e^{-2h_m \mathrm{Re}\ s} A_m vv^* A_m^T = 0 \tag{4}$$

Proof: This can be verified by expanding the first product. The expansion yields terms which all but one contain $\mathbb{M}(s)v$. □

For the one delay critical case, i.e., $m = 1$, $s \in i\mathbb{R}$, the exponential terms cancel and (4) reduces to

$$(sI - A_0)vv^*(sI + A_0^T) + A_1 vv^* A_1^T = 0.$$

In Louisell (2001) this is vectorized to form a quadratic eigenvalue problem and solved with the companion form (see Section 3.2). We note however that, unlike the method presented here, this type of reduction is, in the current form, only applicable for one delay systems.

We now return to Theorem 4 and provide two ways to apply the theorem. In Section 3.1 we consider the commensurate case, which (as stated earlier) is closely related to the method in Chen *et al.* (1995). In Section 3.2 we consider the general (incommensurate) problem and make a substitution which turns the Lyapunov-condition into a quadratic eigenproblem.

3.1 Commensurate delays

Theorem 6. Let $h \in \mathbb{R}_+$ and

$$\bar{h} = (h_1, \ldots, h_m) = (hn_1, hn_2, \ldots, hn_k),$$

be a point in delay-parameter space, with $n_k \in \mathbb{N}_+$, $n_0 = 0$ and let $m' = \max n_k$. The point $\bar{h}$ lies on the critical surface if and only if, for some $\varphi \in \mathbb{R}$, $v \in \mathbb{C}^n$ and $\omega \in \mathbb{R}$, $v^*v = 1$, $\zeta = e^{i\varphi}$,

$$\sum_{k=0}^{m} \left(A_k vv^* \zeta^{m'-n_k} + vv^* A_k^T \zeta^{m'+n_k} \right) = 0, \tag{5}$$

$$i\omega = v^* \left(\sum_{k=0}^{m} A_k \zeta^{-n_k} \right) v$$

and

$$h_k = n_k \frac{\varphi}{\omega},\ k = 1 \ldots m.$$

Proof: This follows from Theorem 4 if we let $\zeta = e^{ih\omega}$, and multiply (3) with $\zeta^{m'}$. □

We now note that the vectorized version of (5) is a *polynomial eigenvalue problem* of degree $2m'$.

[2] The critical curves (surfaces) are called *offspring curves* and *kernel curves* in Sipahi and Olgac (2005*b*) and *Hopf bifurcation curves (surfaces)* in Hale (1991).

Polynomial eigenvalue problems can be solved by a transformation to a linear eigenvalue problem, the most common transformation being the companion form. The companion form and generalizations thereof are analyzed in Mackey *et al.* (2005).

We also note the following: The commensurate case, i.e., $h_k = hk$, is included in Theorem 6. In this case, the first companion form of the vectorised version of (5) is exactly the eigenvalue problem occurring in Chen *et al.* (1995) and Niculescu *et al.* (2005). However, in that context it is not clear that the eigenvector of the polynomial eigenvalue problem u is the vectorization of an Hermitian rank one matrix, i.e. $u = \text{vec } vv^*$, which should be used when constructing an efficient method for large dimensional problems.

3.2 A quadratic eigenproblem approach

Theorem 7. Let $\bar{h} = (h_1, \ldots, h_m)$ be a point in delay-parameter space. $\bar{h}$ lies on the critical surface if and only if there are some $\varphi_k \in [-\pi, \pi], k = 1, \ldots, m-1$, $z \in \mathbb{C}$ on the unit circle, $\omega \in \mathbb{R}$, $v^*v = 1$ such that (z, v) is a solution of the equation

$$z^2 vv^* A_m^T + z\left(\sum_{k=0}^{m-1} A_k vv^* e^{-i\varphi_k} + vv^* A_k^T e^{i\varphi_k}\right) + A_m vv^* = 0 \quad (6)$$

$$\omega = -iv^*\left(A_m z^{-1} + \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}\right) v,$$

and $h_m\omega = \text{Arg } z + 2p_m\pi$, $h_k\omega = \varphi_k + 2p_k\pi$, $k = 1, \ldots, m-1$ for some $p_k \in \mathbb{Z}, k = 1 \ldots m$.

Proof: This follows from Theorem 4 by choosing $z = e^{ih\omega}$ and multiplying (3) with z. □

From this theorem we can parameterize the critical surface using the $m-1$ free parameters $\varphi_k \in [-\pi, \pi]$, $k = 1, \ldots, m-1$. Similar to the previous section (6) can be vectorized to a quadratic eigenvalue problem

$$\left(z^2 A_m \otimes I + z\sum_{k=0}^{m-1} L_k(\varphi_k) + I \otimes A_m\right) u = 0, \quad (7)$$

where where $u = \text{vec } vv^*$ and $L_k(\varphi_k) = I \otimes A_k e^{-i\varphi_k} + A_k \otimes I e^{i\varphi_k}$. Again, quadratic eigenvalue problem can be solved by a transformation to a generalized eigenvalue problem using the companion form. For a survey on quadratic eigenvalue problems see Tisseur and Meerbergen (2001). For completeness we state the first companion form:

All solutions of equation (6) can be found by solving the following equivalent eigenvalue problem

$$\begin{pmatrix} 0 & I \\ I \otimes A_m & \sum_{k=0}^{m-1} L_k(\varphi_k) \end{pmatrix} \begin{pmatrix} u \\ zu \end{pmatrix} = z \begin{pmatrix} I & 0 \\ 0 & -A_m \otimes I \end{pmatrix} \begin{pmatrix} u \\ zu \end{pmatrix}. \quad (8)$$

The dimension of this eigenproblem is $2n^2 \times 2n^2$ which can be very large even for problems of moderate size. By exploiting the structure the of the matrices one can reduce the computational cost considerably (see Jarlebring (2006)).

For the one-dimensional case, the quadratic eigenvalue problem reduces to the problem of finding roots of a quadratic equation. The theorem can then be simplified to

Corollary 8. Let $\bar{h} = (h_1, \ldots, h_m)$ be a point in delay-parameter space for the one-dimensional TDS with $A_k = a_k$. $\bar{h}$ lies on the critical surface if and only if there are some $\varphi_k \in [-\pi, \pi], k = 1, \ldots, m-1$, and

$$h_m = \frac{\text{atan}\left(\frac{\omega + \sum_{k=1}^{m-1} a_k \sin\varphi_k}{a_0 + \sum_{k=1}^{m-1} a_k \cos\varphi_k}\right) + 2p_m\pi}{\omega}$$

$$h_k = \frac{\varphi_k + 2p_k\pi}{\omega}, \; k = 1, \ldots, m-1,$$

$$\omega = \pm\sqrt{a_m^2 - \left(a_0 + \sum_{k=1}^{m-1} a_k \cos\varphi_k\right)^2} - \sum_{k=1}^{m-1} a_k \sin\varphi_k$$

for some $p_k \in \mathbb{Z}, k = 1 \ldots m$.

Here, $\text{atan}\left(\frac{a}{b}\right) = \text{Arg } (b + ai)$, i.e., the four quadrant inverse tangent.

For the single delay, scalar case (principal branch), the corollary reduces to the following expression for the stability margin

$$h = \frac{\text{atan}\left(\frac{\sqrt{b^2-a^2}}{a}\right)}{\sqrt{b^2 - a^2}} = \frac{\text{acos}\left(-\frac{a}{b}\right)}{\sqrt{b^2 - a^2}}.$$

This is exactly the expression in Niculescu (2001) section 3.4.1.

3.3 Method to generate critical surfaces

One interpretation of Theorem 6 is that it generates the critical delays for a commensurate time-delay system, exactly the way it is computed in Chen *et al.* (1995).

We now focus on how we can use Theorem 7 to generate the critical surfaces, which contains potential boundaries of the stability region in delay space.

Theorem 7 is an equivalence theorem between $\varphi_1, \ldots, \varphi_{m-1}$ and $h_1, \ldots, h_m$. Hence, we can see it as a parametrization of the critical surface. With this in mind, we state in pseudo-code the code to generate the critical curves for the two-delay system.

```
1.  FOR φ = −π : Δ : π
2.      Find eigenpairs (z_k, u_k) of (8)
3.      FOR k = 1 : length(z)
4.          IF z_k is on unit circle
5.              Compute v_k such that u_k = vec v_k v_k^*
6.              Compute ω_k = −i v_k^* (A_2 z_k^{-1} + A_0 + A_1 e^{−iφ}) v_k
7.              Accept critical points (h_1, h_2)
```

$$h_1 = \frac{\varphi + 2p\pi}{\omega_k},\ p = -p_{max}, \ldots, p_{max}$$
$$h_2 = \frac{\text{Arg } z_k + 2q\pi}{\omega_k},\ q = -p_{max}, \ldots, p_{max}$$

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8.          END
9.      END
10. END
```

In step 1, Δ is the stepsize of the parameter φ. In step 7, p_{max} is the number of branches which should be included in the computation. Step 7 is not computationally demanding. We can therefore select p_{max} so large that the computation contains all relevant branches. The generalization to more than two delays is straighforward. It involves a nesting of the outer iteration (step 1) with for-loops of the new free variables φ_k and computing the other delays in step 7 similar to h_1.

4. EXAMPLES

Example ($n = 1$, $m = 2$)
Consider the the 1-dimensional two delay system,

$$\dot{x}(t) = a_0 x(t) + a_1 x(t - h_1) + a_2 x(t - h_2),$$

studied in for instance Nussbaum (1978), Hale and Huang (1993) and Gu and Niculescu (2005). The application of Corollary 8 this system yields critical curves with the following parameterization

$$h_{p,q}(\varphi) = \begin{pmatrix} \dfrac{\varphi + 2q\pi}{\sqrt{a_2^2 - (a_0 + a_1\cos\varphi)^2} - a_1 \sin\varphi} \\ \dfrac{\text{atan}\left(\frac{\sqrt{a_2^2-(a_0+a_1\cos\varphi)^2}}{a_0+a_1\cos\varphi}\right) + 2p\pi}{\sqrt{a_2^2 - (a_0 + a_1\cos\varphi)^2} - a_1 \sin\varphi} \end{pmatrix}$$

where $\varphi \in [-\pi, \pi]$, $a_2^2 - (a_0 + a_1 \cos\varphi)^2 > 0$, $p, q \in \mathbb{Z}$. Fig. 1 shows the critical curves for $a_0 = 0.5$, $a_1 = -0.9$ and $a_2 = -1.5$.

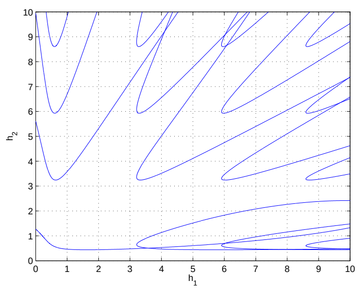

Fig. 1. Critical curves for Example 1

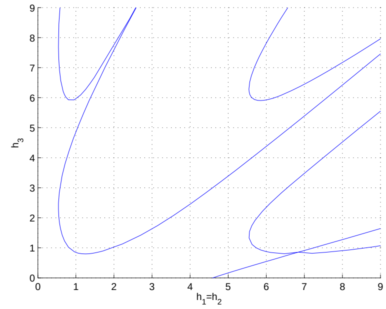

Fig. 2. Critical curves for Example 2

Example (n=4, m=3)
In Chen and Latchman (1995) the following TDS is considered

$$\dot{x}(t) = Ax(t) + B_1 x(t-h_1) + B_2 x(t-h_2) - B_3 x(t-h_3)$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{pmatrix},\ B_1 = \frac{1}{200}\begin{pmatrix} -10 & 1 & 50 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -200 & 0 & -100 & 0 \end{pmatrix},$$

$$B_2 = \frac{1}{2000}\begin{pmatrix} 10 & 5 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \\ -2000 & -1000 & -1000 & 0 \end{pmatrix},$$

$$B_3 = \frac{1}{400}\begin{pmatrix} 15 & 0 & 30 & 50 \\ 0 & 20 & 20 & 0 \\ 20 & 20 & 0 & 0 \\ 0 & -1000 & 0 & -400 \end{pmatrix}.$$

For visualization purposes we consider $h_1 = h_2$. The critical curves of the two-delay system are plotted in Fig. 2, where the only stable region is easily identified as the (unbounded) region containing the origin.

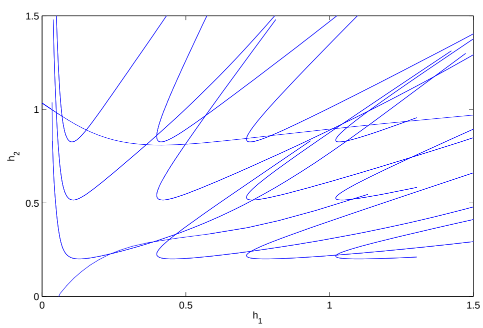

Fig. 3. Critical curves for Example 3

Example ($n = 3$, $m = 2$ from Sipahi and Olgac (2005b))

Consider the 3-dimensional system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2)$$

where

$$A_0 = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix}, \; A_1 = \begin{pmatrix} -5.9 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 7.1 & -70.3 \\ 0 & -1 & 5 \\ 0 & 0 & 6 \end{pmatrix},$$

The critical curves are shown in Fig. 3. Here, the region containing the origin is stable and easily identified. There is however a second stable region (around $h_1 = 0.4, h_2 = 0.3$) which is not trivially identifiable (see Sipahi and Olgac (2005*b*)).

5. CONCLUSIONS

We have studied the conditions on the delay parameters of retarded time delay systems such that the system is critical. This was done by introducing a condition on a Lyapunov type operator. The condition is transformed into conditions on the delay parameters using polynomial eigenvalue problems, which is solved using the companion form. The Lyapunov type condition is shown to reduce to existing methods for several special cases.

For the scalar time delay system we find a closed expression for the critical curves. In the examples section we apply the method to several previously published examples.

ACKNOWLEDGMENT

The author wish to thank Nejat Olgac for providing constructive comments and completion of the references.

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