COMPUTING CRITICAL DELAYS FOR TIME DELAY SYSTEMS WITH MULTIPLE DELAYS

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Abstract
In this work we present a method to analyze the robustness of stability of a *time-delay system* (TDS) with respect to the delays. This is done by computing the delays for which the system has a purely imaginary eigenvalue. These delays, called *critical delays*, generate potential points for a stability switch, i.e., the point where the system switches from a stable to unstable. To derive the method, we find a *Lyapunov-type equation*, equivalent to the characteristic equation of the TDS. Unlike the characteristic equation, the Lyapunov-type equation does not have any non-exponential terms if the eigenvalue is imaginary. This allows us to solve the Lyapunov-type equation by rewriting it to a *quadratic eigenvalue problem* for which there are efficient numerical methods. For the scalar case, we find a new explicit expression for the curves in the stability chart. The method is applied to previously solved examples as well as previously unsolved problems of larger dimension.

Keywords: Multiple time delay, Critical delays, Hopf bifurcation, Robustness, Stability, Lyapunov operators, Polynomial eigenvalue problems

1. INTRODUCTION

In this work we consider linear time-delay systems with multiple constant delays. These types of systems occur in models where there is a non-negligible delay, normally originating from some physical limitation, for instance finite switching times in controllers, unavailability of the current state of the system, etc. The stability properties of the system are highly dependent on the size and the relations of the delays. We are therefore led to analyze the stability region in delay space.

The analysis of the stability region in delay space and delay-dependent stability conditions has, for special cases, received a lot of attention the last decades, for instance the treatment of two delay scalar equation [HH93] and references therein. The multi-dimensional one-delay case is treated by Louisell in [Lou01]. Chen et al [CGN95] compute the stability region for the commensurate case, i.e., the case where the delays are integer multiples of each other. The large amount of literature about conservative bounds of the stability region, often using *linear matrix inequalities*, is well described in [Nic01] and [GKC03] which also contain more thorough descriptions of methods to analyze stability. For more special cases, see the survey article [SO05a]. The only existing method finding the boundaries of the stability region for multi-dimensional, incommensurate case is presented in the recent paper by Sipahi and Olgac [SO05b].

Here, similar to [SO05b], we address the general problem to find the conservative bounds for the stability region for the multi-dimensional multi-delay case. But the approach here is completely different. In [SO05b] the characteristic equation is solved by making a Rekasius substitution and treating the resulting (high order) parameterized polynomial with a Routh's array. In the method presented here, we define a Lyapunov type matrix operator which, under simple conditions, share roots with the characteristic equation. Moreover, the operator turns out to have a particularly simple structure on the boundary of the stability region, which makes it possible to rewrite it into a quadratic eigenvalue problem. This allows us to apply the rich theories on eigenvalue problems, and in particular computationally efficient iterative methods for eigenvalue problems.

We also see that the method is consistent with the theory for the one delay scalar case, for which the (known) explicit expression is found. As a byproduct we also find an explicit parameterization of the boundary of the stability region for the scalar case.

This document is organized as follows. Section 2 defines the problem and some the concepts characteristic eigenproblem and critical curves. Section 3 contains the definition of the Lyapunov type operator as well as the main theorems allowing parameterization of the boundary of the stability region. In Section 4 we provide three examples, one with an analytic solution, one from [SO05b] and one problem with larger dimension.

2. BASIC CONCEPTS

The retarded linear m-delay TDS is described by

$$\Sigma = \begin{cases} \dot{x}(t) = \sum_{k=0}^{m} A_k x(t-h_k),\ t > 0 \\ x(t) = \varphi(t),\ t \in [-h_m, 0] \end{cases} \tag{1}$$

with $h_0 = 0 < h_1 < \ldots < h_m$, $x : [-h_m, \infty) \mapsto \mathbb{R}^n$ and $A_k \in \mathbb{R}^{n \times n}$. We will sometimes denote the system Σ with $\Sigma(h_1, \ldots, h_m)$ in order to stress the dependence on the delays.

Definition 1. The *characteristic eigenvalue problem* of Σ is

$$\mathbb{M}(s)v := \left(-sI_n + \sum_{k=0}^{m} A_k e^{-h_k s}\right) v = 0,\ ||v|| = 1, \tag{2}$$

where $v \in \mathbb{C}^n$ is called *eigenvector* and $s \in \mathbb{C}$ an *eigenvalue*. The set of all eigenvalues $\sigma(\Sigma)$ is called the *spectrum*.

Similar to the delay-free case, a system is exponentially stable if and only if all eigenvalues lie in the open left complex half-plane, i.e. $\sigma(\Sigma) \subset \mathbb{C}_-$. An essential difference is that, unlike the (delay-free) dynamical systems, the spectrum contains a countably infinite number of eigenvalues. Fortunately, it can be proven (see for instance [Hal77]) that there are only a finite number of eigenvalues in $\mathbb{C}_+$.

From continuity it is clear that the TDS at the boundary of any stability region in the delay-parameter space $h_1, h_2, \ldots, h_m$, has at least one purely imaginary eigenvalue. This justifies the following definitions, inspired by the use of the word *critical* in for instance [Pli05] and [GN00].

Definition 2.

(1) Σ is called *critical*[1] if and only if $\sigma(\Sigma) \cap i\mathbb{R} \neq \emptyset$.
(2) The set of all points in delay-parameter space $(h_1, h_2, \ldots, h_m)$ for which $\Sigma(h_1, \ldots, h_m)$ is critical are called the *critical curves* $(m = 2)$ and *critical surfaces* $(m > 2)$.[2]

The stability region in the delay-parameter space is bounded by critical surfaces. The rest of this article will, for that reason, deal with the computation of critical surfaces.

3. RESULTS

The computation of the critical surfaces is done by introducing the following operator of Lyapunov-type.

Definition 3. Let

$$\begin{aligned} \mathbb{L}(X, s) &:= \mathbb{M}(s)X + X\mathbb{M}(s)^* = \\ &= \sum_{k=0}^{m} \left(A_k X e^{-h_k s} + X A_k^T e^{-h_k \bar{s}}\right) - 2X\mathrm{Re}\ s, \end{aligned} \tag{3}$$

where * denotes complex conjugate transpose.

Note that for the critical case Re $s = 0$, the linear term disappears and the $\mathbb{L}$ operator reduces to a sum of exponential functions.

The following theorem characterizes eigenpairs using the Lyapunov operator.

Theorem 4. Given $s \in \mathbb{C}$ and $v \in \mathbb{C}^n$, $v^*v = 1$ the following are equivalent.

$$\mathbb{M}(s)v = 0 \tag{4}$$

$$\mathbb{L}(vv^*, s) = 0 \wedge v^*\mathbb{M}(s)v = 0 \tag{5}$$

Proof: The forward implication is trivial from definitions, i.e., (2) and (3). The backward implication is clear from the following equalities.

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[1] In the context of nonlinear differential equations a critical system is normally called *non-hyperbolic* (at a fixed point).

[2] The critical curves (surfaces) are called *offspring curves* and *kernel curves* in [SO05b] and *Hopf bifurcation curves (surfaces)* in [Hal91].

$$\mathbb{M}(s)v = \mathbb{M}(s)vv^*v = \\ = (\mathbb{L}(vv^*, s)vv^* - vv^*\mathbb{M}(s)^*vv^*)\, v = 0$$

□

Note 1. It is shown in previous work by the author [Jar05] that the method of Chen, et al [CGN95] is in fact the application of Theorem 4 to a commensurate time delay system.

3.1 Quadratic eigenproblem

As stated earlier, we will now consider the critical case, i.e $s = i\omega$. By making the substitutions $z = e^{i\omega}$ and $\varphi_k = \omega h_k$, $k = 0, \ldots, m-1$ equation (5) reduces to a quadratic relation which turns out to be a quadratic eigenvalue problem. This is demonstrated in the following theorem which allows parameterization of the $(m-1)$-dimensional critical surface.

Theorem 5. Let $\bar{h} = (h_1, \ldots, h_m)$ be a point in delay-parameter space. $\bar{h}$ lies on the critical surface if and only if there are some $\varphi_k \in [-\pi, \pi]$, $k = 1, \ldots, m-1$, $z \in \mathbb{C}$ on the unit circle, $\omega \in \mathbb{R}$ such that z and v is a solution to the equation

$$z^2 vv^* A_m^T + z\left(\sum_{k=0}^{m-1} A_k vv^* e^{-i\varphi_k} + vv^* A_k^T e^{i\varphi_k}\right) + A_m vv^* = 0 \quad (6)$$

$$\omega = -iv^*\left(A_m z^{-1} + \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}\right) v,$$

and $h_m = \frac{\text{Arg } z + 2p_m\pi}{\omega}$, $h_k = \frac{\varphi_k + 2p_k\pi}{\omega}$, $k = 1, \ldots, m-1$ for some $p_k \in \mathbb{Z}, k = 1 \ldots m$.

Proof: This follows from Theorem 4 by letting $\varphi_k = \omega h_k$ and $z = e^{i\varphi}$. □

From this theorem we can parameterize the critical surface using the $m-1$ free parameters $\varphi_k \in [-\pi, \pi]$, $k = 1, \ldots, m-1$. We now note that (6) can be vectorized into the following vector equation

$$0 = \Bigg(z^2 A_m \otimes I + z\left(\sum_{k=0}^{m-1} I \otimes A_k e^{-i\varphi_k} + A_k \otimes I e^{i\varphi_k}\right) + I \otimes A_m\Bigg) u$$

where $u = \text{vec } vv^* = \bar{v} \otimes v$. This is a *quadratic eigenvalue problem*. For a survey on quadratic eigenvalue problems see [TM01]. Quadratic eigenvalue problems can be solved using the companion form or generalization thereof (see [MMMM05]), which transforms it to a (generalized) eigenvalue problem.

For the one-dimensional case, the quadratic eigenvalue problem reduces to the problem of finding roots of a quadratic equation. The theorem can then be simplified to

Corollary 6. Let $\bar{h} = (h_1, \ldots, h_m)$ be a point in delay-parameter space for the one-dimensional TDS with $A_k = a_k$. $\bar{h}$ lies on the critical surface if and only if there are some $\varphi_k \in [-\pi, \pi], k = 1, \ldots, m-1$, and

$$h_m = \frac{\text{atan}\binom{\omega + \sum_{k=1}^{m-1} a_k \sin\varphi_k}{a_0 + \sum_{k=1}^{m-1} a_k \cos\varphi_k} + 2p_m\pi}{\omega}$$

$$h_k = \frac{\varphi_k + 2p_k\pi}{\omega}, \; k = 1, \ldots, m-1,$$

$$\omega = \pm\sqrt{a_m^2 - \left(a_0 + \sum_{k=1}^{m-1} a_k \cos\varphi_k\right)^2} - \sum_{k=1}^{m-1} a_k \sin\varphi_k$$

for some $p_k \in \mathbb{Z}, k = 1 \ldots m$.

Here, $\text{atan}\binom{a}{b} = \text{Arg } (b + ai)$, i.e., the four quadrant inverse tangent.

For the first positive branch of the one delay case, the corollary reduces to the following well known bound for stability region (see for instance [Nic01] Section 3.4.1)

$$h = \frac{\text{atan}(\frac{\sqrt{b^2-a^2}}{a})}{\sqrt{b^2 - a^2}} = \frac{\text{acos}(-\frac{a}{b})}{\sqrt{b^2 - a^2}}.$$

3.2 Notes on numerical methods

The computationally demanding part when applying the theorem above is finding the solutions of the quadratic eigenvalue problem (6). From a computational point of view, we have the following properties to make use of, when solving the quadratic eigenvalue problem.

- The eigenvalues of interest lie on the unit circle.
- The matrices resulting from the companion form are sparse.
- Only eigenvectors of the form $u = \text{vec } vv^*$, i.e. a vectorization of an hermitian rank one matrix, are of interest.
- The eigenvalues and eigenvectors are continuous with respect to the parameters.
- The quadratic eigenvalue problem has an eigenvalue pairing similar to a *palindromic eigenvalue problem* [MMM05].

In the implementations for the examples in this work, only the first two properties are explicitly used. The continuity property is used to approximate the critical surface by interpolation of points on the surface.

Searching for eigenvalues along the real axis is a much more investigated problem than finding eigenvalues on the unit circle. We therefore make the a *Cayley transformation*, $z = \frac{1+i\sigma}{1-i\sigma}$ which has the inverse transformation $\sigma = \tan(\frac{\mathrm{Arg}\ z}{2})$ on the real axis. Note that $\sigma \in \mathbb{R}$ iff z is on the unit circle. The transformed eigenvalue problem is, $(A-B)v = \sigma(iA+iB)v$, where A and B are the matrices in the generalized eigenvalue problem (on companion form).

We stress that the system is very large but has very few non-zero components, which suggests that sparse representation of the matrix and iterative eigensolver. Finding all real eigenvalues can be done by, for instance rational Krylov type scans along the real axis. One implementation of that is the Matlab-command `sptarn`.

It is shown in other work by the author that the matrix vector of the Caley-transformed companion-form eigenvalue problem can be efficiently computed by solving a *Lyapunov-equation*. Similar results holds for the shift-and-invert operation. This can be exploited to construct an efficient iterative solver.

4. EXAMPLES

Example ($n=1$, $m=2$)
Applying Corollary 6 on the 1-dimensional two delay system,

$$\dot{x}(t) = ax(t) + bx(t-h_1) + cx(t-h_2),$$

studied in for instance [Nus78] and [HH93], yields critical curves with the following parameterization

$$h_{p,q}(\varphi) = \begin{pmatrix} \dfrac{\varphi + 2q\pi}{\sqrt{c^2-(a+b\cos\varphi)^2} - b\sin\varphi} \\ \dfrac{\mathrm{atan}\left(\frac{\sqrt{c^2-(a+b\cos\varphi)^2}}{a+b\cos\varphi}\right) + 2p\pi}{\sqrt{c^2-(a+b\cos\varphi)^2} - b\sin\varphi} \end{pmatrix}$$

where $\varphi \in [-\pi, \pi]$, $c^2 - (a+b\cos\varphi)^2 > 0$, $p, q \in \mathbb{Z}$. Fig. 1 shows the critical curves for the two delay *hot shower problem* $a = 0$, $b = -1$, $c = -2$.

Example ($n=3$, $m=2$ from [SO05b])
Consider the 3-dimensional system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2)$$

where

$$A_0 = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix},\ A_1 = \begin{pmatrix} -5.9 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 7.1 & -70.3 \\ 0 & -1 & 5 \\ 0 & 0 & 6 \end{pmatrix},$$

which is stable for $h_1 = h_2 = 0$. The critical curves for this system are plotted in Fig. 2. The plot is the result of application of Theorem 5 evaluated using sptarn at 100 equally distributed points on $\varphi \in [-\pi, \pi]$. For visibility the resulting points are connected with lines. The production of the plot took less than 1 second on a computer running Linux and Matlab 7 on a 2.4 GHz Intel Pentium 4 processor with 512 Mb RAM. Using the method of Sipahi and Olgac to produce a plot with visually equivalent steplenths requires 19.2 s on a faster computer, see [SO05b]. However, we note that the method of Sipahi and Olgac also computes the number of unstable eigenvalues for each region. The comparison is hence not completely fair.

Example ($n=24$, $m=3$)
In comparison other methods, the method presented here is expected to scale with dimension and number of delays much better. We demonstrate that with the discretization of the three delay partial differential equation,

$$\begin{cases} u_t'(x,t) = u_{xx}''(x,t) + \beta(1+\sin(3\pi x))u(x,t) \\ \qquad - \kappa_0\delta(x-x_0)u(0,t-h_1) \\ \qquad - \kappa_1\delta(x-x_1)u(x_1,t-h_2) \\ \qquad - \kappa_2\delta(x-x_2)u(1,t-h_3) \\ u_x(0,t) = 0 \\ u_x(1,t) = 0, \end{cases} \tag{7}$$

where we pick $\kappa_0 = \kappa_2 = 4$, $\kappa_1 = 10$, $x_0 = 1/3$, $x_1 = 1/2$, $x_2 = 3/4$ and $\beta = 10$. The physical interpretation of equation (7) is the heat equation on a rod with length one with heat production over the whole rod causing instability and three delayed stabilizing pointwise feedbacks. This system is discretized with central difference in space with n equally distributed intervals, yielding a system of the form $\dot{x}(t) = A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2) + A_3x(t-h_3)$, where $x(t) \in \mathbb{R}^n$, i.e., a three delay TDS.

The critical surface closest to the origin of the discretized system is plotted in Fig. 3. The method applies Theorem 5 for 1530 different combinations of the parameters φ_1 and φ_2 with $n = 24$ discretization points.

We have not presented any error estimation. Therefore, we cannot say anything about how the stability region changes when discretized. As we only know the critical surface in some points, connecting the points with planes as done in Fig. 3 may not be an accurate interpolation.

5. CONCLUSIONS

We study the conditions on the delay parameters for multiple delay time delay systems such that the system has an imaginary eigenvalue. This is done by introducing a condition on a Lyapunov type operator. The condition is transformed into conditions on the delay parameters using a quadratic eigenvalue problems, which is solved using the companion form and the iterative eigenproblem solver rational Krylov. For the one dimensional multiple delay system, we find a closed expression for the condition on the delay parameters. In the numerical examples we solve a problem considered in previous literature faster than previous methods and we solve a previously unsolved problem of larger dimension. It is hence demonstrated that exact bounds for stability region in delay parameter space are computable for moderate size problems. We also expect that more adapted numerical methods will allow us to analyze large dimensional problems.

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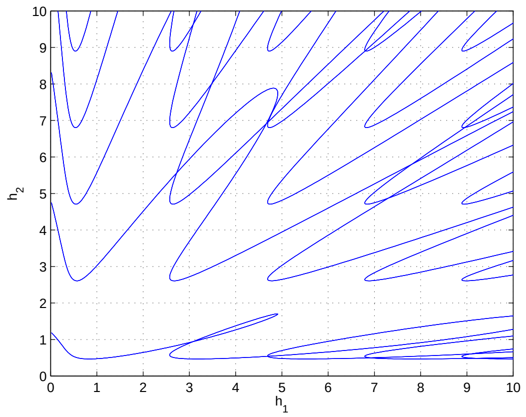

Figure 1. Critical curves for Example 1

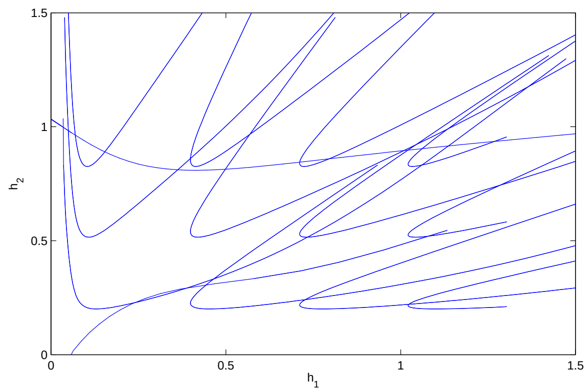

Figure 2. Critical curves for Example 2

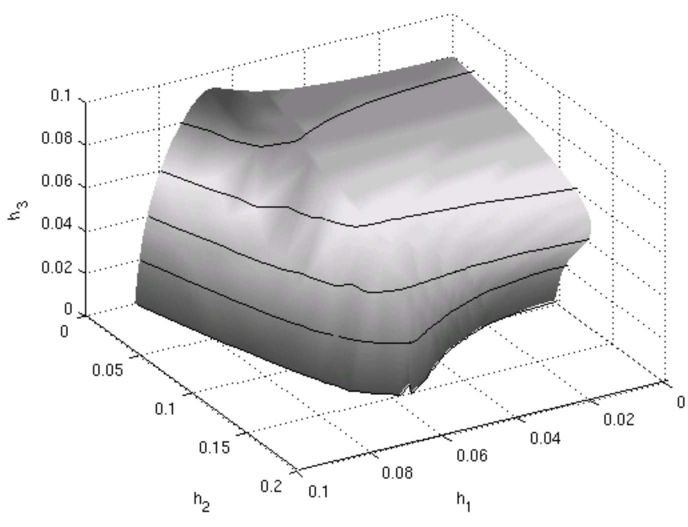

Figure 3. Boundary of stability region for Example 3