The ρ-Loewner Energy

Large deviations, minimizers, and alternative descriptions

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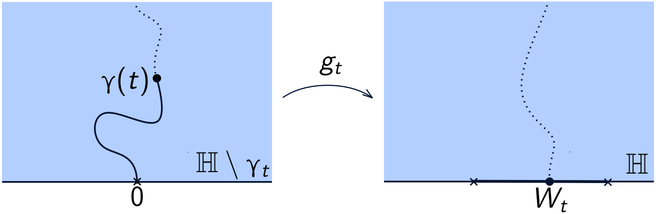
The Chordal Loewner Energy

Chordal Loewner Evolution



- Chord γ in $(\mathbb{H}; 0, \infty)$. Denote $\gamma_t := \gamma([0, t])$.
- **Mapping-out functions** $g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ satisfying, $g_t(z) = z + \frac{a(t)}{z} + o(\frac{1}{z})$ as $z \to \infty$. Parametrize γ s.t. $a(t) = 2t$.
- The **driving function**, $W_t := g_t(\gamma(t))$, encodes γ via the chordal **Loewner differential equation**:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z, \tag{1}$$

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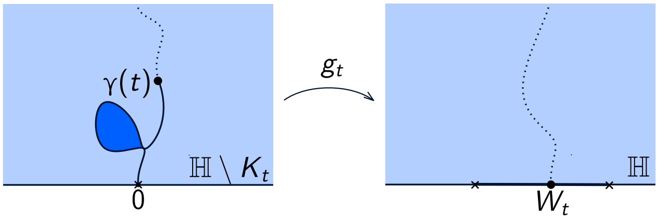

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- Solving (1) for an arbitrary continuous function $t \mapsto W_t$ gives family of locally growing compact sets $(K_t)_{t \geq 0}$.

Schramm-Loewner Evolution

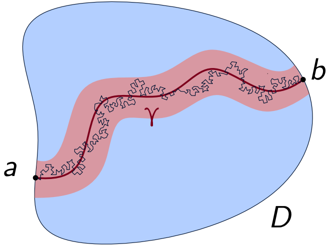
- Chordal **Schramm-Loewner evolution**, SLE_κ, in $(\mathbb{H}; 0, \infty)$ is the random curve driven by $W_t = \sqrt{\kappa} B_t$, $\kappa \geq 0$, where B_t standard Brownian motion [Schramm '00]. A.s. simple when $\kappa \leq 4$.
- Chordal SLE in $(D; a, b)$ is defined by conformal mapping $(D; a, b) \to (\mathbb{H}; 0, \infty)$.
- SLE_κ is the scaling limit of interfaces in planar critical lattice models for several values of $\kappa > 0$.
- SLE_0 is the hyperbolic geodesic in $(D; a, b)$. ($W \equiv 0 \ \rightsquigarrow \ \gamma = i\mathbb{R}^+$.)

Schramm-Loewner Evolution

Chordal SLE_κ satisfies the large deviation principle as $\kappa \to 0+$

$$\text{“}\mathbb{P}[\mathrm{SLE}_\kappa \text{ stays close to } \gamma] \approx \exp\left(-\frac{I^{(D;a,b)}(\gamma)}{\kappa}\right), \text{ as } \kappa \to 0+\text{”}$$

where $I^{(D;a,b)}$ is the **chordal Loewner energy**.



[Wang '19, Peltola-Wang '23, Guskov '23, Abuzaid-Peltola '24]

Chordal Loewner Energy

The **chordal Loewner energy** of a chord γ in $(\mathbb{H}; 0, \infty)$ is defined by

$$I^{(\mathbb{H};0,\infty)}(\gamma) = \frac{1}{2}\int_0^\infty (\dot{W}_t)^2 dt,$$

if W is absolutely continuous, and $I^{(\mathbb{H};0,\infty)}(\gamma) = \infty$ otherwise [Wang '19].

- The unique minimizer of $I^{(\mathbb{H};0,\infty)}$ is SLE_0, that is $i\mathbb{R}^+$.
- $I^{(D;a,b)}$ is defined on chords in $(D; a, b)$ by conformal mapping $(D; a, b) \to (\mathbb{H}; 0, \infty)$.
- Other description(s)?

Dirichlet Energy Formula

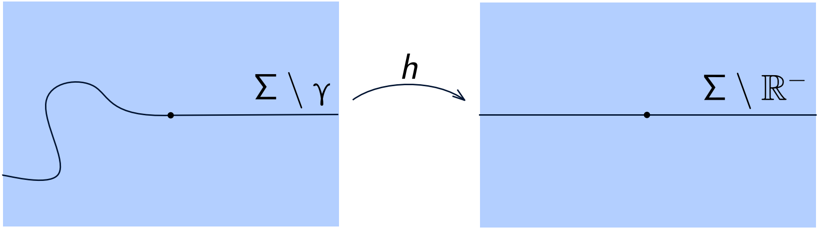
Let $\Sigma = \mathbb{C} \setminus \mathbb{R}^+$ and consider a chord γ in $(\Sigma; 0, \infty)$.

Theorem (Wang '19)

The chordal Loewner energy can be expressed as,

$$I^{(\Sigma;0,\infty)}(\gamma) = \frac{1}{\pi} \int_{\Sigma \setminus \gamma} |\nabla \log |h'||^2 dz^2,$$

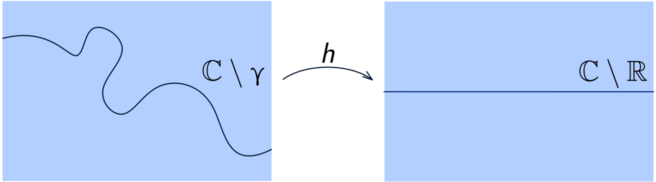
where $h : \Sigma \setminus \gamma \to \Sigma \setminus \mathbb{R}^-$ is conformal and $h(\infty) = \infty$.



Loop Loewner Energy

The **loop Loewner energy** of a Jordan curve γ rooted at ∞ on $\hat{\mathbb{C}}$ is

$$I^L(\gamma) = \frac{1}{\pi}\int_{\hat{\mathbb{C}}\setminus\gamma} |\nabla \log|h'||^2 dz^2.$$



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- I^L is invariant of the root [Rohde-Wang '21].
- Link to **Teichmüller theory**: I^L coincides (up to a multiplicative constant) with the **universal Liouville action** [Wang '19].
- The class of finite Loewner energy loops = The class of **Weil-Petersson quasicircles** [Wang '19].
- Loewner energy also shows up in $n \to \infty$ limit of partition functions of **Coulomb gas** on a Jordan curve/domain [Johansson '22, Johansson-Viklund '23].

$\text{SLE}_\kappa(\rho)$

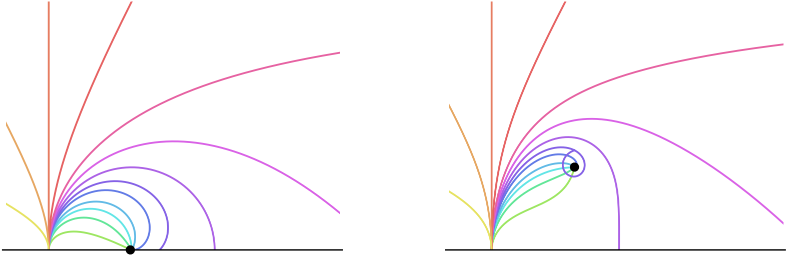
$\mathrm{SLE}_\kappa(\rho)$

- **Chordal $\mathbf{SLE}_\kappa(\rho)$**, $\rho \in \mathbb{R}$, in $(\mathbb{H}; 0, \infty)$, with force point $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$, is the random curve whose driving function satisfies the SDE:

$$dW_t = \mathrm{Re}\frac{\rho}{W_t - z_t}dt + \sqrt{\kappa}dB_t, \quad dz_t = \frac{2}{z_t - W_t}dt,$$

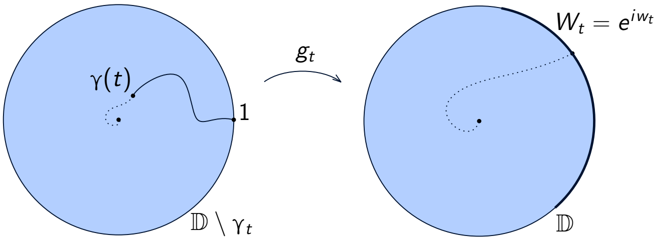
  where $z_t = g_t(z_0)$. Defined up to $\tau = \lim_{\varepsilon \to 0} \inf\{t : |W_t - z_t| \leq \varepsilon\}$.

- $\kappa = 0$:



$\rho =$ -16 -8 -6 -5 -4 -3.5 -3 -2.5 -2 -1 0 2 10

Radial Loewner Evolution



- Slit γ in $(\mathbb{D}; 1, 0)$, parametrized by conformal radius.
- **Mapping-out functions** $g_t : \mathbb{D} \setminus \gamma_t \to \mathbb{D}$, $g_t(0) = 0$, $g_t'(0) = e^t$.
- Radial **Loewner differential equation**,

$$\partial_t g_t(z) = g_t(z) \frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z,$$

  where $W_t = e^{iw_t} = g_t(\gamma(t))$.
- $w_t = \sqrt{\kappa} B_t \rightsquigarrow$ radial SLE_κ in $(\mathbb{D}; 1, 0)$.

$\mathrm{SLE}_\kappa(\rho)$

- **Radial $\mathbf{SLE}_\kappa(\rho)$**, in $(\mathbb{D};1,0)$, with force point $z_0 = e^{iv_0}$, $v_0 \in (0,2\pi)$ is the random curve whose driving function $W_t = e^{iw_t}$ satisfies the SDE:

$$dw_t = \frac{\rho}{2}\cot\left(\frac{w_t - v_t}{2}\right)dt + \sqrt{\kappa}dB_t, \quad dv_t = \cot\left(\frac{v_t - w_t}{2}\right)dt,$$

  where $e^{iv_t} = g_t(e^{iv_0})$.
- First appearance: $\mathrm{SLE}_{8/3}(\rho)$, $\rho > -2$, is the outer boundary of sets satisfying a conformal restriction property [Lawler-Schramm-Werner '03].
- Main player in imaginary geometry: generalized flow-line of GFF [Miller-Sheffield '16, '17].
- Coordinate change property: An $\mathrm{SLE}_\kappa(\rho)$ in $(D;a,b)$ with force point c, is an $\mathrm{SLE}_\kappa(\kappa-6-\rho)$ in $(D;a,c)$ with force point b [Schramm-Wilson '05].

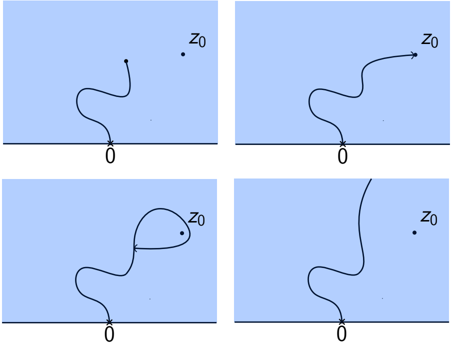
The ρ-Loewner Energy

Chordal ρ-Loewner Energy

- Fix $\rho \in \mathbb{R}$ and $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$. The **chordal ρ-Loewner energy** of a simple curve γ starting at 0 in $\mathbb{H} \setminus \{z_0\}$ is defined as

$$I_{\rho,z_0}^{(\mathbb{H};0,\infty)}(\gamma) = \frac{1}{2}\int_0^T \left(\dot{W}_t - \mathrm{Re}\frac{\rho}{W_t - z_t}\right)^2 dt,$$

if W_t is absolutely continuous, and $I_{\rho,z_0}^{(\mathbb{H};0,\infty)}(\gamma) = \infty$ otherwise.



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  if W_t is absolutely continuous, and $I_{\rho,z_0}^{(\mathbb{H};0,\infty)}(\gamma) = \infty$ otherwise.
- Consistent with Freidlin–Wentzell theory ($dW_t = \mathrm{Re}\frac{\rho}{W_t - z_t}dt + \sqrt{\kappa}dB_t$).
- If $z_0 \in \mathbb{H}$ and $T < \tau$ (i.e. $\inf_{t \le T} |W_t - z_t| > 0$)

$$I_{\rho,z_0}^{(\mathbb{H};0,\infty)}(\gamma) = I^{(\mathbb{H};0,\infty)}(\gamma) + \rho \log \frac{\sin \theta_T}{\sin \theta_0} - \frac{\rho(8+\rho)}{8} \log \frac{|g_T'(z_0)| y_T}{y_0},$$

  where $y_T = \mathrm{Im}(z_T)$ and $\theta_T = \arg(z_T)$.
- If $T < \tau$, then $I_{\rho,z_0}^{(\mathbb{H};0,\infty)}(\gamma) < \infty$ iff $I^{(\mathbb{H};0,\infty)}(\gamma) < \infty$.
- Similar formula when $z_0 \in \mathbb{R} \setminus \{0\}$.

Radial ρ-Loewner energy

- Fix $\rho \in \mathbb{R}$ and $v_0 \in (0, 2\pi)$. The **radial ρ-Loewner energy** of a simple curve starting at 1 in $\mathbb{D} \setminus \{0\}$ is defined as

$$I^{(\mathbb{D};1,0)}_{\rho,e^{iv_0}}(\gamma) = \frac{1}{2}\int_0^T \left(\dot{w}_t - \frac{\rho}{2}\cot\left(\frac{w_t - v_t}{2}\right)\right)^2 dt,$$

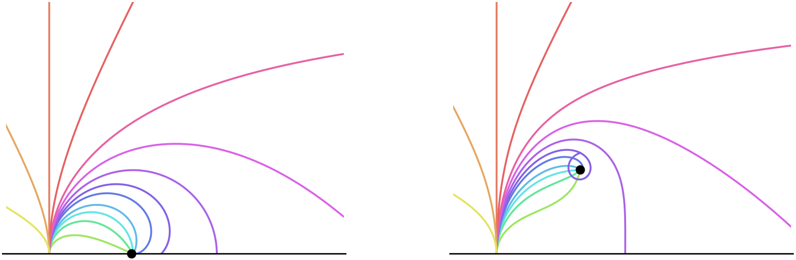
  if w_t is absolutely continuous, and $I^{(\mathbb{D};1,0)}_{\rho,e^{iv_0}}(\gamma) = \infty$ otherwise.
- $I^{(D;a,b)}_{\rho,c}$ is defined via conformal map to $\mathbb{H}$ or $\mathbb{D}$.
- Coordinate change property:

$$I^{(D;a,b)}_{\rho,c}(\gamma) = I^{(D;a,c)}_{-6-\rho,b}(\gamma).$$

- If $T < \tau$, then $I^{(\mathbb{D};1,0)}_{\rho,e^{iv_0}}(\gamma) < \infty$ iff $I^{(\mathbb{D};1,e^{iv_0})}(\gamma) < \infty$.

Minimizers

The unique minimizer of the ρ-Loewner energy is the $SLE_0(\rho)$ curve.



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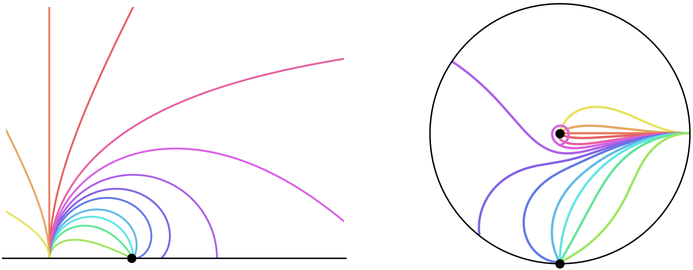

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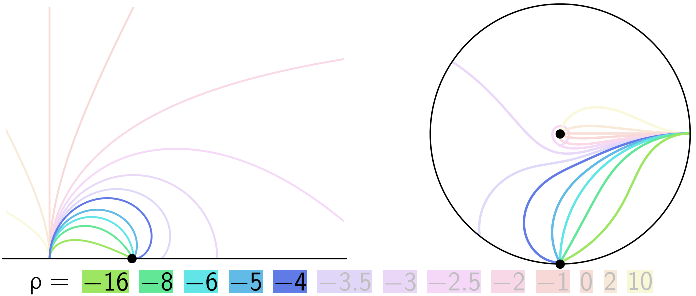

Three phases:

- When $\rho \in (-\infty, -4]$: $\mathrm{SLE}_0(\rho)$ hits the force point.

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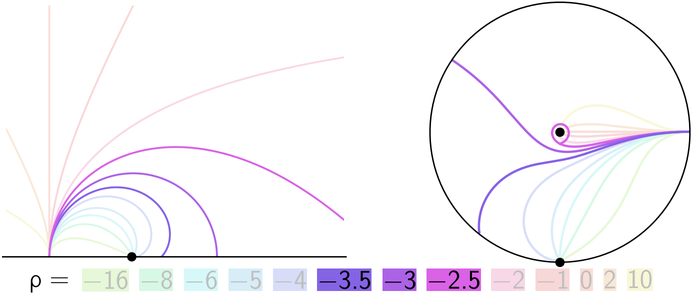

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- When $\rho \in (-\infty, -4]$: $SLE_0(\rho)$ hits the force point.
- When $\rho \in (-4, -2)$: $SLE_0(\rho)$ separates the reference point and force point.

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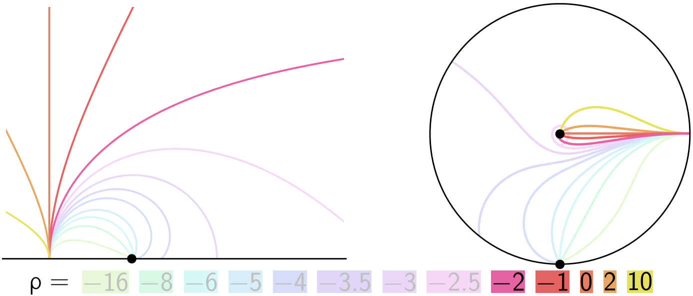

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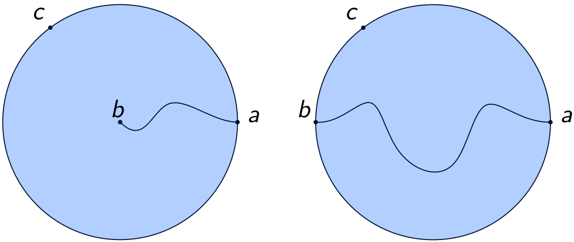
- When $\rho \in (-4, -2)$: $SLE_0(\rho)$ separates the reference point and force point.
- When $\rho \in [-2, \infty)$: $SLE_0(\rho)$ approaches the reference point.

Large Deviation Principle

Fix $\rho > -2$, $a \in \partial\mathbb{D}$, $b \in \overline{\mathbb{D}} \setminus \{a\}$, and $c \in \partial\mathbb{D} \setminus \{a, b\}$.

Theorem (K '24)

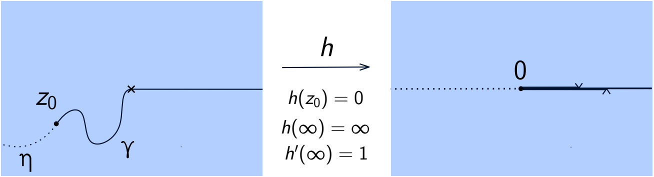
The $SLE_\kappa(\rho)$ in $(\mathbb{D}; a, b)$ with force point c satisfies a large deviation principle as $\kappa \to 0+$, with respect to the Hausdorff topology on the space of simple curves γ in $(\mathbb{D}; a, b)$, with good rate function $I_{\rho,c}^{(\mathbb{D};a,b)}$.



- Proof follows the same outline as in [Peltola-Wang '23]: LDP on driving process on $[0, T]$ $\rightsquigarrow$ LDP on finite time curves $\rightsquigarrow$ LDP on infinite time curves.

Dirichlet Energy Formulas

Radial Case



Lemma (K '24)

If $I^{(\Sigma;0,\infty)}(\gamma) := I^{(\Sigma;0,\infty)}(\gamma \cup \eta) < \infty$, *then*

$$|h'(z_0)|_\eta := \lim_{\varepsilon \to 0+} \frac{|h(\eta(\varepsilon)) - h(z_0)|}{|\eta(\varepsilon) - z_0|}$$

exists and is positive.

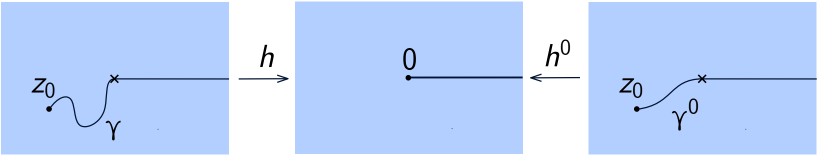
Radial Case

Theorem (K '24)

Fix $\rho > -2$. *Let* γ^0 *denote the* $SLE_0(\rho)$ *in* $(\Sigma; 0, z_0)$ *with force point at* ∞, *and let* h^0 *be the corresponding conformal map. A simple curve* γ *in* $(\Sigma; 0, z_0)$ *has finite* ρ*-Loewner energy w.r.t.* ∞ *if and only if* $\mathcal{D}(\log|h'|) < \infty$, *in which case,*

$$I^{(\Sigma;0,z_0)}_{\rho,\infty}(\gamma) = \mathcal{D}(\log|h'|) - \mathcal{D}(\log|(h^0)'|) - \frac{(\rho+6)(\rho-2)}{8}\log|H'(z_0)|_\eta$$

where $H = (h^0)^{-1} \circ h$, *and* $\mathcal{D}(f) = \frac{1}{\pi}\int |\nabla f|^2 dz^2$.



Radial Case

Theorem (K '24)

Fix $\rho > -2$. Let γ^0 denote the $SLE_0(\rho)$ in $(\Sigma; 0, z_0)$ with force point at ∞, and let h^0 be the corresponding conformal map. A simple curve γ in $(\Sigma; 0, z_0)$ has finite ρ-Loewner energy w.r.t. ∞ if and only if $\mathcal{D}(\log|h'|) < \infty$, in which case,

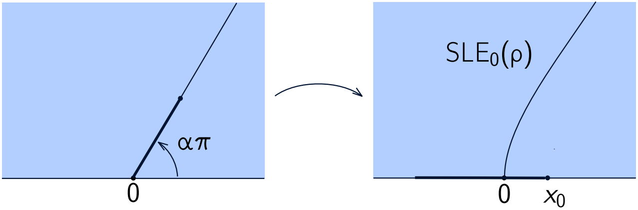
$$I_{\rho,\infty}^{(\Sigma;0,z_0)}(\gamma) = \mathcal{D}(\log|h'|) - \mathcal{D}(\log|(h^0)'|) - \frac{(\rho+6)(\rho-2)}{8}\log|H'(z_0)|_\eta$$

where $H = (h^0)^{-1} \circ h$, and $\mathcal{D}(f) = \frac{1}{\pi}\int |\nabla f|^2 dz^2$.

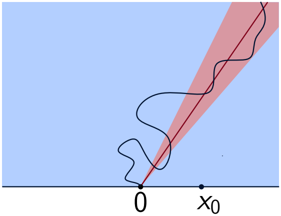
Proof uses coordinate change property, integrated formula, and the Dirichlet energy formula for the chordal Loewner energy.

Chordal Case

- $\mathrm{SLE}_0(\rho)$, $\rho > -2$, with force point $x_0 > 0$, is a "mapped out ray" of angle $\alpha\pi$, where $\alpha = \alpha(\rho) = \frac{\rho+2}{\rho+4} \in (0, 1)$.



- Curves of finite ρ-Loewner energy, $\rho > -2$, approach ∞ within cones around angle $\alpha\pi$



Chordal Case

For a chord γ in $(\Sigma; 0, \infty)$, let $h : \Sigma \setminus \gamma \to \Sigma \setminus \mathbb{R}^-$ be conformal with $h(\infty) = \infty$. We define, whenever the limit exists,

$$\mathcal{D}_\beta(\log |h'|) = \lim_{R\to\infty} \left(\frac{1}{\pi} \int_{B(0,R)} |\nabla \log |h'||^2 dz^2 - c_\beta \log R \right), \quad c_\beta = \frac{(1-2\beta)^2}{2\beta(1-\beta)}.$$

Theorem (K '24)

Fix $\rho > -2$ and $x_0 > 0$. If γ is a chord in $(\Sigma; 0, \infty)$, such that there exists T for which $\gamma([T, \infty))$ is the ρ-Loewner energy optimal extension of γ_T, then

$$I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\gamma) = \mathcal{D}_\alpha(\log |h'|) - \mathcal{D}_\alpha(\log |(h^0)'|) - \frac{\rho(\rho+4)}{4} \log |H'(x_0)|,$$

where H is the conformal map from the upper component of $\Sigma \setminus \gamma$ to the upper component of $\Sigma \setminus \gamma^0$ fixing ∞ and x_0 and satisfying $H'(\infty) = 1$.

Determinants of Laplacians

Determinants of Laplacians

On smooth compact Riemannian surfaces with boundary, (M, g), we may define the ζ-regularized determinant of the Dirichlet Laplace-Beltrami operator, $\det_\zeta \Delta_{(M,g)}$ [Ray-Singer '71]:

- If $\partial M \neq \varnothing$, then $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lim_{n\to\infty} \lambda_n \to \infty$.
- Define $\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^s$, for $\mathrm{Re}\, s > 1$. Analytic cont. to nbhd of 0.
- We define $\det_\zeta \Delta_{(M,g)} := e^{-\zeta'(0)}$ justified by

$$\zeta'(s) = -\sum_{n=1}^{\infty} \log \lambda_n \lambda_n^{-s}, \quad \mathrm{Re}\, s > 1 \quad \rightsquigarrow \quad \text{``}\zeta'(0) = -\log(\prod_{n=1}^{\infty} \lambda_n)\text{''}.$$

- $\det_\zeta \Delta_{(M,g)}$ can be defined in the same way for curvilinear polygonal domains (boundary is p.w. smooth with corner angles in $(0, \infty)$) [Aldana-Kirsten-Rowlett '20, K '24].

Determinants of Laplacians

Let K be a compact subset of M and set (whenever the RHS is defined)

$$\mathcal{H}_{(M,g)}(K) := \log \det_\zeta \Delta_{(M,g)} - \sum_{C_i} \log \det_\zeta \Delta_{(C_i,g)}$$

where the sum runs over connected components of $M \setminus K$.

Theorem (Wang '19)

Let γ be a C^∞ Jordan curve on S^2, and let g be a metric conformally equivalent to the spherical metric g_0, that is, $g = e^{2\sigma} g_0$, $\sigma \in C^\infty(S^2)$. Then,

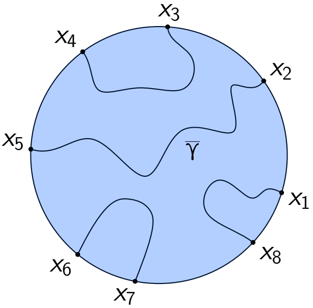
$$I^L(\gamma) = 12(\mathcal{H}_{(S^2,g)}(\gamma) - \mathcal{H}_{(S^2,g)}(S^1)).$$

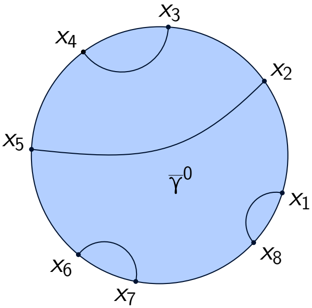
Determinants of Laplacians

In [Peltola-Wang '23] it is proved that the **multichordal Loewner energy** can be expressed as

$$I^{\alpha}_{\mathbb{D};x_1,\ldots,x_{2n}}(\overline{\gamma}) = 12(\mathcal{H}_{(\mathbb{D},g)}(\overline{\gamma}) - \mathcal{H}_{(\mathbb{D},g)}(\overline{\gamma}^0)),$$

for a smooth and finite energy multichord $\overline{\gamma}$ with link-pattern α and $g = e^{2\sigma}dz^2$, $\sigma \in C^\infty(\overline{\mathbb{D}})$, where $\overline{\gamma}^0$ is the geodesic multichord with the same link-pattern.

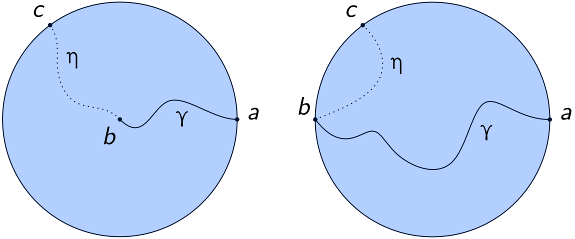




Fix $\rho > -2$, $a, c \in \partial\mathbb{D}$ and $b \in \overline{\mathbb{D}}$. Let γ be a curve in $(\mathbb{D}; a, b)$ and let $\eta = \eta(\gamma)$ denote the hyperbolic geodesic from b to c in $\mathbb{D} \setminus \gamma$. The ρ-**Loewner potential** of γ with respect to a metric $g = e^{2\sigma} dz^2$, is defined as

$$\mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma; g) := \beta \mathcal{H}_{(\mathbb{D},g)}(\gamma) + (1-\beta)\mathcal{H}_{(\mathbb{D},g)}(\gamma \cup \eta),$$

where $\beta = \frac{(2-\rho)(\rho+6)}{12}$, whenever the RHS is well-defined.



Fix $\rho > -2$, $a, c \in \partial\mathbb{D}$ and $b \in \overline{\mathbb{D}}$. Let γ be a curve in $(\mathbb{D}; a, b)$ and let $\eta = \eta(\gamma)$ denote the hyperbolic geodesic from b to c in $\mathbb{D} \setminus \gamma$. The ρ-**Loewner potential** of γ with respect to a metric $g = e^{2\sigma} dz^2$, is defined as

$$\mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma; g) := \beta \mathcal{H}_{(\mathbb{D},g)}(\gamma) + (1-\beta)\mathcal{H}_{(\mathbb{D},g)}(\gamma \cup \eta),$$

where $\beta = \frac{(2-\rho)(\rho+6)}{12}$, whenever the RHS is well-defined.

- If $\rho = 0$, then $\beta = 1$.
- If $\rho = 2$, then $\beta = 0$. ($\rho = 2$ corresponds to one arm of a 2-radial or 2-chordal, and η is the "optimal second arm".)

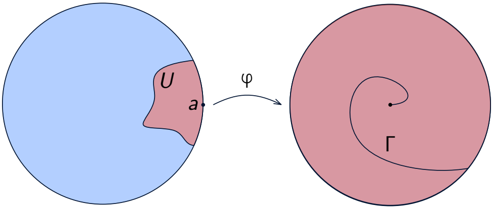
Determinants of Laplacians

Proposition (K '24)

For all $g \in \mathcal{G}(a)$ and $\gamma_1, \gamma_2 \in \mathcal{X}(g, a, b, c, \rho)$ we have that

$$I_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_1) - I_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2) = 12(\mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_1; g) - \mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2; g)).$$

- $\mathcal{G}(a)$ consists of metrics $g = e^{2\sigma} dz^2$, $\sigma \in C^\infty(\overline{\mathbb{D}} \setminus \{a\})$ for which there is a coordinate $\varphi : U \to \mathbb{D} \setminus \Gamma$ in a nbhd of a, for which $\varphi^* g = e^{2\tilde{\sigma}(w)} dw^2$ and $\tilde{\sigma}$ extends smoothly to both sides of the slit Γ.



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- $\mathcal{G}(a)$ consists of metrics $g = e^{2\sigma} dz^2$, $\sigma \in C^\infty(\overline{\mathbb{D}} \setminus \{a\})$ for which there is a coordinate $\varphi : U \to \mathbb{D} \setminus \Gamma$ in a nbhd of a, for which $\varphi^* g = e^{2\tilde{\sigma}(w)} dw^2$ and $\tilde{\sigma}$ extends smoothly to both sides of the slit Γ.
- $\mathcal{X}(g, a, b, c, \rho)$ consists of C^∞-smooth curves γ in $(\mathbb{D}; a, b)$ which are smoothly attached at a, and
    - if $b \in \mathbb{D}$, then $\gamma \cup \eta$ is C^∞-smooth.
    - if $b \in \partial\mathbb{D}$, then there exists T s.t. $\gamma([T, \infty))$ is the ρ-optimal extension of γ_T.
- If $b \in \partial\mathbb{D}$ or $b = 0$ and a and c are antipodal, then there exists $g \in \mathcal{G}(a)$ so that the minimizer γ^0 belongs to $\mathcal{X}(g, a, b, c, \rho)$.

Thank you for your attention!

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