The ρ-Loewner Energy: Large Deviations, Minimizers, and Alternative Descriptions

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Abstract

We introduce and study the ρ-Loewner energy, a variant of the Loewner energy with a force point on the boundary of the domain. We prove a large deviation principle for $\mathrm{SLE}_\kappa(\rho)$, as $\kappa \to 0+$ and $\rho > -2$ is fixed, with the ρ-Loewner energy as the rate function in both radial and chordal settings. The unique minimizer of the ρ-Loewner energy is the $\mathrm{SLE}_0(\rho)$ curve. We show that it exhibits three phases as ρ varies and give a flow-line representation. We also define a whole-plane variant for which we explicitly describe the trace.

We further obtain alternative formulas for the ρ-Loewner energy in the reference point hitting phase, $\rho > -2$. In the radial setting we give an equivalent description in terms of the Dirichlet energy of $\log |h'|$, where h is a conformal map onto the complement of the curve, plus a point contribution from the tip of the curve. In the chordal setting, we derive a similar formula under the assumption that the chord ends in the ρ-Loewner energy optimal way. Finally, we express the ρ-Loewner energy in terms of ζ-regularized determinants of Laplacians.

1 Introduction and main results

1.1 Introduction

A chord is a simple curve connecting two distinct boundary points of a domain in the complex plane. The chordal Loewner differential equation gives a way of encoding an appropriately parametrized chord in a simply connected domain by a continuous and real-valued function. Consider the following reference setting: Suppose γ is a chord from 0 to ∞ in $\mathbb{H}$, the upper half-plane, (parametrized appropriately on $[0, \infty)$). Then, the family $(g_t)_{t \geq 0}$ of conformal maps $g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$, normalized so that $g_t(z) = z + O(z^{-1})$ at ∞, where $\gamma_t = \gamma([0, t])$, satisfies the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z, \tag{1}$$

with driving function $W_t = g_t(\gamma(t))$. The function W_t encodes γ in the sense that one can, given W_t, retrieve $(g_t)_{t \geq 0}$ by solving (1) and then obtain γ by

$$\gamma(t) = \lim_{y \to 0+} g_t^{-1}(W_t + iy).$$

In general, solving the chordal Loewner equation for an arbitrary real-valued and continuous function W_t, yields a family $(g_t)_{t \geq 0}$ of conformal maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$, where $(K_t)_{\geq 0}$ is a family of continuously growing compact subsets of $\overline{\mathbb{H}}$.

The Loewner equation can be used to describe (candidates for) scaling limits of interfaces in planar critical lattice models. Schramm [36] observed that such random curves should satisfy two important properties, conformal invariance and a domain Markov property, and that a random curve has these properties if and only if its driving function is of the form $W_t = \sqrt{\kappa}B_t$, where $\kappa \geq 0$ and B_t is a standard one-dimensional Brownian motion. These curves, called chordal Schramm-Loewner evolution, SLE_κ, have, indeed, been shown to be the scaling limit of interfaces in lattice models for several values of κ, and play a major role in random conformal geometry [37, 40, 39].

In [45], Wang showed that chordal SLE_κ satisfies a large deviation principle (LDP) as $\kappa \to 0+$. The rate function $I^{(\mathbb{H};0,\infty)}$ is the chordal Loewner energy, which for chords γ from 0 to ∞ in $\mathbb{H}$ is defined by

$$I^{(\mathbb{H};0,\infty)}(\gamma) = \frac{1}{2}\int_0^\infty \dot{W}_t^2 dt,$$

when the driving function W_t of γ is absolutely continuous on $[0,T]$, for all $T > 0$, and $I^{(\mathbb{H};0,\infty)}(\gamma) = \infty$ otherwise. Heuristically, this means that

$$\text{“}\mathbb{P}[\text{SLE}_\kappa \text{ stays close to } \gamma] \sim \exp\left(-\frac{I^{(\mathbb{H};0,\infty)}(\gamma)}{\kappa}\right), \quad \text{as } \kappa \to 0+.\text{”}$$

(See also [13].) The chordal Loewner energy has a natural extension, called the loop Loewner energy, to the space of Jordan curves on the Riemann sphere [35]. In later work, Wang showed that the loop Loewner energy has unexpected ties to Teichmüller theory: the loop Loewner energy coincides (up to a multiplicative constant) with the universal Liouville action, a Kähler potential on the Weil-Petersson universal Teichmüller space [44]. In particular, the class of finite Loewner energy loops coincides with the class of Weil-Petersson quasicircles, which has many different characterizations [6]. The links between random conformal geometry and Teichmüller theory revealed by Wang's result are not yet well-understood. It was also shown in [44] that the Loewner energy of a smooth Jordan curve can be expressed in terms of ζ-regularized determinants of Laplacians.

These discoveries sparked interest in large deviations for SLE type processes, as $\kappa \to 0+$, which was studied in the subsequent papers [33, 15, 1, 2]. Parallel to the LDP development, several articles have been devoted to investigating the Loewner energy and its properties, see, e.g., [13, 35, 42, 43, 25, 41, 7, 24]. See also the survey [46].

In the present paper, we study large deviations on $\text{SLE}_\kappa(\rho)$ curves and the corresponding large deviations rate function, which we call the ρ-Loewner energy. The $\text{SLE}_\kappa(\rho)$ curve is a natural generalization of SLE_κ where one puts an attractive or repelling force ($\rho \in \mathbb{R}$) at one or several marked points, called force points. In this paper we deal with the case of one force point, which will, with a few exceptions, be located on the boundary of the domain. The chordal $\text{SLE}_\kappa(\rho)$ curve in $\mathbb{H}$ starting at 0, with reference point ∞ and force point $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$, is the random curve whose Loewner driving function satisfies the SDE

$$dW_t = \text{Re}\frac{\rho}{W_t - z_t}dt + \sqrt{\kappa}dB_t, \quad W_0 = 0, \tag{2}$$

where $z_t = g_t(z_0)$, and B_t is a one-dimensional standard Brownian motion. Using the radial Loewner equation

$$\partial_t g_t(z) = g_t(z)\frac{W_t + g_t(z)}{W_t - g_t(z)} \tag{3}$$

one can, in a similar way, define the radial $\text{SLE}_\kappa(\rho)$ curve in $\mathbb{D}$, the unit disk, starting at 1, with reference point 0 and force point $z_0 = e^{iv_0} \in \partial\mathbb{D} \setminus \{1\}$. It is the random curve whose driving function $W_t = e^{iw_t}$ satisfies the SDE

$$dw_t = \frac{\rho}{2}\cot\frac{w_t - v_t}{2}dt + \sqrt{\kappa}dB_t, \quad w_0 = 0, \tag{4}$$

where $e^{iv_t} = g_t(e^{iv_0})$, and B_t is a one-dimensional standard Brownian motion.

The first appearance of $\text{SLE}_\kappa(\rho)$ was in [20], where it was shown that $\text{SLE}_{8/3}(\rho)$ (in the chordal setting with a boundary force point) is the outer boundary of random sets that satisfy a conformal restriction property. In [47, 10], this work was further developed. In the latter it was shown that a chordal $\mathrm{SLE}_\kappa(\kappa-4)$, with force point at $x_0>0$ and $\kappa \geq 4$, is a chordal SLE_κ conditioned not to hit $[x,\infty)$. Two further special cases of chordal $\mathrm{SLE}_\kappa(\rho)$ processes, now with a force point $z_0 \in \mathbb{H}$, are $\mathrm{SLE}_\kappa(\kappa-6)$, $\kappa \leq 4$, which (after re-parametrization) is a radial SLE_κ from 0 to z_0, and $\mathrm{SLE}_\kappa(\kappa-8)$, $\kappa \leq 4$, which is (the first part) of a chordal SLE_κ "conditioned to pass through z_0" [38]. The $\mathrm{SLE}_\kappa(\rho)$ curves are also main players in Imaginary Geometry, (see, e.g., [26, 27, 28, 30]), where they appear as generalized flow-lines of the Gaussian free field (GFF) with suitable boundary conditions.

1.2 Main results

The ρ-Loewner energy The goal of the present paper is to study the ρ-Loewner energy, the large deviations rate function for $\mathrm{SLE}_\kappa(\rho)$ as $\kappa \to 0+$, which is defined in the following way.

Definition 1 (ρ-Loewner energy)**.** Fix $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$ and $\rho \in \mathbb{R}$. Let γ be a curve in $\mathbb{H} \setminus \{z_0\}$ parametrized by half-plane capacity, starting at 0. The chordal ρ-Loewner energy of γ with respect to the reference point ∞ and force point z_0, is defined by

$$I^{(\mathbb{H};0,\infty)}_{\rho,z_0}(\gamma) = \frac{1}{2}\int_0^T \left(\dot{W}_t - \rho \mathrm{Re}\frac{1}{W_t - z_t}\right)^2 dt,$$

where $z_t = g_t(z_0)$, if W is absolutely continuous, and set to ∞ otherwise. Similarly, fix $e^{iv_0} \in \partial\mathbb{D}$ and $\rho \in \mathbb{R}$. Let γ be a curve in $\mathbb{D} \setminus \{0\}$ parametrized by conformal radius, starting at 1. The radial ρ-Loewner energy of γ with respect to the reference point 0 and force point e^{iv_0}, is defined by

$$I^{(\mathbb{D};1,0)}_{\rho,e^{iv_0}}(\gamma) = \frac{1}{2}\int_0^T \left(\dot{w}_t - \frac{\rho}{2}\cot\frac{w_t - v_t}{2}\right)^2 dt,$$

where $e^{iv_t} = g_t(e^{iv_0})$, if w is absolutely continuous, and set to ∞ otherwise. The radial Loewner energy of γ with respect to the reference point 0 is $I^{\mathbb{D};1,0}(\gamma) = I^{\mathbb{D};1,0}_{0,e^{iv_0}}(\gamma)$.

Remark 1. Specialists familiar with Freidlin-Wentzell theory (see, e.g., Section [9, Section 5.6]) might immediately recognize that the formulas for the ρ-Loewner energy are consistent with that theory. That is, if one, heuristically, applies the the Freidlin-Wentzell theorem to the driving processes of $\mathrm{SLE}_\kappa(\rho)$, as $\kappa \to 0+$, then one obtains the ρ-Loewner energy as the rate function. Note however, that one can not obtain a large deviation principle on the driving process directly in this way since the drift terms in (2) and (4) are not uniformly Lipschitz.

Remark 2. For curves γ which are bounded away from both the reference point and the force point, one can derive an integrated formula for the ρ-Loewner energy, comparing it to the regular Loewner energy. E.g., in the chordal setting with $z_0 \in \mathbb{H}$ we have

$$I^{(\mathbb{H};0,\infty)}_{\rho,z_0}(\gamma) = I^{(\mathbb{H};0,\infty)}(\gamma) + \rho \log\frac{\sin\theta_T}{\sin\theta_0} - \frac{\rho(8+\rho)}{8}\log\frac{|g_T'(z_0)|y_T}{y_0}, \tag{5}$$

where $z_t = g_t(z_0)$, $y_t = \mathrm{Im} z_t$, and $\theta_t = \arg z_t \in (0,\pi)$. The integrated formulas in the other settings are presented in Proposition 2.

We define the ρ-Loewner energy of a curve γ in a simply connected domain D, starting at a point $a \in \partial D$, with respect to a reference point $b \in \overline{D} \setminus \{a\}$ and force point $c \in \partial D \setminus \{a,b\}$ using a conformal map from D to $\mathbb{H}$ or $\mathbb{D}$ mapping a and b appropriately. In [11, 38], it was shown that an $\mathrm{SLE}_\kappa(\rho)$ starting at a, with reference point b and force point c is (after re-parametrization) an $\mathrm{SLE}_\kappa(\kappa-\rho-6)$ starting at a, with reference point c and force point b. In the deterministic setting we have the analog

$$I^{(D;a,b)}_{\rho,c}(\gamma) = I^{(D;a,c)}_{-6-\rho,b}(\gamma). \tag{6}$$

This provides a way to translate facts about the chordal Loewner energy to the radial setting. In particular, we obtain upper and lower bounds for the radial Loewner energy in terms of the chordal Loewner energy (see Corollary 5). The integrated formulas and the coordinate change property (6) are proved in Section 3.

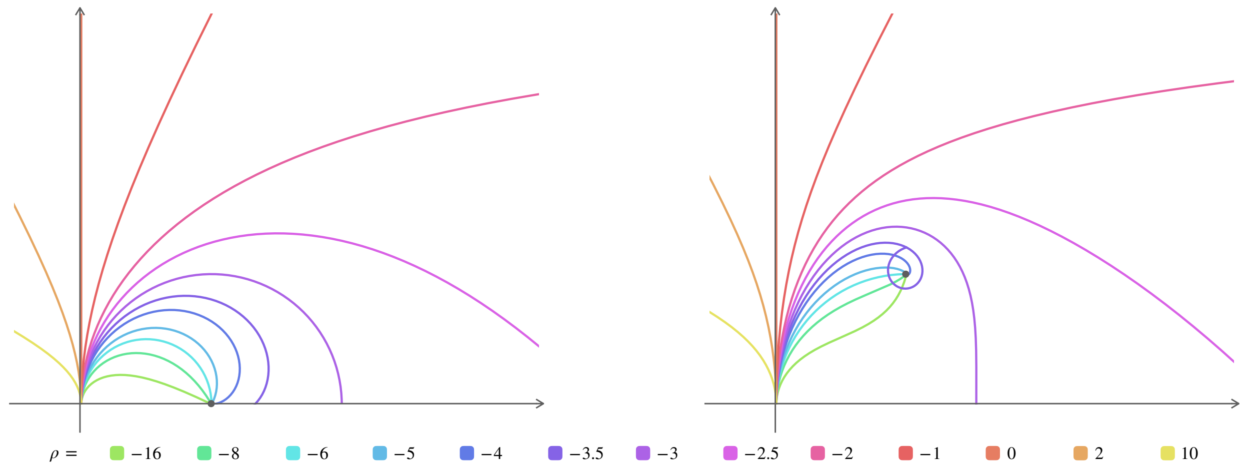

Figure 1: Chordal $\text{SLE}_0(\rho)$ curves. On the left, the force point is on the boundary, and on the right, the force point is in the interior with $\arg z_0 = \pi/2$.

Large deviation principle. Our first main result, which is proved in Section 4, is a large deviation principle (LDP) of $\text{SLE}_\kappa(\rho)$, as $\kappa \to 0+$.

Theorem 1 (LDP)**.** *Fix $a \in \partial\mathbb{D}$, $b \in \overline{\mathbb{D}} \setminus \{a\}$, $c \in \partial\mathbb{D} \setminus \{a, b\}$, and $\rho > -2$. Let $\mathcal{X}^{(D;a,b)}$ denote the space of simple curves from a to b in $\mathbb{D}$ equipped with the Hausdorff topology. In this topology, the $\text{SLE}_\kappa(\rho)$ processes starting at a, with reference point b and force point c, satisfy the large deviation principle with good rate function $I_{\rho,c}^{(\mathbb{D};a,b)}$ as $\kappa \to 0+$.*

The LDP result also holds for the family $\text{SLE}_\kappa(\kappa + \rho)$, as $\kappa \to 0+$ and $\rho > -2$ is fixed. Hence, the $\text{SLE}_\kappa(\tilde{\rho})$ starting at a, with reference point c and force point b, satisfies the LDP, as $\kappa \to 0+$ and $\tilde{\rho} > -4$ is fixed, with rate function $I_{\rho,b}^{(\mathbb{D};a,c)}$. One should note here that, when $\tilde{\rho} > -4$ and κ is small, the $\text{SLE}_\kappa(\tilde{\rho})$ is a.s. stopped within finite time at $t = \tau_{0+}$, where $\tau_{0+} = \lim_{\varepsilon \to 0+} \inf\{t : |W_t - x_t| \le \varepsilon\}$.

Remark 3. The forthcoming article [2] proves an LDP for radial SLE_{0+} in the topology of uniform convergence on the space of simple curves modulo re-parametrization.

Remark 4. The assumption that $\rho > -2$ is a technical one. Our proof of Theorem 1 relies on an LDP for the driving processes on finite time intervals $[0, T]$, which we then carry to an LDP on the curves. Since, the driving process of $\text{SLE}_\kappa(\rho)$ is terminated within time T with positive probability (for all small κ) when $\rho \le -2$, our proof does not work beyond $\rho > -2$.

Minimizers and flow-lines. It follows directly from the definition of the ρ-Loewner energy that its unique minimizer is the $\text{SLE}_0(\rho)$ curve. Indeed, there is a unique driving function for which the integrand in the ρ-Loewner energy is zero for all t, and this is, by definition, the driving function of $\text{SLE}_0(\rho)$. In Section 5, we study the $\text{SLE}_0(\rho)$ curves. In particular, we show that the Imaginary Geometry interpretation of $\text{SLE}_\kappa(\rho)$ as generalized flow-lines of the Gaussian free field (see, e.g., [26, 30]) has the expected analog when $\kappa = 0$. That is, the $\text{SLE}_0(\rho)$ curve is a (classical) flow-line of the appropriate harmonic field (see Proposition 7). It can be seen in Figure 1, which depicts $\text{SLE}_0(\rho)$ curves for some values of ρ, that the behavior of $\text{SLE}_0(\rho)$ changes drastically with ρ. If $\rho \in (-\infty, -4]$, then the $\text{SLE}_0(\rho)$ terminates at the force point. If instead $\rho \in (-4, -2)$, then the $\text{SLE}_0(\rho)$ terminates by hitting back on itself or the boundary. Lastly, if $\rho \in [-2, \infty)$, then the $\text{SLE}_0(\rho)$ approaches the reference point. This (and Remark 4) exemplifies a change of behavior in the ρ-Loewner energy when varying ρ. After Section 5, we therefore restrict our attention to the cases covered by Theorem 1. In Section 5, we also define a whole-plane variant of radial $\text{SLE}_0(\rho)$ for $\rho \ge -2$ and give a flow-line characterization of them. The whole-plane $\text{SLE}_0(\rho)$ curves are portrayed, for some values of $\rho > -2$, in Figure 2. As

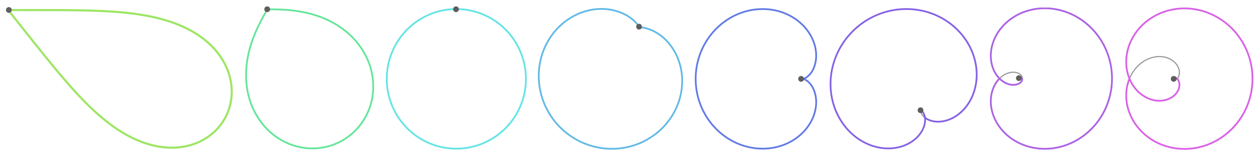

Figure 2: Positively oriented whole-plane $\mathrm{SLE}_0(\rho)$ curves started at 0 in the direction 1 with force point 0 and reference point ∞. From left to right $\rho = -16,\ -8,\ -6,\ -5,\ -4,\ -3.5,\ -3$, and -2.5. The origin is marked by a gray dot, and in the three right-most figures, the continuation of the flow-line, after self-intersection, is shown in gray. In particular, the $\mathrm{SLE}_0(-6)$ is a circle through 0 and $\mathrm{SLE}_0(-4)$ is a cardioid with cusp at 0.

expected, given the phase transition of radial $\mathrm{SLE}_0(\rho)$ at $\rho = -2$, the whole-plane $\mathrm{SLE}_0(-2)$ is different from the whole-plane $\mathrm{SLE}_0(\rho)$, $\rho > -2$, in that it does not terminate within finite time. In fact, the whole-plane $\mathrm{SLE}_0(-2)$ is a logarithmic spiral, where the spiraling rate depends on a parameter which can be interpreted as the infinitesimal starting position of the force point, see Remark 16.

Dirichlet energy formulas. In Section 6, we prove Dirichlet energy formulas for the ρ-Loewner energy, in the chordal and radial setting respectively, with a boundary force point and $\rho > -2$.

Let $\Sigma = \mathbb{C} \setminus \mathbb{R}^+$. It was shown in [44] that, for a chord γ from 0 to ∞ in Σ,

$$I^{(\Sigma;0,\infty)}(\gamma) = \frac{1}{\pi}\int_{\Sigma\setminus\gamma} |\nabla \log|h'(z)||^2 dz^2, \tag{7}$$

where $h : \Sigma \setminus \gamma \to \Sigma \setminus \mathbb{R}^-$ is conformal, fixes ∞, and maps the upper and lower component of $\Sigma \setminus \gamma$ onto $\mathbb{H}$ and $\mathbb{H}^*$ respectively (the latter denoting the lower half-plane). As opposed to the original formula in terms of the driving function, this formula considers the curve as a whole, and in that sense offers a static, rather than dynamic, point of view. Denote by

$$\mathcal{D}(h) = \frac{1}{\pi}\int_D |\nabla \log|h'(z)||^2 dz^2 \tag{8}$$

for a conformal map h defined on $D \subset \mathbb{C}$.

Theorem 2 (Dirichlet energy formula, radial setting)**.** *Fix $\rho > -2$ and $z_0 \in \Sigma$. Consider a simple curve $\gamma \subset \Sigma$ starting at 0 and ending at z_0. Let $h : \Sigma \setminus \gamma \to \Sigma$ be the conformal map satisfying $h(z_0) = 0$, $h(\infty) = \infty$ and $|h'(\infty)| = 1$. Denote by γ^0 the $\mathrm{SLE}_0(\rho)$ from 0 to z_0 with force point at ∞ and let $h^0 : \Sigma \setminus \gamma^0 \to \Sigma$ be the conformal map with the same normalization as h. Then γ has finite ρ-Loewner energy if and only if $\mathcal{D}(h) < \infty$, in which case*

$$I^{(\Sigma;0,z_0)}_{\rho,\infty}(\gamma) = \mathcal{D}(h) - \mathcal{D}(h^0) - \frac{(\rho+6)(\rho-2)}{8}\log|H'(z_0)|_\eta, \tag{9}$$

where $H = (h^0)^{-1} \circ h$ and

$$|H'(z_0)|_\eta := \lim_{s\to 0+}\frac{|H(\eta(s)) - H(z_0)|}{|\eta(s) - z_0|}, \tag{10}$$

where η is the hyperbolic geodesic from z_0 to ∞ in $\Sigma \setminus \gamma$.

Remark 5. The proof uses the Dirichlet energy formula (7) and the integrated formula for the ρ-Loewner energy (5). The main part of the proof is to show that (10) exists whenever $D(h) < \infty$. Note that $D(h) < \infty$ if and only if $\gamma \cup \eta \cup \mathbb{R}^+$ is a Weil-Petersson quasicircle. Hence, γ, with $D(h) < \infty$, is not necessarily C^1 at z_0.

Let $x_0 > 0$ and denote the upper prime end at x_0 of Σ by x_0^+. In Proposition 16 we show that, for $\rho > -2$, chords $\gamma \subset \Sigma$ from 0 to ∞ of finite ρ-Loewner energy, with respect to the force point x_0^+, approach ∞ with an angle $2\alpha\pi$, where $\alpha = \frac{\rho+2}{\rho+4}$, in a certain sense. The corner at ∞ causes the Dirichlet energy of $\log|h'|$ to diverge. To combat this, we introduce a re-normalized Dirichlet energy. Consider a chord γ from 0 to ∞ in Σ. Let $h : \Sigma \setminus \gamma \to \Sigma \setminus \mathbb{R}^-$ be a conformal map as in (7). For $\beta \in (0,1)$ we define, provided that the limit exists,

$$\mathcal{D}_\beta(h) := \lim_{R\to\infty} \left(\frac{1}{\pi} \int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} |\nabla \log|h'|\|^2 dz^2 - c_\beta \log R \right),\ c_\beta = \frac{(1-2\beta)^2}{2\beta(1-\beta)}. \qquad (11)$$

In light of (7), $D_\beta(h)$ can be viewed as a re-normalization of the chordal Loewner energy of γ when $\gamma \cup \mathbb{R}^+$ forms a $2\beta\pi$ corner at ∞. This type of re-normalized Loewner energy has been studied previously in the case of piece-wise linear Jordan curves in [8]. In a similar spirit, but using a different re-normalization procedure, the divergence of the Loewner energy in the presence of corners, has been studied in connection to Coulomb gas in a Jordan domain with piece-wise analytic boundary [16].

We write $\gamma_{[T,\infty)} = \gamma([T,\infty))$, for $T \geq 0$.

Theorem 3 (Dirichlet energy formula, chordal setting)**.** *Fix $\rho > -2$ and $x_0 > 0$, and let $\alpha = \frac{\rho+2}{\rho+4}$. Suppose $\gamma \subset \Sigma$ is a chord from 0 and to ∞ and that there is a $T \geq 0$ such that $\gamma_{[T,\infty)}$ is the $I^{(\Sigma,0,\infty)}_{\rho,x_0^+}$-optimal extension of γ_T. Then,*

$$I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\gamma) = D_\alpha(h) - D_\alpha(h^0) - \frac{\rho(\rho+4)}{4} \log|H'(x_0)|, \qquad (12)$$

where γ^0 denotes the SLE$_0(\rho)$ from 0, with reference point ∞ and force point x_0^+, h^0 is its corresponding conformal map, and H is the conformal map from the upper component of $\Sigma \setminus \gamma$ to the upper component of $\Sigma \setminus \gamma^0$ with $H(x_0) = x_0$, $H(\infty) = \infty$ and $H'(\infty) = 1$.

We expect (12) to hold for a wider class of curves, that do not necessarily approach ∞ in the optimal way, at least under some assumptions on the regularity close to ∞.

ζ-regularized determinants. Finally, in Section 7, we relate the ρ-Loewner energy to ζ-regularized determinants of Laplacians by introducing a ρ-Loewner potential, in the same spirit as in [44, 33]. For a domain $D \subset \mathbb{C}$ endowed with a Riemannian metric g, we let $\Delta_{(D,g)}$ denote the Friedrichs extension of the Dirichlet Laplace-Beltrami operator on (D,g).

Consider a conformal metric $g = e^{2\sigma}dz^2$, $\sigma \in C^\infty(\mathbb{D})$, and fix $a \in \partial\mathbb{D}$, $b \in \overline{\mathbb{D}} \setminus \{a\}$, and $c \in \partial\mathbb{D} \setminus \{a,b\}$. For each curve $\gamma \in \mathbb{D}$ from a to b we let $\eta = \eta(\gamma)$ denote the hyperbolic geodesic from b to c in $\mathbb{D} \setminus \gamma$. We define, for $\rho > -2$, the ρ-Loewner potential of γ with respect to g by

$$\mathcal{H}^{(\mathbb{D};a,b)}_{\rho,c}(\gamma; g) := \beta\mathcal{H}^{\mathbb{D}}(\gamma; g) + (1-\beta)\mathcal{H}^{\mathbb{D}}(\gamma \cup \eta; g), \qquad (13)$$

provided that the right-hand side is defined, where

$$\mathcal{H}^D(K,g) = \log\det{}_\zeta \Delta_{(D,g)} - \sum_{D_i} \log\det{}_\zeta \Delta_{(D_i,g)},$$

and D_i are the connected components of $D \setminus K$. Here the coefficients on the right are given by

$$\beta := \frac{(2-\rho)(\rho+6)}{12}, \quad 1-\beta = \frac{\rho(\rho+4)}{12},$$

which are not positive for all $\rho > -2$, so this is not quite a convex combination.

In order to guarantee that the ρ-Loewner potential is defined, and to relate it to the ρ-Loewner energy, we impose a few regularity assumptions on the metric g and curve γ. To this end, let $\Gamma \subset \mathbb{D}$ be a curve from 1 to 0, smooth up to its endpoints, and fix a conformal map

$\varphi : U \cap D \to \mathbb{D} \setminus \Gamma$, for $a \in U \subset \overline{\mathbb{D}}$ relatively open, with φ extending continuously to U and $\varphi(a) = 0$. We obtain a smooth slit structure, denoted by $(\overline{\mathbb{D}}, \varphi)$, on $\overline{\mathbb{D}}$ by declaring that the z-coordinate is smooth on $\overline{\mathbb{D}} \setminus \{a\}$ and that the $w = \varphi(z)$-coordinate is smooth on U. We say that a $(\overline{\mathbb{D}}, \varphi)$-smooth curve $\gamma \in \overline{\mathbb{D}}$, starting at a, is smoothly attached at a if $\varphi(\gamma) \cup \Gamma$ is smooth at a. For a fixed choice of a, b, c, Γ and φ as above, we let $\hat{\mathcal{X}}$ denote the class of $(\overline{\mathbb{D}}, \varphi)$-smooth curves $\gamma \subset \mathbb{D} \cup \{a, b\}$ from a to b, smoothly attached at a such that

- if $b \in \mathbb{D}$, then $\gamma \cup \eta$ is smooth at b,
- if $b \in \partial\mathbb{D}$, then there exists a $T \geq 0$ such that $\gamma_{[T,\infty)}$ is the $I^{(\mathbb{D},a,b)}_{\rho,c}$-optimal extension of γ_T.

Proposition 1 (Determinants of Laplacians)**.** *Fix $\rho > -2$, and a, b, c, Γ and φ as above. For all $(\overline{\mathbb{D}}, \varphi)$-smooth conformal metrics g, and $\gamma_1, \gamma_2 \in \hat{\mathcal{X}}$ we have*

$$I^{(\mathbb{D},a,b)}_{\rho,c}(\gamma_1) - I^{(\mathbb{D},a,b)}_{\rho,c}(\gamma_2) = 12\Big(\mathcal{H}^{(\mathbb{D},a,b)}_{\rho,c}(\gamma_1; g) - \mathcal{H}^{(\mathbb{D},a,b)}_{\rho,c}(\gamma_2; g)\Big).$$

Remark 6. If $b \in \partial\mathbb{D}$, or if $b = 0$ and c is antipodal to a, then there is a choice of Γ and φ such that the $I^{(\mathbb{D},a,b)}_{\rho,c}$-minimizer belongs to $\hat{\mathcal{X}}$. See Remark 21.

Remark 7. The proof of Proposition 1 uses Theorem B, a Polyakov-Alvarez type formula for a smooth conformal change of metric proved in our companion paper [19]. The regularity assumptions at a, that is, that $\overline{\mathbb{D}}$ locally has the geometry of a smooth slit domain and that γ is smoothly attached at a, are imposed to guarantee smoothness of the change of metric used in the proof.

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2 Preliminaries

2.1 The chordal and radial Loewner equations

We here provide some of the details regarding the chordal and radial Loewner equations that were omitted in Section 1.1. For further details, we refer the reader to [21] and [18]. A compact set $K \subset \overline{\mathbb{H}}$ is called a half-plane hull if $\mathbb{H} \setminus K$ is simply connected and $K = \overline{K \cap \mathbb{H}}$. For each half-plane hull there is a unique conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$, called the mapping-out function of γ_t, satisfying hydrodynamic normalization, that is

$$g_K(z) = z + a_K z^{-1} + o(z^{-1}), \text{ as } z \to \infty.$$

A family (K_t) of continuously growing half-plane hulls is said to be parametrized by half-plane capacity if $a_{K_t} = 2t$ for all t. Solving (1) for an arbitrary real-valued and continuous function $t \mapsto W_t$ yields a family of conformal maps $g_t := g_{K_t} : \mathbb{H} \setminus K_t \to \mathbb{H}$, satisfying hydrodynamic normalization, where (K_t) is a family of continuously and locally growing half-plane hulls parametrized by half-plane capacity.

In a similar manner, a compact set $K \subset \overline{\mathbb{D}}$ is called a disk hull if $0 \notin K$, $\mathbb{D} \setminus K$ is simply connected, and $K = \overline{K \cap \mathbb{D}}$. The mapping-out function of a disk hull K is the unique conformal map $g_K : \mathbb{D}\setminus K \to \mathbb{D}$ satisfying $g_K(0) = 0$ and $g_K'(0) > 0$. A family (K_t) of continuously growing disk hulls is said to be parametrized according to conformal radius if $g_{K_t}'(0) = e^t$ for all t. Solving the radial Loewner equation (3) for an arbitrary continuous function $t \mapsto W_t = e^{iw_t} \in \partial\mathbb{D}$ gives a family of mapping-out functions $(g_t := g_{K_t})$ corresponding to a family (K_t) of continuously and locally growing disk hulls parametrized by conformal radius. The radial SLE$_\kappa$, $\kappa \geq 0$, from 1 to 0 in $\mathbb{D}$ is the random curve with radial driving process $w_t = \sqrt{\kappa}B_t$, where B_t is a one-dimensional standard Brownian motion.

We will use $\cdot$ to denote time-derivatives, e.g., $\dot{g}_t(z) = \partial_t g_t(z)$.

2.2 SLE$_\kappa(\rho)$ processes

Let $\kappa \geq 0$ and $\rho \in \mathbb{R}$.

Definition 2. The chordal SLE$_\kappa(\rho)$ in $\mathbb{H}$, starting at 0, with reference point ∞ and force point $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$ is the random curve whose driving process W_t satisfies the SDE

$$dW_t = \text{Re}\frac{\rho}{W_t - z_t}dt + \sqrt{\kappa}dB_t, \quad W_0 = 0,$$

where $z_t = g_t(z_0)$, defined up to the time $\tau_{0+} = \lim_{\varepsilon\to 0+}\tau_\varepsilon$, where $\tau_\varepsilon = \inf\{t > 0 : |W_t - z_t| = \varepsilon\}$, and B_t is standard one-dimensional Brownian motion.

Definition 3. The radial SLE$_\kappa(\rho)$-processes in $\mathbb{D}$, starting at 1, with reference point 0 and force point $z_0 \in \partial\mathbb{D}$ is the random curve whose driving process W_t satisfies the SDE

$$dW_t = -\left(\frac{\rho}{2}W_t\frac{W_t + z_t}{W_t - z_t} + \frac{\kappa}{2}W_t\right)dt + i\sqrt{\kappa}W_t dB_t, \quad W_0 = 1, \tag{14}$$

where $z_t = g_t(z_0)$, defined up to $\tau_{0_+} = \lim_{\varepsilon\to 0^+}\tau_\varepsilon$, where $\tau_\varepsilon = \inf\{t : |W_t - z_t| = \varepsilon\}$, and B_t is a standard one-dimensional Brownian motion.

Remark 8. Note that, if one changes coordinates by $W_t = e^{iw_t}$ and $z_t = e^{iv_t}$, where w_t and v_t are the unique continuous functions with $w_0 = 0$ and $v_0 \in (0, 2\pi)$, then, using Itô's formula, (14) transforms into (4).

Remark 9. One can, for some values of κ and ρ, define the SLE$_\kappa(\rho)$ process past τ_{0+}, up until the so-called continuation threshold. However, when κ is small and $\rho > -2$, as in Theorem 1, we have $\tau_{0+} = \infty$ a.s. Hence, the definition above will suffice for our purposes.

Remark 10. Often, the random curve in Definition 2 is called the chordal SLE$_\kappa(\rho)$ in $\mathbb{H}$ from 0 *to* ∞ with force point z_0. However, for some values of κ and ρ, the chordal SLE$_\kappa(\rho)$ is a.s. bounded. To avoid confusion, we refer to ∞ as the reference point rather than the end-point. The same convention is used in the radial setting.

2.3 Harmonic measure

We recall some of the basic properties of harmonic measure. The harmonic measure of a Borel set $E \subset \partial\mathbb{D}$ with respect to 0 and $\mathbb{D}$ is

$$\omega(0, E, \mathbb{D}) = \frac{m(E)}{2\pi},$$

where $m(E)$ is the "Lebesgue measure" on $\partial\mathbb{D}$. For a simply connected domain D, $z \in \partial D$ and $E \subset \partial D$ (here we consider ∂D as the set of prime ends of D) we define

$$\omega(z, E, D) := \omega(0, \varphi(E), \mathbb{D}),$$

where $\varphi : D \to \mathbb{D}$ is a conformal map with $\varphi(z) = 0$. Then the harmonic measure is, by construction, a conformally invariant probability measure on ∂D. The harmonic measure can be considered as a completely deterministic object, but it also has a useful probabilistic interpretation: $\omega(z, E, D)$ is the probability that a planar Brownian motion, started at z, exits D at E. This interpretation also applies when D is not simply connected. We will sometimes abuse notation in the following way: if K is a compact subset of $\mathbb{C} \setminus D$, then we will write $\omega(z, K, D)$ for $\omega(z, K \cap \partial D, D)$.

We will sometimes refer to monotonicity of harmonic measure. By this we mean the following: Suppose that D_1 and D_2 are two domains with $z \in D_1 \cap D_2$, and that $E_1 \subset \partial D_1$ and $E_2 \subset \partial D_2$ are such that any path in $\mathbb{C}$ starting at z which exits D_1 at E_1 also exits D_2 at E_2. Then, the probabilistic interpretation of harmonic measure gives

$$\omega(z, E_1, D_1) \leq \omega(z, E_2, D_2).$$

We will also encounter the harmonic measure with respect to a boundary point. If D is a domain with smooth boundary at $x \in \partial D$ and $E \subset \partial D$ we define

$$\omega_x(E, D) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \omega(x + \varepsilon n_x, E, D)$$

where n_x is the inner unit normal at x. If $\varphi : D \to D'$ is a conformal map where D' is a domain with smooth boundary at $x' = \varphi(x)$ then

$$\omega_{x'}(\varphi(E), D') = \omega_x(E, D)|\varphi'(x)|^{-1}.$$

If D is a domain such that $0 \in \partial(1/D)$, where $1/D := \{1/z : z \in D\}$, and $\partial(1/D)$ is smooth at 0, we define

$$\omega_\infty(E, D) := \omega_0(1/E, 1/D),$$

for $\partial E \subset \partial D$. If D and D' are two such domains and $\varphi : D \to D'$ is a conformal map fixing ∞ this gives

$$\omega_\infty(\varphi(E), D') = \omega_\infty(E, D)|\varphi'(\infty)|^{-1},$$

where we use the convention $\varphi'(\infty) := \psi'(0)$, with $\psi(z) = 1/\varphi(1/z)$ (we use this convention for derivatives at ∞ throughout). Finally, we mention, that if $I = [a, b] \subset \mathbb{R}$, then

$$\begin{aligned}
\omega(z, I, \mathbb{H}) &= \frac{1}{\pi} \arg\left(\frac{z-a}{z-b}\right), \qquad z \in \mathbb{H}, \\
\omega_x(I, \mathbb{H}) &= \frac{1}{\pi} \frac{b-a}{(x-a)(x-b)}, \quad x \in \mathbb{R} \setminus I, \\
\omega_\infty(I, \mathbb{H}) &= \frac{1}{\pi}(b-a).
\end{aligned}$$

2.4 Chordal Loewner energy

The chordal Loewner energy of a curve $\gamma : (0, T) \to \mathbb{H}$, $T \in (0, \infty]$, with $\gamma(0+) = 0$ and reference point ∞, parametrized by half-plane capacity, is the Dirichlet energy of its driving function W,

$$I^C(\gamma) = \begin{cases} \frac{1}{2}\int_0^T \dot{W}_t^2 dt, & W \text{ abs. cont. on } (0, t] \text{ for all } t \in (0, T), \\ \infty, & \text{otherwise.} \end{cases}$$

Here C denotes the chordal setting $(\mathbb{H}; 0, \infty)$. If $I^C(\gamma) < \infty$, then γ is simple. One could, in principle, also talk about the chordal Loewner energy of a family of continuously and locally growing half-plane hulls $(K_t)_t$. However, unless $(K_t)_t$ corresponds to a simple curve, we have $I^C((K_t)_t) = \infty$, so we can restrict to studying (simple) curves. It is not hard to show that I^C is scale invariant. This allows one to define the Loewner energy of a curve γ in a simply connected domain D from $a \in \partial D$ with reference point $b \in \partial D \setminus \{a\}$ by using a conformal map $\varphi : D \to \mathbb{H}$, with $\varphi(a) = 0$, $\varphi(b) = \infty$

$$I^{(D;a,b)}(\gamma) := I^{(\mathbb{H};0,\infty)}(\varphi(\gamma)) = I^C(\varphi(\gamma)).$$

The unique minimizer of I^C is $i\mathbb{R}^+$ (since the corresponding driving function is $W \equiv 0$) and hence the unique minimizer of $I^{(D;a,b)}$ is the hyperbolic geodesic from a to b. The Loewner energy has the additive property

$$I^{(\mathbb{H};0,\infty)}(\gamma) = I^{(\mathbb{H};0,\infty)}(\gamma_t) + I^{(\mathbb{H}\setminus\gamma_t;\gamma(t),\infty)}(\gamma_{[t,T)}).$$

Therefore the Loewner energy of a bounded curve $\gamma : (0,T] \to \mathbb{H}$ equals the Loewner energy of the "completed curve" $\gamma \cup \eta$, that is,

$$I^C(\gamma) = I^C(\gamma \cup \eta),$$

where η is the hyperbolic geodesic in $\mathbb{H} \setminus \gamma$ from $\gamma(T)$ to ∞.

Consider $z_0 \in \mathbb{H}$. By [45, Proposition 3.1]

$$\inf_{\gamma \ni z_0} I^C(\gamma) = -8 \log \sin \arg(z_0) = -8 \log \sin(\pi\omega(z_0, (-\infty, 0], \mathbb{H})) \tag{15}$$

where the infimum is taken over all simple curves passing through z_0. The infimum is attained for the curve $\mathrm{SLE}_0(-8)$ with force point at z_0.

2.5 Large deviation principles

We give an overview of the concepts from large deviations theory that we will use, and refer the reader to [9] for details. Let $\mathcal{Y}$ be a regular Hausdorff topological space. A function $J : \mathcal{Y} \to [0,\infty]$ is called a rate function if it is lower-semicontinuous, that is, if the sub-level sets $\{y \in \mathcal{Y} : J(y) \leq c\}$, $c \geq 0$, are closed. If all sub-level sets are compact we say that J is a good rate function. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathcal{Y}$ and let $(P_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on $(\mathcal{Y},\mathcal{B})$. We say that $(P_\varepsilon)_{\varepsilon>0}$ satisfies the large deviations principle with rate function J if

$$\liminf_{\varepsilon\to 0+} \varepsilon \log P_\varepsilon(O) \geq -\inf_{y\in O} J(y), \quad \limsup_{\varepsilon\to 0+} \varepsilon \log P_\varepsilon(F) \leq -\inf_{y\in F} J(y),$$

for all open sets O and closed sets F.

When showing the large deviation principle for the $\mathrm{SLE}_\kappa(\rho)$ driving process we will use theory of exponential approximations. The main idea is that one can obtain an LDP on a family of measures by first showing LDPs for families of measures which are sufficiently good approximations of the original family.

Definition 4. Let $(\mathcal{Y}, d)$ be a metric space and $(P_{m,\varepsilon})_{\varepsilon>0}$, $m \in \mathbb{N}$, and $(P_\varepsilon)_{\varepsilon>0}$ be families of probability measures on $(\mathcal{Y},\mathcal{B})$. We say that $((P_{m,\varepsilon})_\varepsilon)_m$ is an exponentially good approximation of $(P_\varepsilon)_\varepsilon$ if there, for every $\varepsilon > 0$, exists a probability space $(\Omega, \mathcal{F}, \mu_\varepsilon)$ and random variables (Z_ε), $(Z_{m,\varepsilon})$, $m \in \mathbb{N}$, on $(\Omega, \mathcal{F}, \mu_\varepsilon)$, with marginal distributions P_ε and $P_{m,\varepsilon}$ respectively, such that

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mu_{\varepsilon,m}(d(Z_\varepsilon, Z_{\varepsilon,m}) > \delta) = -\infty,$$

for each $\delta > 0$. If the approximating sequence is constant, that is, $P_{m,\varepsilon} = \tilde{P}_\varepsilon$ for all m, then we say that (P_ε) and $(\tilde{P}_\varepsilon)$ are exponentially equivalent.

Theorem A ([9, Theorem 4.2.16]). *Suppose that for every m, the family of measures $(P_{m,\varepsilon})$ satisfies the LDP with rate function J_m and that $(P_{m,\varepsilon})$ is an exponentially good approximation of (P_ε). If J is a good rate function and for every closed set F,*

$$\inf_{y\in F} J(y) \leq \limsup_{m\to\infty} \inf_{y\in F} J_m(y),$$

then (P_ε) satisfies the LDP with rate function J.

2.6 ζ-regularized determinants of Laplacians

Let (M,g) be a Riemannian 2-manifold with $\partial M \neq 0$. The (positive) Dirichlet Laplace-Beltrami operator on M, originally defined on smooth functions with compact support, can be extended to a self-adjoint operator using a Friedrichs extension. Denote this operator by $\Delta_{(M,g)}$. The spectrum of $\Delta_{(M,g)}$ is discrete and positive, and the eigenvalues, which can be ordered in non-decreasing order,

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...,$$

satisfy Weyl's law

$$\lambda_n \sim \frac{n}{4\pi \operatorname{Vol}(M,g)}, \quad \text{as } n \to \infty. \tag{16}$$

Hence, the determinant of $\Delta_{(M,g)}$ is not defined in the classical sense. The ζ-regularized determinant of $\Delta_{(M,g)}$ is defined using the spectral ζ-function

$$\zeta_{(M,g)}(s) = \sum_{n\geq 1} \lambda_n^{-s}, \ \operatorname{Re} s > 1.$$

Note that the right-hand side converges by (16). A computation shows that the spectral ζ-function can be rewritten as

$$\zeta_{(M,g)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(e^{-t\Delta_{(M,g)}})dt, \ \operatorname{Re} s > 1.$$

Under certain regularity assumptions on (M,g) (which we will make precise below), one can, using a short-time asymptotic expansion of the heat trace $\operatorname{Tr}(e^{-t\Delta_{(M,g)}})$, show that $\zeta_{(M,g)}$ can be analytically continued to a neighborhood of the origin. This allows one to define the ζ-regularized determinant of $\Delta_{(M,g)}$ by

$$\det\nolimits_\zeta \Delta_{(M,g)} := e^{-\zeta'_{(M,g)}(0)}$$

motivated by the formal computation

$$\zeta'_{(M,g)}(s) = -\sum_{n\geq 1} \log \lambda_n \lambda^{-s}, \ \operatorname{Re} s > 1 \rightsquigarrow \text{“}\zeta'_{(M,g)}(0) = -\log \prod_{n\geq 1} \lambda_n.\text{”}$$

Definition 5. Let M be a compact surface with boundary $\partial M \neq \varnothing$, with finitely many distinct marked points $p_1, ..., p_n \in \partial M$, a smooth structure and a smooth Riemannian metric g on $M \setminus \{p_1, ..., p_n\}$. We say that $(M, g, (p_j), (\beta_j))$ is a curvilinear polygonal domain with corners $\beta_j \pi \in (0, 2\pi]$, $j = 1, ..., n$, if there exists, for each $j = 1, ..., n$ an open neighborhood $U_j \ni p_j$ and a homeomorphism $\varphi_j : U_j^\circ \to V_j \subset \mathbb{C}$, extending continuously to U_j with $\varphi_j(p_j) = 0$ satisfying the following:

(i) The boundary $\varphi_j(\partial M \cap U_j)$ (viewed in the sense of prime ends) is rectifiable. Let $\gamma : (a,b) \to \mathbb{C}$ be the arc-length parametrization of $\varphi_j(\partial M \cap U_j)$ which is positively oriented and satisfies $\gamma(0) = 0$. Then $\gamma|_{(a,0]}$ and $\gamma|_{[0,b)}$ are smooth and form an interior angle $\beta_j \pi$ at 0.

(ii) The pull-back $(\varphi_j^{-1})^* g$ can be expressed as $e^{2\sigma_j} dz^2$ where $\sigma_j \in C^\infty(V_j^\circ)$, and all partial derivatives extend continuously to $\gamma|_{(a,0]}$ and $\gamma|_{[0,b)}$.

(iii) There is a smooth Jordan curve extension Γ_1 of $\gamma|_{(a,0]}$ such that Γ_1 is positively oriented with respect to the bounded component of $\mathbb{C} \setminus \Gamma_1$. Moreover, there exists an extension $\sigma_{j,1} \in C^\infty(V_{j,1})$ of σ_j, where $V_{j,1}$ is the closure of the bounded component of $\mathbb{C} \setminus \Gamma_1$.

(iv) The analogous to the previous condition holds for $\gamma_{[0,b)}$ giving a Jordan curve Γ_2, a smoothly bounded domain $V_{j,2}$, and $\sigma_{j,2} \in C^\infty(V_{j,2})$.

Remark 11. In Definition 5 "smooth structure" refers to a smooth structure in the usual differential geometry sense. That is, a smooth structure on $M \setminus \{p_1, ..., p_n\}$ is an atlas of smoothly compatible charts $(U_\alpha, \varphi_\alpha)_\alpha$, where $\varphi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism and $V_\alpha \subset \overline{\mathbb{H}}$ relatively open.

Remark 12. Definition 5 can be generalized to allow for corners with interior angles $\beta_j\pi \in (0,\infty)$, see [19]. Since we will only encounter curvilinear polygonal domains where the interior angles $\beta_j\pi \in (0, 2\pi]$, Definition 5 will suffice for the purposes of this paper.

Definition 6. Let $(M, g, (p_j), (\beta_j))$ be a curvilinear polygonal domain with angles $\beta_j\pi \in (0, 2\pi]$. We say that $\psi : M \to \mathbb{R}$ is smooth, $\psi \in C^\infty(M, g, (p_j), (\beta_j))$, if $\psi \in C^\infty(M \setminus \{p_1, ..., p_n\})$ and if there, for every $j = 1, ..., n$, is a choice of (φ_j, U_j), $V_{j,1}$, and $V_{j,2}$ as in Definition 5 such that all partial derivatives of $\psi \circ \varphi^{-1}$ extend continuously to $\gamma_{(a,0]}$ and $\gamma_{[0,b)}$ (with γ as in Definition 5) and there exists extensions $\psi_{j,1} \in C^\infty(V_{j,1})$ and $\psi_{j,2} \in C^\infty(V_{j,2})$ of $\psi \circ \varphi_j^{-1}$.

In [31], Nursultanov, Rowlett, and Sher obtained a short-time asymptotic expansion of the heat trace on curvilinear polygonal domains with corners of angles $\beta_j\pi \in (0, 2\pi)$, showing that $\det_\zeta \Delta_{(M,g)}$ can be defined for such domains. In the companion paper [19], we obtain a short-time asymptotic expansion for curvilinear polygonal domains with corners of arbitrary positive angles ($\beta_j\pi \in (0,\infty)$), showing that $\det_\zeta \Delta_{(M,g)}$ can be defined in such cases as well.

The short-time asymptotic expansion of the heat trace is not sufficient to compute the regularized determinant. However, using a short-time asymptotic expansion of $\mathrm{Tr}(\sigma e^{-t\Delta_{(M,g)}})$ one can obtain an explicit formula for $\log\det_\zeta \Delta_{(M,g)} - \log\det_\zeta \Delta_{(M,g_0)}$ where $g = e^{2\sigma} g_0$, for smooth σ. The first comparison formulas of this type were obtained by Polyakov, in the case of manifolds without boundary [34], and Alvarez, in the case of manifolds with a smooth boundary [5]. The comparison formulas of Polyakov and Alvarez were proved by Osgood, Phillips, and Sarnak in [32]. In [4], a Polyakov-Alvarez type formula was obtained for curvilinear polygonal domains with angles $\beta_j\pi \in (0, 2\pi)$ and in [19] we show that the same formula holds when $\beta_j\pi \in (0,\infty)$.

For a Riemannian surface (M, g), Vol_g and ℓ_g denote the volume and arc-length measures with respect to g. Moreover, K_g denotes the Gaussian curvature, k_g the geodesic curvature, and ∂_n the outer unit derivative. We have the following Polyakov-Alvarez comparison formula.

Theorem B ([19, Theorem 2])**.** *Let $(M, g_0, (p_j), (\beta_j))$ be a curvilinear polygonal domain and $\sigma \in C^\infty(M, g_0, (p_j), (\beta_j))$. Define $g = e^{2\sigma} g_0$. Then*

$$\begin{aligned}\log\det_\zeta \Delta_{(M;g_0)} - \log\det_\zeta \Delta_{(M,g)} =& \frac{1}{6\pi}\left(\frac{1}{2}\int_M |\nabla_{g_0}\sigma|^2 d\,\mathrm{Vol}_{g_0} + \int_M \sigma K_{g_0} d\,\mathrm{Vol}_{g_0} + \int_{\partial M} \sigma k_{g_0} d\ell_{g_0}\right)\\ &+ \frac{1}{4\pi}\int_{\partial M} \partial_{n_{g_0}}\sigma d\ell_{g_0} + \frac{1}{12}\sum_{j=1}^n \Big(\frac{1}{\beta_j} - \beta_j\Big)\sigma(p_j).\end{aligned}$$

When working with conformal changes of metric, that is, $g = e^{2\sigma} g_0$ for some smooth σ, the following transformation rules are useful

$$\begin{aligned} \Delta_g &= e^{-2\sigma}\Delta_{g_0}, & \partial_{n_g} &= e^{-\sigma}\partial_{n_{g_0}},\\ d\,\mathrm{Vol}_g &= e^{2\sigma} d\,\mathrm{Vol}_{g_0}, & d\ell_g &= e^{\sigma} d\ell_{g_0},\\ \Delta_{g_0}\sigma &= e^{2\sigma}K_g - K_{g_0}, & \partial_{n_{g_0}}\sigma &= e^{\sigma}k_g - k_{g_0}. \end{aligned} \tag{17}$$

For ease of notation, we will drop the subscript specifying the metric when we consider the Euclidean metric, i.e., $\Delta = \Delta_{dz^2}$.

3 Basic properties of the ρ-Loewner energy

In this section, we define the ρ-Loewner energy and give some of its basic properties. For ease of notation we let $C = (\mathbb{H}; 0, \infty)$ and $R = (\mathbb{D}; 1, 0)$ denote the two reference settings.

Definition 7. Fix $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$ and $\rho \in \mathbb{R}$. Let $(K_t)_{t\in[0,T]}$, $T \in [0,\infty)$, be a family of half-plane hulls, parametrized by half-plane capacity and driven by a continuous function W_t, with $W_0 = 0$, such that $z_0 \notin K_T$. We define the ρ-Loewner energy of $(K_t)_{t\in[0,T]}$ in $\mathbb{H}$ from 0, with reference point ∞ and force point z_0, as

$$I^C_{\rho,z_0}((K_t)) = \frac{1}{2}\int_0^T \left(\dot{W}_t - \rho \mathrm{Re}\frac{1}{W_t - z_t}\right)^2 dt,$$

where $z_t = g_t(z_0)$, if W_t is absolutely continuous and otherwise we set $I^C_{\rho,z_0}((K_t)) = \infty$. For a family, $(K_t)_{t\in[0,T)}$, $T \in (0,\infty]$, where $(K_t)_{t\in[0,\tau]}$ is as above, for all $\tau \in [0,T)$, we define

$$I^C_{\rho,z_0}((K_t)_{t\in[0,T)}) = \lim_{\tau\to T-} I^C_{\rho,z_0}((K_t)_{t\in[0,\tau]}).$$

Definition 8. Fix $v_0 \in (0, 2\pi)$ and $\rho \in \mathbb{R}$. Let $(K_t)_{t\in[0,T]}$, $T \in [0,\infty)$, be a family of disk hulls, parametrized by conformal radius and driven by $W_t = e^{iw_t}$, where w_t is continuous with $w_0 = 0$, such that $e^{iv_0} \notin K_T$. The ρ-Loewner energy of $(K_t)_{t\in[0,T]}$ in $\mathbb{D}$ from 1, with reference point 0 and force point e^{iv_0}, is defined as

$$I^R_{\rho,e^{iv_0}}((K_t)) = \frac{1}{2}\int_0^T \left(\dot{w}_t - \frac{\rho}{2}\cot\left(\frac{w_t - v_t}{2}\right)\right)^2 dt,$$

where v_t is the unique continuous function with $e^{iv_t} = g_t(e^{iv_0})$, if w_t is absolutely continuous and otherwise we set $I^R_{\rho,e^{iv_0}}((K_t)) = \infty$. The definition is extended to families $(K_t)_{t\in[0,T)}$, $T \in (0,\infty]$ as in Definition 7.

The integrated formulas for the ρ-Loewner energy in the following proposition allows us to compare the ρ-Loewner energy to the chordal (or radial) Loewner energy for finite times *strictly before* the force point is hit.

Proposition 2. *For families $(K_t)_{t\in[0,T]}$ of half-plane or disk hulls as in Definitions 7 and 8 we have the following*

$$I^C_{\rho,z_0}((K_t)) = I^C((K_t)) + \rho\log\frac{\sin\theta_T}{\sin\theta_0} - \frac{\rho(8+\rho)}{8}\log\frac{|g_T'(z_0)|y_T}{y_0}, \qquad z_0 \in \mathbb{H}, \tag{18}$$

$$I^C_{\rho,z_0}((K_t)) = I^C((K_t)) - \rho\log\frac{|W_T - z_T|}{|z_0|} - \frac{\rho(4+\rho)}{4}\log|g_T'(z_0)|, \qquad z_0 \in \mathbb{R}\setminus\{0\}, \tag{19}$$

$$I^R_{\rho,z_0}((K_t)) = I^R((K_t)) - \rho\log\frac{|W_T - z_T|}{|W_0 - z_0|} - \frac{\rho(4+\rho)}{8}\log(|g_T'(z_0)|^2|g_T'(0)|), \; z_0 \in \partial\mathbb{D}, \tag{20}$$

where I^C denotes the chordal Loewner energy and

$$I^R((K_t)) := \begin{cases} \frac{1}{2}\int_0^T \dot{w}_t^2 dt & \textit{if } w_t \textit{ abs. cont.,} \\ \infty & \textit{otherwise,} \end{cases}$$

is the radial Loewner energy. For $z_0 \in \mathbb{H}$ we write $z_t = x_t + iy_t$, $x_t, y_t \in \mathbb{R}$, and $\theta_t = \arg(z_t)$.

Note that (18-20) hold even when the ρ-Loewner energy is infinite.

Proof. We show (18). Identities (19) and (20) can be shown using the same technique.

Fix $z_0 \in \mathbb{H}$. If W_t is not absolutely continuous, then $I^C(\gamma) = \infty$ and $I^C_{\rho,z_0}(\gamma) = \infty$, by definition. So, we may assume that W_t is absolutely continuous. We then have

$$\frac{1}{2}\int_0^T \left(\dot{W}_t - \rho\mathrm{Re}\frac{1}{W_t - z_t}\right)^2 dt = I^C(\gamma) + \int_0^T \left(\frac{\rho^2}{2}\frac{(W_t - x_t)^2}{|W_t - z_t|^4} - \rho\dot{W}_t\frac{W_t - x_t}{|W_t - z_t|^2}\right)dt.$$

Hence, it suffices to show that

$$\int_0^T \left(\frac{\rho^2}{2}\frac{(W_t - x_t)^2}{|W_t - z_t|^4} - \rho\dot{W}_t\frac{W_t - x_t}{|W_t - z_t|^2}\right)dt = \rho\log\frac{\sin\theta_T}{\sin\theta_0} - \frac{\rho(8+\rho)}{8}\log\frac{|g_T'(z_0)|y_T}{y_0}. \tag{21}$$

The Loewner equation gives

$$\dot{x}_t = 2\frac{x_t - W_t}{|W_t - z_t|^2}, \quad \dot{y}_t = -2\frac{y_t}{|W_t - z_t|^2}, \quad \dot{g}_t'(z_0) = -\frac{2g_t'(z_0)}{(g_t(z_0) - W_t)^2}.$$

which in turn yields

$$\partial_t(\log y_t) = -\frac{2}{|W_t - z_t|^2}, \quad \partial_t(\log|g_t'(z_0)|) = \mathrm{Re}\partial_t \log g_t'(z_0) = -2\frac{(W_t - x_t)^2 - y_t^2}{|z_t - W_t|^4},$$

Since W_t is absolutely continuous, $z_t \in C^1$, and $|W_t - z_t|$ is bounded below, we have that $\log|W_t - z_t|$ is absolutely continuous with

$$\partial_t \log|W_t - z_t| = \dot{W}_t \frac{W_t - x_t}{|W_t - z_t|^2} + 2\frac{(W_t - x_t)^2 - y_t^2}{|W_t - z_t|^4} \text{ a.e.}$$

By combining the above and integrating we find (21). □

Proposition 2 immediately shows that $I^C_{\rho,z_0}((K_t)_{t\in[0,T]}) < \infty$ implies that $(K_t)_{t\in[0,T]}$ corresponds to a simple curve (note that this assumes $z_0 \notin K_T$). The same conclusion can be drawn in the disk setting in view of the following proposition.

Proposition 3. *Fix $v_0 \in (0, 2\pi)$ and let $(K^R_t)_{t\in[0,T]}$ be a family of disk hulls as in Definition 8. Let $\varphi_0 : \mathbb{H} \to \mathbb{D}$ be a conformal map with $\varphi_0^{-1}(e^{iv_0}) = \infty$, $\varphi_0^{-1}(1) = 0$. Then $(K^C_{t(s)}) := (\varphi_0^{-1}(K^R_{t(s)}))$, where $s \mapsto t(s)$ is the appropriate re-parametrization, is a family of half-plane hulls as in Definition 7 with respect to the point $z_0^C := \varphi_0^{-1}(0)$. The reverse also holds, i.e., every such family of half-plane hulls induces a family of disk hulls. Moreover, under this change of coordinates*

$$I^R_{\rho,e^{iv_0}}((K^R_t)_t) = I^C_{-6-\rho,z_0}((K^C_s)_s).$$

This can be seen as a deterministic analog of [38, Theorem 3], and the proof is very similar.

Proof. Throughout this proof, superscripts C and R will be used to distinguish between variables, functions, etc., belonging to the chordal and radial settings respectively. Most of these computations are given in [38], but for the sake of completeness we present them here as well. Let (g^R_t) be the family of mapping-out functions corresponding to (K^R_t). Since (K^C_t) is a family of continuously growing half-plane hulls ($e^{iv_0} \notin K^R_T$), there is a re-parametrization $t = t(s) : [0, S] \to [0, T]$ such that $(K^C_{t(s)})$ is parametrized by half-plane capacity. Let g^C_s be the mapping-out function of $K^C_{t(s)}$. We denote $\varphi_s := g^R_{t(s)} \circ \varphi_0 \circ (g^C_s)^{-1} : \mathbb{H} \to \mathbb{D}$. It can be seen that φ_s must be of the form, $\varphi_s(z) = \lambda_s \frac{z - z_s}{z - \overline{z}_s}$, where $z_s = g^C_s(z_0)$, for some $\lambda_s \in \partial\mathbb{D}$. Note that $\lambda_s = \varphi_s(\infty) = g^R_{t(s)}(e^{iv_0})$. Loewner's theorem shows that (g^C_s) satisfies the chordal Loewner equation with $W^C_s = \varphi_s^{-1}(e^{iw_{t(s)}})$. Since $(g^R_t)'(0) = e^t$ we have

$$\frac{dt}{ds} = \partial_s \log|(\varphi_0^{-1})'(0)(g^C_s)'(z_0)\varphi_s'(z_s)| = \partial_s \log|(g^C_s)'(z_0)| - \partial_s \log y_s = \frac{4y_s^2}{|z_s - W^C_s|^4},$$

where the third equality follows from the chordal Loewner equation. Since

$$w_{t(s)} = -i\log(\varphi_s(W^C_s)),$$

we have that w_t is absolutely continuous if and only if W^C_s is absolutely continuous, and if they are then

$$\partial_s w_{t(s)} = -i(\partial_s \log\lambda_s + \partial_s \log(W^C_s - z_s) - \partial_s \log(W^c_s - \overline{z}_s)) \text{ a.e.}$$

From the radial Loewner equation, we have $\partial_s \lambda_s = \lambda_s \frac{W^R_{t(s)} + \lambda_s}{W^R_{t(s)} - \lambda_s} \frac{dt}{ds}$. Thus,

$$\partial_s \log\lambda_s = \frac{\varphi_s(W^C_s) + \lambda_s}{\varphi_s(W^C_s) - \lambda_s} \frac{dt}{ds} = i\frac{W^C_s - x_s}{y_s} \frac{dt}{ds}.$$

Moreover, the chordal Loewner equation gives

$$\partial_s \log(W_s^C - z_s) = \frac{\dot{W}_s^C}{W_s^C - z_s} + \frac{2}{(W_s^C - z_s)^2},$$

and the same holds when z_s is replaced by $\overline{z}_s$. Finally, observe that

$$\cot\left(\frac{w_{t(s)} - v_{t(s)}}{2}\right) = -\frac{W_s^C - x_s}{y_s}.$$

When combining all of the above, we find that

$$\int_0^T \left(\dot{w}_t - \rho \cot\left(\frac{w_s - v_s}{2}\right)\right)^2 ds = \int_0^S \left(\dot{W}_s^C + (6+\rho)\frac{W_s^C - x_s}{|W_s^C - z_s|^2}\right)^2 ds,$$

whenever W_s^C and w_t are absolutely continuous. This completes the proof. □

Since we now know that any hull family, on a closed interval $[0,T]$, with infinite chordal Loewner energy also has infinite ρ-Loewner energy, we may from now on restrict our attention to hull families corresponding to simple curves.

In most cases, curves in $\mathbb{H}$ ($\mathbb{D}$) will be parametrized according to half-plane capacity (conformal radius) but in general, we will consider curves γ up to re-parametrization. When we write $I^C_{\rho,z_0}(\gamma)$ ($I^R_{\rho,e^{iv_0}}(\gamma)$) it is implicit that the curve γ is (re-)parametrized by half-plane capacity (conformal radius).

For a force point $x_0 \in \mathbb{R} \setminus \{0\}$ we have the following analog of Proposition 3.

Proposition 4. *Fix $x_0 \in \mathbb{R} \setminus \{0\}$. Let $\gamma : (0,T] \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ parametrized according to half-plane capacity and let $\varphi : \mathbb{H} \to \mathbb{H}$ be a conformal map satisfying $\varphi(0) = 0$ and $\varphi(x_0) = \infty$. Then*

$$I^C_{\rho,x_0}(\gamma) = I^C_{-6-\rho,\varphi(\infty)}(\varphi(\gamma)). \tag{22}$$

The proof is similar to that of Proposition 3 and is left to the reader.

Let $D \subsetneq \mathbb{C}$ be a simply connected domain, $a \in \partial D$, $b \in \partial D \setminus \{a\}$, and $c \in \overline{D} \setminus \{a,b\}$ (here we consider ∂D in terms of prime ends). We define the ρ-Loewner energy of a simple curve $\gamma \in D \setminus \{c\}$, starting at a, with reference point b and force point $c \in \overline{D} \setminus \{a,b\}$, as

$$I^{(D;a,b)}_{\rho,c}(\gamma) := I^C_{\rho,\varphi(c)}(\varphi(\gamma)),$$

where $\varphi : D \to \mathbb{H}$ is a conformal map with $\varphi(a) = 0$ and $\varphi(b) = \infty$. This is well-defined since $I^C_{\rho,z_0}(\gamma) = I^C_{\rho,\lambda z_0}(\lambda\gamma)$ for any $\lambda > 0$. Suppose instead that $a \in \partial D$, $b \in D$, and $c \in \partial D \setminus \{a\}$. Then we define the ρ-Loewner energy of a simple curve γ starting at a, with reference point b and force point c, as

$$I^{(D;a,b)}_{\rho,c}(\gamma) := I^R_{\rho,\varphi(c)}(\varphi(\gamma)),$$

where $\varphi : D \to \mathbb{D}$ is *the* conformal map with $\varphi(a) = 1$ and $\varphi(b) = 0$. Propositions 3 and 4 can now be summarized by (6). The ρ-Loewner energy satisfies the same type of additive property as the chordal Loewner energy, that is

$$I^{(D;a,b)}_{\rho,c}(\gamma_T) = I^{(D;a,b)}_{\rho,c}(\gamma_t) + I^{(D\setminus\gamma_t;\gamma(t),b)}_{\rho,c}(\gamma_{[t,T]}),\ t \in (0,T).$$

4 A large deviation principle for $\text{SLE}_\kappa(\rho)$

The goal of this section is to prove Theorem 5, which includes Theorem 1. The proof follows the same outline as the proof of the LDP for chordal SLE_κ in [33]. Using an LDP on the driving process on a finite time interval (obtained in Section 4.1) and certain continuity properties of the Loewner map we obtain an LDP on $\text{SLE}_\kappa(\rho)$ stopped at a finite time (see Section 4.2). We then show, in Section 4.3, the large deviation principle on the full $\mathrm{SLE}_\kappa(\rho)$ curve by using estimates on return probabilities (which are shown in Appendix A). These steps can be carried out, more or less exactly as in [33]. Therefore, the bulk of the work will lie in showing the LDP for the driving process of $\mathrm{SLE}_\kappa(\rho)$, and establishing the return probability estimates for $\mathrm{SLE}_\kappa(\rho)$.

Throughout this section $\mathbb{P}$ denotes the standard Wiener measure and B_t denotes a one-dimensional standard Brownian motion.

4.1 LDP on driving process

The radial and chordal settings will be treated in very similar ways. Whenever we wish to distinguish the two cases we will use a symbol $X \in \{C, R\}$, often as a superscript, where C and R denote $(\mathbb{H}, 0, \infty)$ and $(\mathbb{D}, 1, 0)$ respectively. We define

$$H^C(x) := \frac{2}{x} \quad \text{and} \quad H^R(x) := \cot\left(\frac{x}{2}\right).$$

Fix $\rho > -2$, $T > 0$, and $v_0 \in \mathbb{R} \setminus \{0\}$ for $X = C$ and $v_0 \in (0, 2\pi)$ for $X = R$. Let $C_0([0,T])$ denote the space of real-valued continuous functions $t \mapsto f_t$ on $[0,T]$ satisfying $f_0 = 0$, endowed with the supremum norm. For each $f \in C_0([0,T])$, there is a unique pair (w^ρ, v^ρ) of continuous functions satisfying

$$w_t^\rho = \frac{\rho}{2} \int_0^t H^X(w_s^\rho - v_s^\rho) ds + f_t, \tag{23}$$

$$v_t^\rho = -\int_0^t H^X(w_s^\rho - v_s^\rho) ds + v_0, \tag{24}$$

for $t < \tau_{0+} \wedge T$ where $\tau_{0+} = \tau_{0+}(f) = \lim_{\varepsilon \to \infty} \tau_\varepsilon$ and $\tau_\varepsilon := \inf\{t : |H^X(w_t^\rho - v_t^\rho)| \geq 1/\varepsilon\}$. Indeed, since H^X is Lipschitz on $\{x : |H^X(x)| \leq 1/\varepsilon\}$ there is a unique solution to (23, 24) up to time τ_ε, for every $\varepsilon > 0$. We define, for $\varepsilon > 0$,

$$C_{\varepsilon,\rho}^X = \{f \in C_0([0,T]) : \tau_\varepsilon = \infty\}, \quad C_{0+,\rho}^X = \{f \in C_0([0,T]) : \tau_{0+} = \infty\},$$

and

$$\mathcal{W}^\rho = \mathcal{W}^\rho(T) : C_0([0,T]) \to C_0([0,T])$$

by $\mathcal{W}^\rho(f) = w^\rho(f)$, if $f \in C_{0+,\rho}^X$, and by $\mathcal{W}^\rho(f) \equiv 0$ if $f \in C_0([0,T]) \setminus C_{0+,\rho}^X$. One can, using standard arguments, show that $\mathcal{W}^\rho$ is continuous on $C_{0+,\rho}^X$. It follows from Lemma 3, proved below, that $\sqrt{\kappa} B|_{[0,T]} \in C_{0+,\rho}^X$ a.s. for sufficiently small κ. Therefore,

$$\mathcal{W}^\rho(\sqrt{\kappa} B|_{[0,T]}) = w^\rho(\sqrt{\kappa} B|_{[0,T]}), \text{ a.s.}$$

That is, $\mathcal{W}^\rho(\sqrt{\kappa} B|_{[0,T]})$ coincides a.s. with the driving process of $\mathrm{SLE}_\kappa(\rho)$ when κ is small. We introduce a functional $I_{\rho,v_0}^X : C_0([0,T]) \to [0,\infty]$ defined by

$$I_{\rho,v_0}^X(w) = \frac{1}{2} \int_0^T \left(\dot{w}_t - \frac{\rho}{2} H^X(w_t - v_t)\right)^2 dt \tag{25}$$

for all $w \in \mathcal{W}^\rho(C_{0+,\rho}^X)$ which are absolutely continuous, where v is the solution of (24) given w, and otherwise we set $I_{\rho,v_0}^X(w) = \infty$. Observe that, for $f \in C_{0+,\rho}^X$, we have $I_D(f) = I_{\rho,v_0}^X(w^\rho(f))$, where I_D denotes the Dirichlet energy.

Proposition 5. *The laws of $\mathcal{W}^\rho(\sqrt{\kappa} B|_{[0,T]})$ and $\mathcal{W}^{\kappa+\rho}(\sqrt{\kappa} B|_{[0,T]})$ satisfy the large deviation principle with good rate function I_{ρ,v_0}^X as $\kappa \to 0+$ and $\rho > -2$ is fixed.*

Before presenting the proof of Proposition 5 we provide a few lemmas. Define

$$K^C(x) = x^2/4 \quad \text{and} \quad K^R(x) = \sin^2(x/2),$$

and note that

$$(\log K^X(x))' = H^X(x) \quad \text{and} \quad |H^X(x)| \geq 1/\varepsilon \implies K^X(x) \leq \varepsilon^2.$$

Lemma 1. *For every $M > 0$ there exists $\varepsilon > 0$ such that $I^X_{\rho,v_0}(w) < M$ implies $w \in \mathcal{W}^\rho(C^X_{\varepsilon,\rho})$.*

Proof. Since $I^X_{\rho,v_0}(w) < \infty$ implies $w \in \mathcal{W}^\rho(C^X_{0+,\rho})$, it suffices to prove that there exists $\varepsilon > 0$ such that $I^X_{\rho,v_0}(w) \geq M$ for all $w \in \mathcal{W}^\rho(C^X_{0+,\rho} \setminus C^X_{\varepsilon,\rho})$.

Suppose that $f \in C^X_{0+,\rho} \setminus C^X_{\delta,\rho}$, for some $\delta > 0$, is absolutely continuous, and denote by $w = w^\rho(f)$ and $v = v^\rho(f)$. Then $\tau_\delta(f) \leq T$ implying that $K^X(w_{\tau_\delta} - v_{\tau_\delta}) \leq \delta^2$. We may assume that $\tau_\delta = T$ (since $I^X_{\rho,v_0}(w) \geq I^X_{\rho,v_0}(w_{[0,\tau_\delta]})$). Observe that

$$\partial_t \log K^X(w_t - v_t) = \dot{w}_t H^X(w_t - v_t) + (H^X(w_t - v_t))^2.$$

Therefore,

$$(H^X(w_t - v_t))^2 \leq (H^X(w_t - v_t))^2 + (H^X(w_t - v_t) + \dot{w}_t)^2 = \dot{w}_t^2 + 2\partial_t \log K^X(w_t - v_t),$$

which, when plugged into (25), gives

$$\begin{aligned} I^X_{\rho,v_0}(w) &\geq \min(1, \tfrac{\rho+2}{2})\left(\min(1, \tfrac{\rho+2}{2}) I_D(w) - |\rho| \log \frac{K^X(w_T - v_T)}{K^X(w_0 - v_0)} \right) \\ &\geq -|\rho| \min(1, \tfrac{\rho+2}{2}) \log \frac{\delta^2}{K^X(-v_0)}. \end{aligned}$$

By setting $\varepsilon = \delta$ sufficiently small, the right-hand side becomes larger than M, showing that $I^X_{\rho,v_0}(w) \geq M$ for all $w \in \mathcal{W}^\rho(C^X_{0+,\rho} \setminus C^X_{\varepsilon,\rho})$. □

Lemma 2. $I^X_{\rho,v_0} : C_0([0,T]) \to [0,\infty]$ *is a good rate function.*

Proof. Let $M \in [0,\infty)$. We wish to show that the sub-level set

$$E_M := \{w \in C_0([0,T]) : I^X_{\rho,v_0}(w) \leq M\}$$

is compact. Take a sequence $(w_n) \subset E_M \subset \mathcal{W}^\rho(C^X_{0+,\rho})$. Recall that $\mathcal{W}^\rho : C^X_{0+,\rho} \to C_0([0,T])$ is continuous. Let, for each n, $f_n \in C^X_{0+,\rho}$ be such that $w_n = \mathcal{W}^\rho(f_n)$. Then, $I_D(f_n) = I^X_{\rho,v_0}(w_n) \leq M$. Since $I_D : C_0([0,T]) \to [0,\infty]$ is a good rate function there exists a subsequence (f_{n_k}) converging to $f \in C_0([0,T])$ with $I_D(f) \leq M$. Lemma 1 then implies that $f \in C^X_{0+,\rho}$. By continuity of $\mathcal{W}^\rho$ on $C^X_{0+,\rho}$, w_{n_k} converges to $w = \mathcal{W}^\rho(f)$ and since $I^X_{\rho,v_0}(w) = I_D(f) \leq M$, this shows that E_M is compact. □

We will show Proposition 5 using exponentially good approximations. For each $\varepsilon > 0$, let $H^X_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function satisfying $H^X_\varepsilon(x) = H^X(x)$ whenever $|H^X(x)| \leq 1/\varepsilon$. Let L_ε denote the corresponding Lipschitz constant. For each $f \in C_0([0,T])$ there exists a unique solution $(w^{\rho,\varepsilon}, v^{\rho,\varepsilon}) = (w^{\rho,\varepsilon}(f), v^{\rho,\varepsilon}(f)) \in C_0([0,T]) \times C([0,T])$ to

$$w^{\rho,\varepsilon}_t = \frac{\rho}{2} \int_0^t H^X_\varepsilon(w^{\rho,\varepsilon}_s - v^{\rho,\varepsilon}_s) ds + f_t \tag{26}$$

$$v^{\rho,\varepsilon}_t = -\int_0^t H^X_\varepsilon(w^{\rho,\varepsilon}_s - v^{\rho,\varepsilon}_s) ds + v_0. \tag{27}$$

Note that $w^{\rho,\varepsilon} : C_0([0,T]) \to C_0([0,T])$ is continuous and bijective since for each $w \in C_0([0,T])$ there is a unique v solving (27), and for this pair (w,v) we may solve (26) for f. It follows from the contraction principle and Schilder's theorem (see, e.g., [9, Theorem 4.2.1 and Theorem 5.2.3]) that the process $w^{\rho,\varepsilon}(\sqrt{\kappa} B|_{[0,T]})$, $t \in [0,T]$ satisfies the large deviation principle with good rate function

$$I^X_{\rho,v_0,\varepsilon}(w) = \begin{cases} \frac{1}{2} \int_0^T (\dot{w}_t - \frac{\rho}{2} H^X_\varepsilon(w_t - v_t))^2 dt & \text{if } w \text{ abs. cont.,} \\ \infty & \text{otherwise.} \end{cases}$$

Note that, for $f \in C^X_{\varepsilon,\rho}$, we have $\mathcal{W}^\rho(f) = w^{\rho,\varepsilon}(f)$ and $I^X_{\rho,v_0}(\mathcal{W}^\rho(f)) = I^X_{\rho,v_0,\varepsilon}(w^{\rho,\varepsilon}(f))$.

Lemma 3. *Fix $\rho > -2$ and v_0 as above. Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \kappa < \rho+2$ and $0 < \varepsilon < \varepsilon_0$*

$$\mathbb{P}[\sqrt{\kappa}B|_{[0,T]} \notin C^X_{\varepsilon,\rho}] \leq c\varepsilon^{2\frac{\rho+2}{\kappa}-1} \tag{28}$$

where $c = c(\kappa, \rho, v_0, T)$, for which $\lim_{\kappa\to 0+} \kappa \log c(\kappa, \rho, v_0, T)$ exists and is finite. In particular,

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}[\sqrt{\kappa}B|_{[0,T]} \notin C^X_{\varepsilon,\rho}] \to -\infty \ \text{ as } \varepsilon \to 0+\,. \tag{29}$$

Proof. Define $\tilde{w}^{\kappa,\rho}_t := v^\rho_{t/\kappa}(\sqrt{\kappa}B) - w^\rho_{t/\kappa}(\sqrt{\kappa}B)$, $t \in [0, \kappa T]$, so that

$$d\tilde{w}^{\kappa,\rho}_t = \frac{\rho+2}{2\kappa} H^X(\tilde{w}^{\kappa,\rho}_t)dt + d\tilde{B}_t$$

where $\tilde{B}_t = -\sqrt{\kappa}B_{t/\kappa}$ is a standard Brownian motion. We introduce a local martingale $F_{a,v_0}(\tilde{w}^{\kappa,\rho}_t) + t$ where $a = (\rho+2)/\kappa$, and

$$F_{a,v_0}(x) = -2\int_{v_0}^x \int_{v_0}^t (K^X(u))^a du (K^X(t))^{-a} dt$$

for $x \in (0, 2\pi)$, when $X = R$, and, for $\pm x > 0$, when $X = C$ and $\pm v_0 > 0$. Indeed, $F_{a,v_0}(\tilde{w}^{\kappa,\rho}_t)+t$ is a local martingale since

$$\begin{aligned}
F'_{a,v_0}(x) &= -2\int_{v_0}^x (K^X(u))^a du (K^X(x))^{-a},\\
F''_{a,v_0}(x) &= -2 + 2a\int_{v_0}^x (K^X(u))^a du (K^X(x))^{-a-1}(K^X)'(x),
\end{aligned}$$

so that $d(F_{a,v_0}(\tilde{w}^{\kappa,\rho}_t) + t) = F'_{a,v_0}(\tilde{w}^{\kappa,\rho}_t)d\tilde{B}_t$ by Itô's formula. Let $\delta, M > 0$ be such that $K^X(v_0) \in (\delta, M)$. We define a stopping time $\tau^{\kappa,\rho}_{\delta,M} = \kappa T \wedge \tau^{\kappa,\rho}_\delta \wedge \tau^{\kappa,\rho}_M$, where

$$\tau^{\kappa,\rho}_x = \inf\{t : K^X(\tilde{w}^{\kappa,\rho}_t) = x\}.$$

By the condition on δ and M, $F_{a,v_0}(\tilde{w}^{\kappa,\rho}_{t\wedge\tau^{\kappa,\rho}_{\delta,M}}) + t \wedge S^{\kappa,\rho}_{\delta,M}$ is uniformly bounded, and therefore a martingale. Hence,

$$0 = \mathbb{E}[F_{a,v_0}(\tilde{w}^{\kappa,\rho}_0) + 0] \leq \mathbb{E}[F_{a,v_0}(\tilde{w}^{\kappa,\rho}_{\kappa T\wedge S^{\kappa,\rho}_{\delta,M}})] + \kappa T. \tag{30}$$

Observe that $F_{a,v_0}(x) \leq 0$, and that

$$\mathbb{E}[F_{a,v_0}(\tilde{w}^{\kappa,\rho}_{\kappa T\wedge S^{\kappa,\rho}_{\delta,M}})|\tau^{\kappa,\rho}_\delta \leq \kappa T \wedge \tau^{\kappa,\rho}_M] \leq F_{a,\tilde{v}^X_0}(x^X_\delta),$$

where

$$\tilde{v}^X_0 = \begin{cases} |v_0|, & X = C,\\ v_0 \wedge (2\pi - v_0), & X = R,\end{cases} \qquad \text{and} \qquad x^X_\delta = \begin{cases} 2\sqrt{\delta}, & X = C,\\ 2\arcsin\sqrt{\delta}, & X = R.\end{cases}$$

Combining this with (30) we obtain

$$\mathbb{P}[\tau^{\kappa,\rho}_\delta \leq \kappa T \wedge \tau^{\kappa,\rho}_M] \leq \frac{\kappa T}{-F_{a,\tilde{v}^X_0}(x^X_\delta)}.$$

One can see that $\mathbb{P}[\tau^{\kappa,\rho}_M < \tau^{\kappa,\rho}_\delta \wedge \kappa T] \to 0$ as $M \to \infty$ (in the case $X = R$ this is trivial) since the drift of $\tilde{w}^{\kappa,\rho}_t$ is bounded for $t < \tau^{\kappa,\rho}_\delta$. Therefore,

$$\mathbb{P}[\tau^{\kappa,\rho}_\delta \leq \kappa T] \leq \frac{\kappa T}{-F_{a,\tilde{v}_0}(x^X_\delta)}.$$

It remains to bound $-F_{a,\tilde{v}_0}(x_\delta^X)$ from below. For this, we use that

$$\frac{x^2}{\pi^2} \le K^R(x) \le \frac{x^2}{4},\ x \in (0,\pi].$$

A computation shows that for $0 < x < \tilde{v}_0^X/3$ and $a \ge 1$ we have

$$-F_{a,\tilde{v}_0}(x) \ge \frac{c_1^a (\tilde{v}_0^X)^{2a+1} x^{-2a+1}}{4a^2-1},$$

for a constant $c_1 > 0$. Now set $\delta = \varepsilon^2$. Since $|H^X(x)| \le 1/\varepsilon$ implies $K^X(x) \le \delta$ we find

$$\mathbb{P}[\sqrt{\kappa}B_{[0,T]} \notin C_{\varepsilon,\rho}^X] \le \mathbb{P}[\tau_\delta^{\kappa,\rho} \le \kappa T] \le c_2(\kappa,\rho)\frac{T\varepsilon^{2a-1}}{(\tilde{v}_0^X)^{2a+1}}, \tag{31}$$

where $\lim_{\kappa\to 0+} c_2(\kappa,\rho) \in \mathbb{R}$. Here we have used that $x_\delta^X \le c_3\sqrt{\delta} = c_3\varepsilon$ for a constant $c_3 > 0$. This concludes the proof. □

Lemma 4. *Fix $\varepsilon > 0$. There exists $\kappa_0 > 0$ such that $\|w^{\kappa+\rho}(f) - w^\rho(f)\|_\infty \le \frac{|\kappa|T}{2\varepsilon} e^{\frac{|\rho+\kappa|+2}{2}L_{\varepsilon/2}T}$, where L_ε is the Lipschitz constant of H_ε^X, for all $f \in C_{\varepsilon,\rho}^X$ and $\kappa \in (0,\kappa_0)$.*

Proof. Let $\tilde{T}(\kappa) = T \wedge \inf\{t : |H^X(w_t^{\kappa+\rho} - v_t^{\kappa+\rho})| \ge 2/\varepsilon\}$. Then for $t \in [0,\tilde{T}(\kappa)]$

$$\begin{aligned}
&|w_t^\rho - w_t^{\kappa+\rho}| + |v_t^\rho - v_t^{\kappa+\rho}| \\
&\le \frac{|\rho+\kappa|+2}{2}\int_0^t |H^X(w_s^\rho - v_s^\rho) - H^X(w_s^{\kappa+\rho} - v_s^{\kappa+\rho})|ds + \frac{|\kappa|}{2}\int_0^t |H^X(w_s^\rho - v_s^\rho)|ds \\
&\le \frac{|\rho+\kappa|+2}{2}L_{\varepsilon/2}\int_0^t |(w_s^\rho - v_s^\rho) - (w_s^{\rho+\kappa} - v_s^{\rho+\kappa})|ds + \frac{|\kappa|T}{2\varepsilon}.
\end{aligned}$$

Therefore Grönwall's lemma shows

$$|w_t^\rho - w_t^{\rho+\kappa}| + |v_t^\rho - v_t^{\rho+\kappa}| \le \frac{|\kappa|T}{2\varepsilon} e^{\frac{|\rho+\kappa|+2}{2}L_{\varepsilon/2}T}, \tag{32}$$

for $t \in [0,\tilde{T}(\kappa)]$. Since $|H^X(w_t^\rho - v_t^\rho)| \le 1/\varepsilon$ for all $t \in [0,T]$ there exists a κ_0, which may be chosen uniformly over all $f \in C_{\varepsilon,\rho}^X$, such that (32) forces $\tilde{T}(\kappa) = T$ for all $\kappa \in (0,\kappa_0)$. Hence, (32) holds for all $t \in [0,T]$ whenever $\kappa \in (0,\kappa_0)$. □

Proof of Proposition 5. For each $n = 1, 2, ...$, we have

$$\begin{aligned}
\mathbb{P}[\|\mathcal{W}^\rho(\sqrt{\kappa}B|_{[0,T]}) - w^{\rho,1/n}(\sqrt{\kappa}B|_{[0,T]})\|_\infty > \delta] &\le \mathbb{P}[\|\mathcal{W}^\rho(\sqrt{\kappa}B|_{[0,T]}) - w^{\rho,1/n}(\sqrt{\kappa}B|_{[0,T]})\|_\infty > 0] \\
&\le \mathbb{P}[\sqrt{\kappa}B|_{[0,T]} \notin C_{1/n,\rho}^X].
\end{aligned}$$

By applying Lemma 3, we deduce that (the law of) $w^{\rho,1/n}(\sqrt{\kappa}B_t)$ is an exponentially good approximation of (the law of) $\mathcal{W}^\rho(\sqrt{\kappa B_t})$. Moreover, we claim that for any closed set F

$$\inf_{w\in F} I_{\rho,v_0}^X(w) \le \limsup_{n\to\infty} \inf_{w\in F} I_{\rho,v_0,1/n}^X(w).$$

Suppose F were a closed set for which this did not hold. Let $M = \inf_{w\in F} I_{\rho,v_0}^X(w)$ so that

$$\limsup_{n\to\infty} \inf_{w\in F} I_{\rho,v_0,1/n}^X(w) < M.$$

Let $\varepsilon > 0$ be as in Lemma 1, and let $\tilde{F} = F \cap \mathcal{W}^\rho(C_{\varepsilon,\rho}^X)$. Then

$$\limsup_{n\to\infty} \inf_{w\in F} I_{\rho,v_0,1/n}^X(w) = \limsup_{n\to\infty} \inf_{w\in \tilde{F}} I_{\rho,v_0,1/n}^X(w) = \inf_{w\in\tilde{F}} I_{\rho,v_0}^X(w)$$

since $I^X_{\rho,v_0,1/n}(w) = I^X_{\rho,v_0}(w)$, for all $w \in \mathcal{W}^\rho(C^X_{1/n,\rho})$ whenever $n \geq 1/\varepsilon$. This contradicts the assumption on F and establishes the claim. By applying Theorem A, we obtain the LDP for $\mathcal{W}^\rho(\sqrt{\kappa}B_t)$.

To obtain the LDP for $\mathcal{W}^{\kappa+\rho}(\sqrt{\kappa}B|_{[0,T]})$ we, in light of Theorem A, only have to show that it is exponentially equivalent to $\mathcal{W}^\rho(\sqrt{\kappa}B|_{[0,T]})$. Let $\delta > 0$ and fix $\varepsilon > 0$. By Lemma 4 we have that there exists a κ_0, depending on ε but not on δ, such that $\|\mathcal{W}^{\kappa+\rho}(f) - \mathcal{W}^\rho(f)\|_\infty \leq \delta$ for all $\kappa \in (0, \kappa_0)$ and $f \in C^X_{\varepsilon,\rho}$. Thus,

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}[\|\mathcal{W}^{\kappa+\rho}(\sqrt{\kappa}B|_{[0,T]}) - \mathcal{W}^\rho(\sqrt{\kappa}B|_{[0,T]})\|_\infty > \delta] \leq \limsup_{\kappa\to 0+} \kappa \log \mathbb{P}[\sqrt{\kappa}B|_{[0,T]} \notin C^X_{\varepsilon,\rho}].$$

By applying Lemma 3, and then letting $\varepsilon \to 0+$ we find that

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}[\|\mathcal{W}^{\kappa+\rho}(\sqrt{\kappa}B_t) - \mathcal{W}^\rho(\sqrt{\kappa}B_t)\|_\infty > \delta] = -\infty,$$

for all $\delta > 0$. This shows that $\mathcal{W}^{\kappa+\rho}(\sqrt{\kappa}B_t)$ is exponentially equivalent to $\mathcal{W}^\rho(\sqrt{\kappa}B_t)$. □

4.2 LDP on finite time curves

Let $\mathcal{C}^R$ denote the (compact) space of compact subsets of $\overline{\mathbb{D}}$ endowed with the Hausdorff distance and the topology induced by it. Fix a conformal map $\varphi : \mathbb{D} \to \mathbb{H}$. Let $\mathcal{C}^C = \varphi(\mathcal{C}^R)$, endowed with the distance and topology induced by φ. The induced Hausdorff distance on $\mathcal{C}^C$ depends on the choice of φ, however, the induced topology (which we refer to simply as the Hausdorff topology) will not. We will denote the (induced) Hausdorff distance by $d^h : \mathcal{C}^X \times \mathcal{C}^X \to \mathbb{R}$.

Fix $T \in (0, \infty)$. Let $\mathcal{K}^C$ and $\mathcal{K}^R$ denote the space of half-plane hulls and disk hulls respectively. Further, let

$$\mathcal{K}^C_T = \{K \in \mathcal{K}^C : \text{hcap}(K) = 2T\} \quad \text{and} \quad K^R_={K \in \mathcal{K}^R : \text{crad}(\mathbb{D} \setminus K, 0) = e^{-T}\}.$$

We will consider two topologies on $\mathcal{K}^X$: the Hausdorff topology and the topology induced by Carathéodory kernel convergence. The former is simply the subspace topology induced by the inclusion $\iota : \mathcal{K}^X_T \hookrightarrow \mathcal{C}^X$. The latter is the topology where $(K_n) \subset \mathcal{K}^X_T$ converges to $K \in \mathcal{K}^X_T$ if and only if the inverse mapping-out functions $(g^{-1}_{K_n})$ converge uniformly on compacts to (g^{-1}_K).

Throughout this section we fix, in addition to T, $\rho > -2$ and $z_0 = v_0 \in \mathbb{R} \setminus \{0\}$, if $X = C$, and $z_0 = e^{iv_0}$, $v_0 \in (0, 2\pi)$, if $X = R$. The goal of this section is to show the following large deviation principle.

Proposition 6. *The X-SLE$_\kappa(\rho)$ process and X-SLE$_\kappa(\kappa+\rho)$ process, stopped at time T (with standard parametrization), satisfy the large deviation principle with good rate function*

$$I^X_{\rho,z_0} : \mathcal{K}^X_T \to [0, \infty]$$

with respect to the Hausdorff topology on $\mathcal{K}^X_T$.

Since $\tau_{0+} = \infty$ a.s. for the X-SLE$_\kappa(\rho)$ when $\rho > -2$ and κ is sufficiently small (recall Lemma 3), we have that an X-SLE$_\kappa(\rho)$ stopped at time $T \wedge \tau_{0+}$ is a.s. the image of $\mathcal{W}^\rho(\sqrt{\kappa}B|_{[0,T]})$ under the Loewner map

$$\mathcal{L}^X_T : C_0([0,T]) \to \mathcal{K}^X_T,$$
$$w \mapsto K_T.$$

The Loewner map $\mathcal{L}^X_T$ is continuous when $\mathcal{K}^X_T$ is endowed with the Carathéodory topology (see [18, Proposition 6.1] for the chordal case and [29, Proposition 6.1] for the radial case), but is not continuous when $\mathcal{K}^X_T$ is endowed with the Hausdorff topology. Lemmas 5 and 6 were shown in the chordal setting in [33] and as the proofs in the radial setting are almost identical we omit them.

Lemma 5 (Chordal case: [33, Lemma 2.3])**.** *Let $(K_n)_n \in \mathcal{K}^X$ be a sequence of X-hulls converging to $K \in \mathcal{K}^X$ in the Carathéodory topology and to $\tilde{K} \in \mathcal{C}$, with $\infty \notin \tilde{K}$ if $X = C$ and $0 \notin \tilde{K}$ if $X = R$, in the Hausdorff topology. If $X = R$, then $\mathbb{D} \setminus K$ coincides with the connected component of $\mathbb{D} \setminus \tilde{K}$ containing 0. If $X = C$, then $\mathbb{H} \setminus K$ coincides with the unbounded connected component of $\mathbb{H} \setminus \tilde{K}$.*

Lemma 6 (Chordal case: [33, Lemma 2.4])**.** *Let F be a Hausdorff-closed subset of $\mathcal{K}_T^X$. If*

$$w \in \overline{(\mathcal{L}_T^X)^{-1}(F)} \setminus (\mathcal{L}_T^X)^{-1}(F)$$

then $\mathcal{L}_T^X(w)$ has non-empty interior. Similarly, if O is a Hausdorff-open subset of $\mathcal{K}_T^X$ and $w \in (\mathcal{L}_T^X)^{-1}(O) \setminus ((\mathcal{L}_T^X)^{-1}(O))^\circ$ then $\mathcal{L}_T^X(w)$ has non-empty interior.

Lemma 7. *Let $O \subset \mathcal{K}_T^X(z_0)$ be Hausdorff-open and $F \subset \mathcal{K}_T^X(z_0)$ be Hausdorff-closed. Then*

$$\inf_{w \in ((\mathcal{L}_T^X)^{-1}(O))^\circ} I_{\rho,v_0}^X(w) = \inf_{w \in (\mathcal{L}_T^X)^{-1}(O)} I_{\rho,v_0}^X(w),$$

$$\inf_{w \in \overline{(\mathcal{L}_T^X)^{-1}(F)}} I_{\rho,v_0}^X(w) = \inf_{w \in (\mathcal{L}_T^X)^{-1}(F)} I_{\rho,v_0}^X(w).$$

Proof. Any $w \in \overline{(\mathcal{L}_T^X)^{-1}(F)} \setminus (\mathcal{L}_T^X)^{-1}(F)$ (and similarly $w \in (\mathcal{L}_T^X)^{-1}(O) \setminus (\mathcal{L}_T^X)^{-1}(O)^\circ$) corresponds to an X-hull with non-empty interior K by Lemma 6. If $w \notin \mathcal{W}^\rho(C_{0+,\rho}^X)$ then $I_{\rho,v_0}^X(w) = \infty$ by definition. If $w \notin \mathcal{W}^\rho(C_{0+,\rho}^X)$, then $z_0 \notin K_T$ and $I_{\rho,v_0}^X(w) = I_{\rho,v_0}^X((K_t)_t) = \infty$ by the discussion in Section 3. □

Lemma 8. *$I_{\rho,z_0}^X : \mathcal{K}_T^X \to [0,\infty]$ is a good rate function.*

Proof. Fix $M \in [0,\infty)$. We wish to show that the sub-level set

$$E_M := \{K \in \mathcal{K}_T^X : I_{\rho,z_0}^X(K) \le M\}$$

is compact. Note that all $K \in E_M$ are simple curves (this follows from the discussion in Section 3). Take a sequence $(\gamma_n)_n \subset E_M$. Let $w_n = (\mathcal{L}_T^X)^{-1}(\gamma_n) \in \mathcal{W}^\rho(C_{0+,\rho}^X)$ be the corresponding driving functions, and note that $I_{\rho,v_0}^X(w_n) = I_{\rho,z_0}^X(\gamma_n)$. Since $I_{\rho,v_0}^X : C_0([0,T]) \to [0,\infty]$ is a good rate function, there exists a subsequence (w_{n_k}) converging to some $w \in \mathcal{W}^\rho(C_{0+,\rho}^X)$ with $I_{\rho,v_0}^X(w) \le M$. Therefore, $K = \mathcal{L}_T^X(w)$ must be a simple curve $\gamma = K$, with $I_{\rho,z_0}^X(\gamma) = I_{\rho,v_0}^X(w)$. Now, let $F_{k_0} = \overline{\{\gamma_{n_k}\}_{k=k_0}^\infty}$, for $k_0 \in \mathbb{N}$, where the closure is taken in $\mathcal{K}_T^X$ w.r.t. the Hausdorff topology. We now claim that $\gamma \in F_{k_0}$ for all k_0. Suppose the opposite. Then there exists k_0 such that $w \in \overline{(\mathcal{L}_T^X)^{-1}(F_{k_0})} \setminus (\mathcal{L}_T^X)^{-1}(F_{k_0})$. By Lemma 6, γ has non-empty interior, a contradiction. Therefore, since $\gamma \in F_{k_0}$ for all $k_0 \in \mathbb{N}$, there is a further subsequence $(\gamma_{n_{k_l}})$ converging to $\gamma \in E_M$, as desired. □

Proof of Proposition 6. Let $\gamma^{\kappa,\rho}$ denote an X-SLE$_\kappa(\rho)$, and let O be a Hausdorff-open subset of $\mathcal{K}_T^X$. Then by Proposition 5 and Lemma 7 we have

$$\begin{aligned}
\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}[\gamma_T^{\kappa,\rho} \in O] &= \liminf_{\kappa \to 0+} \kappa \log \mathbb{P}[\mathcal{W}^\rho(\sqrt{\kappa} B|_{[0,T]}) \in (\mathcal{L}_T^X)^{-1}(O)] \\
&\ge \liminf_{\kappa \to 0+} \kappa \log \mathbb{P}[\mathcal{W}^\rho(\sqrt{\kappa} B|_{[0,T]}) \in ((\mathcal{L}_T^X)^{-1}(O))^\circ] \\
&\ge - \inf_{w \in ((\mathcal{L}_T^X)^{-1}(O))^\circ} I_{\rho,v_0}^X(w) \\
&= - \inf_{w \in (\mathcal{L}_T^X)^{-1}(O)} I_{\rho,v_0}^X(w) \\
&= - \inf_{\gamma \in O} I_{\rho,z_0}^X(\gamma)
\end{aligned}$$

The closed sets and the case $\gamma^{\kappa,\kappa+\rho}$ can be treated in the same way. This finishes the proof. □

4.3 LDP on infinite time curves

Let $\mathcal{X}^C \subset \mathcal{C}^C$ and $\mathcal{X}^R \subset \mathcal{C}^R$ denote the spaces of simple curves in $\mathbb{H}$ from 0 to ∞ and in $\mathbb{D}$ from 1 to 0 respectively, endowed with the subspace topology. Let

$$D^X = \begin{cases} \mathbb{H}, & X = C, \\ \mathbb{D}, & X = R, \end{cases} \qquad a^X = \begin{cases} 0, & X = C, \\ 1, & X = R, \end{cases} \qquad b^X = \begin{cases} \infty, & X = C, \\ 0, & X = R. \end{cases}$$

Throughout this section we fix $\rho > -2$, and $z_0 \in \partial D^X \setminus \{a^X\}$. For $r > 0$, let

$$C_r = \{z \in D^X : d^h(\{z\}, \{b^X\}) = r\},$$

where d^h is the (induced) Hausdorff distance (see Section 4.2). For a curve $\gamma \in \mathcal{X}^X$, let $\hat{\tau}_r = \inf\{s : \gamma(s) \in C_r\}$. If $\kappa < \min(2(\rho+2), 4)$ then the X-SLE$_\kappa(\rho)$ curve is simple and approaches b^X. The first follows from absolute continuity with respect to SLE$_\kappa$ up to $T \wedge \tau_{0+}$ for all $T \geq 0$, and the second was shown in the chordal and radial settings in [26, Theorem 1.3] and [30, Proposition 3.30] respectively. For $\kappa < \min(2(\rho+2), 4)$, let $\mathbb{P}^{\kappa,\rho}$ be the X-SLE$_\kappa(\rho)$ probability measure on $\mathcal{X}^X$.

The main ingredient needed to extend the large deviation principle from finite to infinite time is the following lemma.

Lemma 9. *Let $R, M > 0$. Then there exists an $r > 0$ such that*

(a) $\inf\{I^X_{\rho,z_0}(\gamma) : \gamma \in \mathcal{X}^X, \gamma_{[\hat{\tau}_r,\infty)} \cap C_R \neq \varnothing\} \geq M$,

(b) $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma_{[\hat{\tau}_r,\infty)} \cap C_R \neq \varnothing] \leq -M$,

(b') $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\kappa+\rho}[\gamma_{[\hat{\tau}_r,\infty)} \cap C_R \neq \varnothing] \leq -M$.

Proof. In the case $X = R$ this coincides exactly with the statement in Proposition 18. In the case $X = C$, this is essentially the same statement as Proposition 17, only expressed in a different metric. □

Corollary 1. *Let $\tilde{R}, M > 0$. Then there exists $T > 0$ such that*

(a) $\inf\{I^X_{\rho,z_0}(\gamma) : \gamma \in \mathcal{X}^X, d^h(\gamma_T, \gamma) \geq \tilde{R}\} \geq M$,

(b) $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[d^h(\gamma_T, \gamma) \geq \tilde{R}] \leq -M$,

(b') $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\kappa+\rho}[d^h(\gamma_T, \gamma) \geq \tilde{R}] \leq -M$.

Proof. Let r be as in Lemma 9 with $R = \tilde{R}/2$. We claim that $T = T_r$ is sufficient, where T_r is a deterministic upper bound on $\hat{\tau}_r$ (such upper bounds exist both in the radial and chordal setting). Suppose $\gamma \in \mathcal{X}^X$ such that $\gamma_{[\hat{\tau}_r,\infty]} \cap C_R = \varnothing$, then

$$d^h(\gamma_T, \gamma) \leq d^h(\gamma_{\hat{\tau}_r}, \gamma) < r + R < \tilde{R}.$$

We therefore conclude that

$$\{\gamma \in \mathcal{X}^X : d^h(\gamma, \gamma_T) \subset \tilde{R}\} \leq \{\gamma \in \mathcal{X}^X : \gamma_{[\hat{\tau}_r,\infty)} \cap C_R \neq \varnothing\}.$$

The result now follows from Lemma 9. □

Theorem 4. *The ρ-Loewner energy $I^X_{\rho,z_0} : \mathcal{X}^X \to [0,\infty]$ is a good rate function.*

Proof. Let $M \in [0,\infty)$ and consider $E_M = \{\gamma \in \mathcal{X}^X : I^X_{\rho,z_0}(\gamma) \leq M\}$. Since $\mathcal{C}^X$ is compact we have that E_M is compact if it is closed as a subset of $\mathcal{C}^X$.

Let $(\gamma_n) \subset E_M$ be a sequence converging to $K \in \mathcal{C}^X$, and let $(w^n) \subset C_0([0,\infty))$ be the sequence of corresponding driving functions. Since $I^X_{\rho,v_0} : C_0([0,T]) \to [0,\infty]$ is a good rate function for all $T > 0$ (see Lemma 2) there is a subsequence (w_{n_k}) converging uniformly on compact subsets of $[0,\infty)$ to some $w \in C_0([0,\infty))$. It follows that $I^X_{\rho,v_0}(w|_{[0,T]}) \leq M$ for all

$T \in (0,\infty)$. Therefore, w encodes a simple curve γ with $I^X_{\rho,z_0}(\gamma) \leq M$. Additionally, Corollary 9 shows that $\gamma(T) \to b^X$ as $T \to \infty$. Hence $\gamma \in E_M \subset \mathcal{X}^X$. Note that, the continuity of $\mathcal{L}_T$ on simple curves (recall Lemma 6) shows that $\gamma^{n_k}_T \to \gamma_T$ as $n \to \infty$ for each fixed $T > 0$ (as in the proof of Lemma 8). We now wish to show that $\gamma^{n_k} \to \gamma$, i.e., $K = \gamma$. If $K \neq \gamma$, then $\varepsilon := d^h(K,\gamma) > 0$. Fix $T > 0$ such that $I^X_{\rho,z_0}(\eta) > M$ for all $\eta \in \mathcal{X}^X$ with $d^h(\eta_T,\eta) \geq \varepsilon/4$ (such a T exists by Corollary 1). Next, fix k such that $d^h(\gamma^{n_k},K) < \varepsilon/4$ and $d^h(\gamma^{n_k}_T,\gamma_T) < \varepsilon/4$. Then,

$$d^h(K,\gamma) \leq d^h(K,\gamma^{n_k}) + d^h(\gamma^{n_k},\gamma^{n_k}_T) + d^h(\gamma^{n_k}_T,\gamma_T) + d^h(\gamma_T,\gamma) < \varepsilon,$$

a contradiction. Therefore, $\gamma_{n_k} \to K = \gamma \in E_M$. We deduce that E_M is closed. □

Theorem 5. *The X-$SLE_\kappa(\rho)$ and X-$SLE_\kappa(\kappa+\rho)$ processes satisfy the large deviation principle, as $\kappa \to 0+$, with respect to the Hausdorff topology, with good rate function I^X_{ρ,z_0}. That is, for any Hausdorff-open subset $O \subset \mathcal{X}^X$ and Hausdorff-closed subset $F \subset \mathcal{X}^X$ we have*

$$\liminf_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma \in O] \geq -\inf_{\gamma\in O} I^X_{\rho,z_0}(\gamma), \tag{33}$$

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma \in F] \leq -\inf_{\gamma\in F} I^X_{\rho,z_0}(\gamma) \tag{34}$$

and the same holds when ρ is replaced by $\kappa + \rho$.

Proof. For $K \in \mathcal{C}^X$ and $r > 0$ we use $B^h(K,r) \subset \mathcal{C}^X$ to denote the Hausdorff-open ball centered at K and of radius r.

We start with the closed sets. Let $F \subset \mathcal{X}^X$ be a Hausdorff-closed set and let $\overline{F}$ denote its closure in $\mathcal{C}^X$. Note that $\overline{F}$ is compact. Moreover, denote

$$N = \inf_{\gamma\in F} I^X_{\rho,z_0}(\gamma) \in [0,\infty].$$

If $N = 0$, then (34) is trivial, so we may assume that $N > 0$. Fix $M \in (0,N)$. Since I^X_{ρ,z_0} is a good rate function on $\mathcal{X}^X$ there exists, for every $K \in \overline{F}$, an $\varepsilon_K > 0$ such that

$$\gamma \in B^h(K,3\varepsilon_K) \cap \mathcal{X}^X \implies I^X_{\rho,z_0}(\gamma) \geq M. \tag{35}$$

Since $\{B^h(K,\varepsilon_K)\}_{K\in\overline{F}}$ is an open cover of the compact set $\overline{F}$ we know that there is a finite sub-cover $\{B^h(K_1,\varepsilon_1),...,B^h(K_n,\varepsilon_n)\}$, where we write $\varepsilon_j = \varepsilon_{K_j}$. We deduce that

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma \in F] \leq \max\{\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma \in \overline{B^h(K_j,\varepsilon_j)} \cap \mathcal{X}^X]\}_{j=1}^n.$$

For every $j = 1,...,n$, let $T_j > 0$ be chosen according to Corollary 1 with M and $\tilde{R} = \varepsilon_j$. By the triangle inequality we see that

$$\gamma \in \overline{B^h(K_j,\varepsilon_j)} \cap \mathcal{X}^X \implies \gamma_{T_j} \in \overline{B^h(K_j,2\varepsilon_j)} \cap \mathcal{K}^X_{T_j} \text{ or } d^h(\gamma,\gamma_{T_j}) \geq \varepsilon_j.$$

Hence $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma \in \overline{B^h(K_j,\varepsilon_j)} \cap \mathcal{X}^X]$ is bounded above by the maximum of

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma_{T_j} \in \overline{B^h(K_j,2\varepsilon_j)} \cap \mathcal{K}^X_{T_j}] \leq -\inf_{\gamma_{T_j}\in\overline{B^h(K_j,2\varepsilon_j)}\cap\mathcal{K}^X_{T_j}} I^X_{\rho,z_0}(\gamma_{T_j}), \tag{36}$$

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[d^h(\gamma,\gamma_{T_j}) \geq \varepsilon_j] \leq -M. \tag{37}$$

The bound (36) follows from Proposition 6 and the fact that $\overline{B^h(K_j,2\varepsilon_j)} \cap \mathcal{K}^X_{T_j}$ is a Hausdorff-closed subset of $\mathcal{K}^X_{T_j}$. The bound (37) follows directly from Corollary 1. We now show that

$$\inf_{\gamma_{T_j}\in\overline{B^h(K_j,2\varepsilon_j)}\cap\mathcal{K}^X_{T_j}} I^X_{\rho,z_0}(\gamma_{T_j}) \geq M. \tag{38}$$

Take $\gamma_{T_j} \in \overline{B^h(K_j, 2\varepsilon_j)} \cap \mathcal{K}_{T_j}^X$ and denote its I_{ρ,z_0}^X-optimal continuation by $\tilde{\gamma}$. If $d^h(\gamma_{T_j}, \tilde{\gamma}) \geq \varepsilon_j$, then the choice of T_j gives that $I_{\rho,z_0}^X(\tilde{\gamma}) \geq M$. If instead $d^h(\gamma_{T_j}, \tilde{\gamma}) < \varepsilon$, then the triangle inequality implies that $d^h(\tilde{\gamma}, K_j) < 3\varepsilon_j$ and the choice of ε_j then implies that $I_{\rho,z_0}^X(\tilde{\gamma}) \geq M$. Since $I_{\rho,z_0}^X(\gamma_{T_j}) = I_{\rho,z_0}^X(\tilde{\gamma})$, this shows (38) and we deduce that

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma^\kappa \in F] \leq -M.$$

Taking the limit $M \to N-$ we obtain (34).

We move on to the open sets. Let O be a Hausdorff-open subset of $\mathcal{X}^X$ and now let

$$N = \inf_{\gamma \in O} I_{\rho,z_0}^X(\gamma).$$

If $N = \infty$, then (33) is trivial, so we may assume that $N < \infty$. Fix $\varepsilon > 0$. There exists a $\gamma^\varepsilon \in O$ such that $I_{\rho,z_0}^X(\gamma^\varepsilon) \leq N + \varepsilon$. Moreover, there exists a $\delta = \delta(\varepsilon) > 0$ such that $B^h(\gamma^\varepsilon, 2\delta) \cap \mathcal{X}^X \subset O$. Let T be as in Corollary 1 with $M = N + 2\varepsilon$ and $\tilde{R} = \delta$. Now, for $\gamma \in \mathcal{X}^X$

$$\gamma_T \in B^h(\gamma^\varepsilon, \delta) \cap \mathcal{K}_T^X \implies \gamma \in B^h(\gamma^\varepsilon, 2\delta) \text{ or } d^h(\gamma_T, \gamma) \geq \delta,$$

by the triangle inequality. Therefore,

$$\mathbb{P}^{\kappa,\rho}[\gamma \in B^h(\gamma^\varepsilon, 2\delta) \cap \mathcal{X}^X] \geq \mathbb{P}^{\kappa,\rho}[\gamma_T \in B^h(\gamma^\varepsilon, \delta) \cap \mathcal{K}_T^X] - \mathbb{P}^{\kappa,\rho}[d^h(\gamma, \gamma_T) \geq \delta].$$

Since $B^h(\gamma^\varepsilon, \delta) \cap \mathcal{K}_T^X$ is a Hausdorff-open subset of $\mathcal{K}_T^X$, and since we must have $d^h(\gamma^\varepsilon, \gamma_T^\varepsilon) < \delta$ (by the choice of T), Proposition 6 shows

$$\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma_T \in B^h(\gamma^\varepsilon, \delta) \cap \mathcal{K}_T^X] \geq -I_{\rho,z_0}^X(\gamma_T^\varepsilon) \geq -(N + \varepsilon).$$

By Corollary 1

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[d^h(\gamma_T, \gamma) \geq \delta] \leq -(N + 2\varepsilon).$$

Combining the above we find,

$$\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma \in O] \geq -(N + \varepsilon),$$

and by taking the limit $\varepsilon \to 0+$ we obtain (33). The same arguments hold when ρ is replaced by $\kappa + \rho$. □

5 The minimizers – $\text{SLE}_0(\rho)$

It is a direct consequence of the definition of the ρ-Loewner energy, that its unique minimizer is the $\text{SLE}_0(\rho)$ curve. In this section, we will study these curves. As we will see in Section 5.2, some of these curves have been studied extensively before, but not, to our knowledge, under the name $\text{SLE}_0(\rho)$ [22, 23]. $\text{SLE}_0(\rho)$ curves have also appeared in [24], in the context of optimization problems for the Loewner energy, and in [3], where multichordal SLE_0 are described as $\text{SLE}_0(\overline{\rho})$.

We will see that in both the radial and chordal setting the $\text{SLE}_0(\rho)$ curves exhibit three phases: a force point hitting phase, a boundary hitting phase, and a reference point hitting phase. This is illustrated in Figure 1.

In Section 5.1 we will show that the interpretation of $\text{SLE}_\kappa(\rho)$, $\kappa > 0$, as generalized flow-lines of the GFF (see, e.g., [26, 30]) has the expected deterministic analog when $\kappa = 0$: $\text{SLE}_0(\rho)$ is a flow-line, in the classical sense, of the appropriate harmonic field. In [3, Section 4.4], a similar statement was shown for multichordal SLE_0, and in [42] the authors proved a finite Loewner energy analog of the SLE-GFF flow-line coupling.

We proceed by studying chordal $\text{SLE}_0(\rho)$ with a boundary force point, and radial $\text{SLE}_0(\rho)$ with a boundary force point (equivalent to chordal $\text{SLE}_0(\rho)$ with an interior force point) separately in Section 5.2 and Section 5.3. In Section 5.3 we also define and study a whole-plane variant of $\text{SLE}_0(\rho)$ starting at ∞, with reference point 0 and force point ∞, when $\rho \leq -2$.

5.1 Flow-line property

For $z_0 \in \mathbb{C}$, let M_{z_0} be the logarithmic Riemann surface centered at z_0. By this we mean that $M_{z_0} = \{(r,\theta) : \ r > 0,\ \theta \in \mathbb{R}\}$ endowed with the complex structure induced by $\pi_{z_0} : M_{z_0} \to \mathbb{C}$, $(r,\theta) \mapsto z_0 + re^{i\theta}$. For convenience we use the notation $z = z_0 + re^{i\theta}$ for $z = (r,\theta) \in M_0$, so that M_0 is (locally) identified with $\mathbb{C}$. The main point is that $\arg(\cdot - z_0) : M_{z_0} \to \mathbb{R}$ is (with the identification above) a single valued function.

We say that a C^1 curve η, parametrized by arc-length, is a flow line of a field h if

$$\eta'(t) = e^{ih(\eta(t))}.$$

Proposition 7. *Let η^{ρ,z_0} denote an $SLE_0(\rho)$ in $\mathbb{H}$, starting at 0, with reference point ∞ and force point $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$, re-parametrized by arc-length. If $z_0 \in \mathbb{R} \setminus \{0\}$, let $h_{\rho,z_0} : \mathbb{H} \to \mathbb{R}$ be the field defined by*

$$h_{\rho,z_0}(z) = \begin{cases} \pi(1+\frac{\rho}{2}) - \arg(z) - \frac{\rho}{2}\arg(z-z_0), & z_0 > 0, \\ \pi - \arg(z) - \frac{\rho}{2}\arg(z-z_0), & z_0 < 0, \end{cases}$$

where the branches of $\arg(z)$ and $\arg(z-z_0)$ are chosen so that they take values in $(0,\pi)$. Then η^{ρ,z_0} a flow-line of h_{ρ,z_0}.

If $z_0 = re^{i\theta} \in \mathbb{H}$, $\theta \in (0,\pi)$, let $h_{\rho,z_0} : \mathbb{H}_{z_0} \to \mathbb{R}$, where $\mathbb{H}_{z_0} := \pi_{z_0}^{-1}(\mathbb{H}) \subset M_{z_0}$, be the field defined by

$$h_{\rho,z_0}(z) = \pi - \arg(z) - \tfrac{\rho}{4}(\arg(z-z_0) + \arg(z-\bar{z}_0)),$$

where the branches of $\arg(z)$ and $\arg(z-\overline{z}_0)$ are chosen so that they take values in $(0,\pi)$. Then the lift of η^{ρ,z_0} by π_{z_0}, starting at $(r,\theta+\pi)$, is a flow-line of h_{ρ,z_0}.

Proof. Let γ^{ρ,z_0} denote η^{ρ,z_0} parametrized by half-plane capacity. Since the driving function of γ^{ρ,z_0} satisfies

$$\dot{W}_t = \mathrm{Re}\frac{\rho}{W_t - z_t}$$

and the right-hand side is continuous in t, we have that γ^{ρ,z_0} is C^1 away from its end-points [49]. Therefore, for any $t < s$

$$\arg((\gamma^{\rho,z_0})'(t)) = \lim_{w \to g_s(\gamma^{\rho,z_0}(t)-)} \arg((g_s^{-1})'(w)),$$

where g_s is the mapping-out function of γ_s^{ρ,z_0}. We have, for $w_0 \in \mathbb{H} \setminus \gamma^{\rho,z_0}$,

$$\partial_t \log(w_t - W_t) = \frac{1}{w_t - W_t}\left(\frac{2}{w_t - W_t} - \mathrm{Re}\frac{\rho}{W_t - z_t}\right), \quad \partial_t \log g_t'(w_0) = -\frac{2}{(w_t - W_t)^2},$$

$$\partial_t \log(w_t - z_t) = -\frac{2}{(w_t - W_t)(z_t - W_t)}, \quad \partial_t \log(w_t - \overline{z}_t) = -\frac{2}{(w_t - W_t)(\overline{z}_t - W_t)},$$

where $z_t = g_t(z_0)$, $w_t = g_t(w_0)$, and W_t is the driving function of γ^{ρ,z_0}. By combining this and recalling that $\arg = \mathrm{Im}\log$, we find

$$\partial_t (h_{\rho,z_t - W_t}(w_t - W_t) - \arg(g_t'(w_0))) = 0. \tag{39}$$

In the case where $z_0 \in \mathbb{H}$ we make sense of $h_{\rho,z_t-W_t}(w_t - W_t)$ by lifting g_t to a conformal map from M_{z_0} to M_{z_t}. Since $\arg(g_0'(w_0)) = 0$, we thus have,

$$g_t'(w_0) = h_{\rho,z_t-W_t}(w_t - W_t) - h_{\rho,z_0}(w_0).$$

Therefore, if $x_0 \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} \arg((\gamma^{\rho,z_0})'(t)) &= \lim_{w_s \to g_s(\gamma^{\rho,z_0}(t)-)} \arg((g_s^{-1})'(w_s)) \\ &= \lim_{w_s \to g_s(\gamma^{\rho,z_0}(t)-)} (h_{\rho,z_0}(w_0) - h_{\rho,z_s - W_s}(w_s - W_s)) \\ &= h_{\rho,z_0}(\gamma^{\rho,z_0}(t)), \end{aligned}$$

since h_{ρ,z_0} vanishes on $\mathbb{R}^-$ if $z_0 > 0$ and on $(z_0, 0)$ if $z_0 < 0$. This shows that

$$(\eta^{\rho,z_0})'(t) = e^{ih_{\rho,z_0}(\eta^{\rho,z_0}(t))},$$

so that η^{ρ,z_0} is a flow-line of h_{ρ,z_0}. If $z_0 = re^{i\theta} \in \mathbb{H}$, let $\tilde{\gamma}^{\rho,z_0}$ denote the lift of γ^{ρ,z_0} starting at $(r, \theta + \pi)$. Then h_{ρ,z_0} vanishes along the lift of $\mathbb{R}^-$ which has $(r, \theta + \pi)$ as an end-point. We therefore deduce, in the same way as above, that

$$\arg((\gamma^{\rho,z_0})'(t)) = \arg((\tilde{\gamma}^{\rho,z_0})'(t)) = h_{\rho,z_0}(\tilde{\gamma}^{\rho,z_0}(t)).$$

We deduce that the arc-length parametrization, $\tilde{\eta}^{\rho,z_0}$, of $\tilde{\gamma}^{\rho,z_0}$ satisfies

$$(\tilde{\eta}^{\rho,z_0})'(t) = e^{ih_{\rho,z_0}(\tilde{\eta}^{\rho,z_0}(t))},$$

and it is therefore a flow-line of h_{ρ,z_0}. □

5.2 Chordal SLE$_0(\rho)$ with boundary force point

The chordal SLE$_\kappa(\rho)$ with a boundary force point exhibits the following (probabilistic) self-similarity property: if one maps out an initial part of the curve and then rescales the picture so that the image of the force point is mapped to its initial location one gets a new SLE$_\kappa(\rho)$ with the same force point. When $\kappa = 0$ this self-similarity property becomes deterministic.

Deterministically self-similar Loewner curves have been studied before in [22] and [23]. In [23] it was shown that the only curves $\gamma \in C^3$ satisfying

$$\gamma = r(t)(g_t(\gamma_t) - W_t),$$

for all t and some $r(t) > 0$, are curves driven by

(i) $W_t = 0$, which corresponds to $\gamma = i\mathbb{R}^+$ (as a set),

(ii) $W_t = ct$, where $c \neq 0$, which corresponds to a curve γ which approaches ∞ horizontally,

(iii) $W_t = c\sqrt{t+\tau} - c\sqrt{\tau}$, where $c \neq 0$ and $\tau > 0$, corresponding to a curve which approaches ∞ at a certain angle $\theta_1(c) \in (0, \pi)$,

(iv) $W_t = c\sqrt{\tau} - c\sqrt{\tau - t}$, where $|c| \geq 4$ and $\tau > 0$, corresponding to a curve which hits the real line at an angle $\theta_2(c) \in [0, \pi)$, and

(v) $W_t = c\sqrt{\tau} - c\sqrt{\tau - t}$, where $|c| \in (0, 4)$ and $\tau > 0$, corresponding to a logarithmic spiral.

These driving functions were also considered in [17].

Proposition 8. *The driving function of SLE$_0(\rho)$ with force point $x_0 > 0$ is*

$$W_t^{\rho,x_0} = \begin{cases} \frac{\rho}{\rho+2}x_0 - \frac{\rho}{\rho+2}\sqrt{x_0^2 + 2(2+\rho)t}, & \rho \neq -2, \\ \frac{2}{x_0}t, & \rho = -2. \end{cases}$$

When $x_0 < 0$ we have $W_t^{\rho,x_0} = -W_t^{\rho,-x_0}$.

Proof. Assume that $x_0 > 0$. By definition, the driving function of SLE$_0(\rho)$ satisfies

$$dW_t^{\rho,x_0} = \frac{\rho}{W_t^{\rho,x_0} - x_t}dt, \quad W_0^{\rho,x_0} = 0. \tag{40}$$

This gives the separable equation

$$d(x_t - W_t^{\rho,x_0}) = \frac{2+\rho}{x_t - W_t^{\rho,x_0}}dt$$

which is solved by

$$x_t - W_t^{\rho,x_0} = \sqrt{x_0^2 + 2(2+\rho)t}.$$

Plugging this into (40) and integrating yields the expression of the driving function. The case when $x_0 < 0$ follows by symmetry. □

Corollary 2. *Fix $x_0 > 0$ and let γ^{ρ,x_0} be the SLE$_0(\rho)$ with force point x_0. Then*

(i) If $\rho \in (-\infty, -4]$, then $\gamma^{\rho,x_0}(\tau) = x_0$ and the (outer) hitting angle is $\alpha\pi = \pi\frac{4+\rho}{2+\rho}$. Moreover the hitting time is $\tau = -\frac{x_0^2}{2(2+\rho)}$.

(ii) If $\rho \in (-4, -2)$, then $\gamma^{\rho,x_0}(\tau) = -\frac{2}{\rho+2}x_0 > x_0$ and the (outer) hitting angle is $\alpha\pi = \pi\frac{4+\rho}{2}$. Moreover the hitting time is $\tau = -\frac{x_0^2}{2(2+\rho)}$.

(iii) If $\rho = -2$, then γ^{ρ,x_0} approaches ∞ asymptotic to $\{x + i\pi x_0 : x \in \mathbb{R}^+\}$.

(iv) If $\rho \in (-2, \infty)$, then γ^{ρ,x_0} approaches ∞ at an angle $\alpha\pi = \pi\frac{2+\rho}{4+\rho}$.

Proof. These statements all follow from [17] after an appropriate translation and scaling. □

Remark 13. For $\rho > -2$ one can consider the situation where the force point is placed infinitesimally close to the origin, at $0+$ (or similarly $0-$). From Proposition 8 we can see that the driving function then becomes

$$W_t^{\rho,0+} = -\rho\sqrt{\frac{2t}{\rho+2}}$$

which corresponds to a ray from 0 to ∞ with angle $\alpha\pi = \pi\frac{2+\rho}{4+\rho}$ (see, e.g., [17]).

5.3 Radial SLE$_0(\rho)$ with boundary force point

The radial SLE$_0(\rho)$ curves are a bit harder to study. However, we can easily obtain their driving functions.

Proposition 9. *The driving function of radial SLE$_0(\rho)$ with force point at e^{iv_0}, $v_0 \in (0, 2\pi)$, is*

$$w_t^{\rho,v_0} = \begin{cases} -\frac{\rho}{\rho+2}\left(2\arccos\left(\cos\frac{v_0}{2}e^{-\frac{\rho+2}{4}t}\right) - v_0\right), & \rho \neq -2, \\ t\cot\frac{v_0}{2}, & \rho = -2. \end{cases} \tag{41}$$

Proof. The driving function of SLE$_0(\rho)$ satisfies

$$dw_t^{\rho,v_0} = \frac{\rho}{2}\cot\left(\frac{w_t^{\rho,v_0} - v_t}{2}\right)dt \tag{42}$$

so that $v_t - w_t^{\rho,v_0}$ satisfies the separable equation

$$d(v_t - w_t^{\rho,v_0}) = \frac{\rho+2}{2}\cot\left(\frac{v_t - w_t^{\rho,v_0}}{2}\right)dt,$$

solved by

$$v_t - w_t^{\rho,v_0} = 2\arccos\left(\cos\frac{v_0}{2}e^{-\frac{\rho+2}{4}t}\right).$$

Plugging this into (42) and integrating gives the expression for (41). □

Proposition 10. *For all $\rho > -2$ the radial SLE$_0(\rho)$ comes arbitrarily close to 0.*

Proof. The expression for the radial driving function tells us that the SLE$_0(\rho)$ is not stopped at a finite time for $\rho > -2$. □

Remark 14. As a consequence of Corollary 4, which will be shown in Section 6.1, this means that, for each $\rho > -2$, the SLE$_0(\rho)$ also approaches 0.

Now consider $\rho = -2$. The proof of Proposition 9 shows that $v_t - w_t^{-2,v_0} = v_0$, for all t. This implies that the radial SLE$_0(-2)$, or equivalently, chordal SLE$_0(-4)$ is self-similar. Therefore we can easily compute its chordal driving function.

Proposition 11. *The chordal SLE*$_0(-4)$ *with force point* $z_0 \in \mathbb{H}$ *has driving function*

$$W_t^{-4,z_0} = 2\cos\theta_0(|z_0| - \sqrt{|z_0|^2 - 4t}).$$

Hence, the chordal SLE$_0(-4)$ *is a logarithmic spiral approaching* z_0 *(unless* $\theta_0 = \pi/2$*).*

Proof. We saw that the driving function of the radial SLE$_0(-2)$, which we denote here by γ^R, satisfies $v_t - w_t^{-2,v_0} = v_0$, for all t. This is equivalent to

$$\omega(0, (\gamma_t^R)^+ \cup a, \mathbb{D} \setminus \gamma_t^R) = \omega(0, a, \mathbb{D}),$$

where $a = \{e^{i\theta} : \theta \in [0, v_0]\}$ for all t. By conformal invariance of harmonic measure, this gives for chordal SLE$_0(-4)$, denoted here by γ^C,

$$\omega(z_0, (\gamma_t^C)^+ \cup [0,\infty), \mathbb{H} \setminus \gamma_t^C) = \omega(z_0, [0,\infty), \mathbb{H}),$$

for all t, which is equivalent to $\sin\theta_t = \sin\theta_0$ for all t. We have, from the definition of the chordal SLE$_0(-4)$ and the chordal Loewner equation, that

$$\dot{W}_t^{-4,z_0} = -4\frac{W_t^{-4,z_0} - x_t}{|W_t^{-4,z_0} - z_t|^2}, \qquad \dot{x}_t = 2\frac{x_t - W_t^{-4,z_0}}{|W_t^{-4,z_0} - z_t|^2}.$$

Using $\sin\theta_t = \sin\theta_0$ we find

$$\dot{W}_t^{-4,z_0} = 4\frac{\cos\theta_0}{|W_t^{-4,z_0} - z_t|}, \qquad \partial_t \log|W_t^{-4,z_0} - z_t| = -\frac{2}{|W_t^{-4,z_0} - z_t|^2},$$

so that

$$|W_t^{-4,z_0} - z_t| = \sqrt{|z_0|^2 - 4t}, \qquad W_t^{-4,z_0} = 2\cos\theta_0(|z_0| - \sqrt{|z_0|^2 - 4t}).$$

In [17], it was shown that the curve driven by $W_t = 2\sqrt{k(1-t)}$, $k \in (0,4)$, is a logarithmic spiral approaching $\sqrt{k} + \sqrt{k-4}$. After an appropriate translation, scaling, and reflection, this shows that the SLE$_0(\rho)$ curve is a logarithmic spiral approaching z_0. ☐

In order to understand radial SLE$_0(\rho)$ with $\rho < -2$ we note that the expression (41) for w_t^{ρ,v_0} can be extended to the interval $(-\infty, T)$ where $T \in \mathbb{R} \cup \{\infty\}$ such that $\pm e^{\frac{\rho+2}{4}T} = \cos\frac{v_0}{2}$ (where the sign depends on whether $v_0 \in (0,\pi)$ or $v_0 \in (\pi, 2\pi)$). That is,

$$w_t^{\rho,v_0}(t) = -\frac{2\rho}{\rho+2}\arccos\Big(\pm e^{\frac{\rho+2}{4}(T-t)}\Big) + C = \frac{2\rho}{\rho+2}\arcsin\Big(\pm e^{\frac{\rho+2}{4}(T-t)}\Big) + \tilde{C}, \; t \in (-\infty, T)$$

where C, and $\tilde{C}$ are constants. (This can also be done for $\rho = -2$. See Remark 16.) With this in mind, it is natural to make the following definition.

Definition 9. Fix $\rho \in (-\infty, -2)$. The whole-plane SLE$_0(\rho)$, started at ∞ in the direction $e^{i\theta}$, of conformal radius $e^T \in (-\infty,\infty)$, positive orientation, and with reference point 0 and force point ∞, is the whole-plane Loewner chain with driving function e^{iw_t} where

$$w_t = \frac{2\rho}{\rho+2}\arcsin(e^{\frac{\rho+2}{4}(T-t)}) + \theta, \quad t \in (-\infty, T).$$

Similarly, the same object but with negative orientation is defined as the whole-plane Loewner chain with driving function

$$w_t = -\frac{2\rho}{\rho+2}\arcsin(e^{\frac{\rho+2}{4}(T-t)}) + \theta, \quad t \in (-\infty, T).$$

Remark 15. This definition is natural since it has the "deterministic domain Markov property" that we expect from a whole-plane SLE$_0(\rho)$, that is, upon mapping out the initial part of the curve, we get a radial SLE$_0(\rho)$ with force point at the image of ∞. It follows from this property that the whole-plane SLE$_0(\rho)$ is a simple curve on $(-\infty, T)$ since we know that the radial SLE$_0(\rho)$ is a simple curve (since it has finite chordal Loewner energy for all times strictly before the force point is swallowed).

Proposition 12. *Fix $\rho \in (-\infty, -2)$. Let η denote an arc-length parametrization of the whole-plane $SLE_0(\rho)$ starting at ∞ in the direction 1, of any conformal radius, positive orientation, and with reference point 0 and force point ∞. Then η^{ρ,z_0} has a lift $\tilde{\eta}$ to M_0, such that $\arg(\tilde{\eta}(t)) \to 0$ as $t \to -\infty$. The curve $\tilde{\eta}(t)$ is a flow-line of $h(z) = \frac{6+\rho}{4}\arg(z) + \pi$. This further implies that η is the image of $\{x + iy_0 : x \in (x_0, \infty)\}$, for some $y_0 > 0$ and $x_0 \in [-\infty, \infty)$, under $z \mapsto z^{-\frac{4}{2+\rho}}$ where the branch is chosen so that $\arg((x + iy_0)^{-\frac{4}{\rho+2}}) \to 0$ as $x \to +\infty$.*

Proof. Let γ denote η parametrized by conformal radius. Let $g_t : \mathbb{C} \setminus \gamma_t \to \mathbb{D}$ with $g_t(0) = 0$, $g_t'(0) = e^t$, be its family of mapping-out functions. From the theory of whole-plane Loewner evolution, we have that $g_t(z)e^{-t} \to z$ as $t \to -\infty$ uniformly on compacts.

There is a unique continuous function $t \mapsto v_t \in \mathbb{R}$, such that, for every $t \in (-\infty, 0)$ we have that $g_t(\gamma_{[t,T)})$ is a radial $\mathrm{SLE}_0(\rho)$ from e^{iw_t}, with reference point 0 and force point e^{iv_t}, and $v_t \to \pi$ as $t \to -\infty$. Once we show that γ is "well behaved" in the limit $t \to -\infty$ we will be able to say that $e^{iv_t} = g_t(\infty)$, but for now, we will simply treat v_t as a continuous function with the given properties. For $t \in (-\infty, T)$, let $\varphi_t : \mathbb{D} \to \mathbb{H}$ be the conformal map with $\varphi_t(e^{iv_t}) = \infty$, $\varphi_t(e^{iw_t}) = 0$, and $|\varphi_t'(0)| = 1$. A computation gives

$$\varphi_t(z) = -e^{i(v_t - w_t)/2}\frac{z - e^{iw_t}}{z - e^{iv_t}}.$$

Denote by $z_t = \varphi(0) = -e^{-i(v_t - w_t)/2}$. Further, let $\psi_t(z) = i(z - z_t)e^{-t}$, so that ψ_t maps $\mathbb{H}$ onto $\{x + iy : x < e^{-t}\sin((v_t - w_t)/2), y \in \mathbb{R}\}$.

Let $\tilde{\rho} = -6 - \rho$. For every $t \in (-\infty, T)$ we know that a lift of $\gamma_{(t,T)}$ to M_0, which we denote by $\tilde{\gamma}_{(t,T)}$, reparametrized by arc-length, is a flow-line of the field

$$h_t(z) = h_{z_t,\tilde{\rho}}(\varphi_t \circ g_t(z))) - \arg((\varphi_t \circ g_t)'(z))$$

defined on $\pi_0^{-1}(\mathbb{C} \setminus \gamma_t)$, where $h_{z_t,\tilde{\rho}}$ is as defined in Proposition 7. Here, as in the proof of Proposition 7, we make sense of $h_t(z)$ by considering a lift of $\varphi_t \circ g_t$, mapping $\pi_0^{-1}(\mathbb{C} \setminus \gamma_t)$ onto $\pi_{z_t}^{-1}(\mathbb{H})$. Moreover, from (39), we obtain that h_t and h_s coincide on the intersection of their domains. We thus obtain a field h defined on all of M_0 coinciding with h_t for each $t \in (-\infty, T)$ wherever the latter is defined. We now show that $h(z) = -\frac{\tilde{\rho}}{4}\arg(z) + \pi$. We have

$$\begin{aligned} h(z) &= h_{z_t,\tilde{\rho}}(\varphi_t \circ g_t(z)) - \arg((\varphi_t \circ g_t)'(z)) \\ &= \lim_{t \to -\infty}(h_{z_t,\tilde{\rho}}(\varphi_t \circ g_t(z)) - \arg((\varphi_t \circ g_t)'(z))) \\ &= \lim_{t \to -\infty}(\hat{h}_t(H_t(z)) - \arg(H_t'(z))), \end{aligned} \tag{43}$$

where $\hat{h}_t(z) = h_{z_t,\tilde{\rho}}(\psi^{-1}(z)) - \arg((\psi^{-1})'(z))$ and $H_t(z) = \psi_t \circ \varphi_t \circ g_t$. Using that $g_t(z)e^t \to z$ uniformly on compacts as $t \to -\infty$ we find that $H_t(z) \to 2z$ uniformly on compacts as $t \to -\infty$. Moreover,

$$\hat{h}_t(z) = \frac{3\pi}{2} - \arg(z_t - ie^t z) - \frac{\tilde{\rho}}{4}(\arg(-ize^t) + \arg(2i\mathrm{Im}z_t - ize^t)),$$

where the first and second arg-terms take values in $(0, \pi)$. Since $H_t(z)e^t \to 0$ and $z_t \to i$ as $t \to -\infty$,

$$\begin{aligned} &\lim_{t \to -\infty} \arg(2i\mathrm{Im}z_t - iH_t(\gamma(s))e^t) = \frac{\pi}{2}, \\ &\lim_{t \to -\infty} \arg(z_t - ie^t H_t(z)) = \frac{\pi}{2}, \\ &\lim_{t \to -\infty} \arg(H_t'(z)) = 0, \\ &\lim_{t \to -\infty} \arg(-iH_t(z)e^t) = \arg(z) - \frac{\pi}{2}. \end{aligned}$$

Hence $h(z) = -\frac{\tilde{\rho}}{4}\arg(z) + \pi$.

To show the second part of the statement, consider $(\tilde{\eta})^{1+\frac{\tilde{\rho}}{4}}$. Then $\tilde{\eta}'(t) = e^{ih(\tilde{\eta}(t))}$ and the chain rule yields

$$\arg(((\tilde{\eta}(s))^{1+\frac{\tilde{\rho}}{4}})') = \pi.$$

Therefore, $(\tilde{\eta})^{1+\frac{\tilde{\rho}}{4}}$ is (as a set) contained in a horizontal line on M_0. Thus, η is the image of $\{x + iy_0 : x \in (x_0, \infty)\}$, for some $y_0 \in \mathbb{R}$ and $x_0 \in [-\infty, \infty)$, under some branch of $z \mapsto z^{-\frac{4}{2+\rho}}$. Since γ starts in the direction 1, meaning that $w_t \to 0$ as $t \to -\infty$, the branch must be chosen so that

$$\arg((x + iy_0)^{-\frac{4}{2+\rho}}) \to 0, \quad \text{as } x \to +\infty.$$

Moreover, if $\rho \in (-\infty, -4]$, then $x_0 = -\infty$, that is, γ is a simple curve both starting and ending at ∞. If $\rho \in (-4, -2)$, then $x_0 > -\infty$, that is, γ is a curve starting at ∞ and ending at a self-intersection. Additionally, since γ has positive orientation, the harmonic measure of the right side of the curve $(v_t - w_t)/(2\pi) \in (0, 1/2)$ for all t. Hence, we must have $y_0 > 0$. ☐

Remark 16. Since the driving function of $\text{SLE}_0(-2)$ is linear, one can similarly define whole-plane $\text{SLE}_0(-2)$ to be a whole-plane Loewner chain with driving function

$$w_t = t \cot \frac{v_0}{2} + \theta, \quad t \in \mathbb{R}$$

for some $v_0 \in (0, 2\pi)$ and $\theta \in [0, 2\pi)$. With Proposition 12 in mind one can guess that whole-plane $\text{SLE}_0(-2)$ should be a flow-line of $h(z) = \arg(z) + C$ for some C (if we set $C = \pi$ the flow-lines will be rays from ∞ to 0, which is clearly not what we want). Indeed, one can quite easily see, using Proposition 11 and [22, Proposition 3.3 and Figure 3], that this is in fact the case. Using the explicit computations of [22, 17] one finds that $C = \frac{3\pi}{2} - \frac{v_0}{2} \in (\frac{\pi}{2}, \frac{3\pi}{2})$. (Note that $C = \pi/2$ produces concentric circular flow-lines which are counter clockwise oriented, circles centered at 0, and that $C = 3\pi/2$ produces circles of the opposite orientation.)

Corollary 3. *Suppose $v_0 \in (0, \pi) \cup (\pi, 2\pi)$. Let $\gamma : (0, T) \to \mathbb{D}$ be the radial $SLE_0(\rho)$ starting at* 0*, with reference point* 0 *and force point e^{iv_0}. Then γ is a simple curve and can be continuously extended to T. If $\rho \in (-4, -2)$, then $\gamma(T-) \in \gamma_{(0,T)} \cup \partial\mathbb{D} \setminus \{e^{iv_0}\}$ such that $\mathbb{D} \setminus \gamma$ separates e^{iv_0} from* 0*. Moreover, the component of $\mathbb{D} \setminus \gamma$ containing* 0 *has interior angle $\pi\frac{4+\rho}{2}$ at $\gamma(T-)$ (unless $\gamma(T-) = 1$, in which case the interior angle is $\pi\frac{4+\rho}{4}$). If $\rho \leq -4$, then $\gamma(T-) = e^{iv_0}$ and the component of $\mathbb{D} \setminus \gamma$ containing* 0 *has an interior angle $\pi\frac{\rho+4}{\rho+2}$ at e^{iv_0}.*

Proof. The topological properties of γ follow directly from Proposition 12 and Remark 15. When $\rho \leq -4$, the size of the intersection angle also follows easily from Proposition 12 since the whole-plane $\text{SLE}_0(\rho)$ forms an angle $2\pi\frac{\rho+4}{\rho+2}$ at ∞. For $\rho \in (-4, -2)$, we can find the intersection angle by using Proposition 7 in the following way. By changing coordinates we may instead consider $\hat{\gamma}$, a chordal $\text{SLE}_0(-6 - \rho)$ in $\mathbb{H}$ with force point z_0. Suppose $\arg(z_0) \in (0, \pi/2)$. Then $\hat{\gamma}$ separates z_0 from ∞ by winding around z_0 clockwise. We may assume that $\hat{\gamma}$ ends upon hitting $\mathbb{R}^+$ (for otherwise we may achieve this by mapping out a portion of $\hat{\gamma}$). Now observe that $h_{-\rho-6,z_0}$ takes the value $-\pi\frac{\rho+4}{2}$ at the endpoint of the lift of $\hat{\gamma}$ starting at $(r, \theta + \pi)$. From this we deduce that the intersection angle is as claimed. ☐

6 Finite energy curves and Dirichlet energy formulas

This section is devoted to studying curves of finite ρ-Loewner energy when the force point is on the boundary and $\rho > -2$ as well as proving Theorems 2 and 3.

In Section 6.1 we study fully grown curves $\gamma \subset \mathbb{D}$ starting at 1, with respect to the reference point 0 and force point $z_0 \in \partial\mathbb{D} \setminus \{1\}$ (we will make precise what this means). We prove that

$I^R_{\rho,z_0}(\gamma) < \infty$, $\rho > -2$, if and only if γ approaches 0 (that is, ends at 0) and $I^{(\mathbb{D};1,e^{iv_0})}(\gamma) < \infty$. Therefore, for a fully grown curve γ and $\rho_1, \rho_2 > -2$, we have

$$I^R_{\rho_1,z_0}(\gamma) < \infty \iff I^R_{\rho_2,z_0}(\gamma) < \infty.$$

In Section 6.2, study fully grown curves $\gamma \subset \mathbb{H}$ starting at 0, with respect to the reference point ∞ and force point $x_0 > 0$. We show that $I^C_{\rho,x_0}(\gamma) < \infty$ only if γ is transient and approaches ∞ at an angle $\alpha\pi = \frac{\rho+2}{\rho+4}\pi$ (in the sense of Proposition 16). Therefore, for a fully grown curve γ and $\rho_1, \rho_2 > -2$, with $\rho_1 \neq \rho_2$, we have

$$I^C_{\rho_1,x_0}(\gamma) < \infty \implies I^C_{\rho_2,x_0}(\gamma) = \infty.$$

Theorems 2 and 3 are proved in Sections 6.1 and 6.2 respectively.

6.1 Radial setting

In order to benefit from the previous knowledge about the chordal Loewner energy (e.g., the bound (15) and the Dirichlet energy formula (7)) we change coordinates and instead consider the chordal setting with an interior force point and $\rho < -4$. All of our findings can be translated back to the radial setting using an appropriate conformal map. Throughout this section we fix $z_0 = x_0 + iy_0 \in \mathbb{H}$. Let $\mathcal{X}^C_{z_0}$ be the class of simple curves $\gamma : (0,T) \to \mathbb{H} \setminus \{z_0\}$, with $\gamma(0+) = 0$, which are maximal in the sense that $T = \tau_{0+} = \lim_{\varepsilon \to 0+} \tau_\varepsilon$ where $\tau_\varepsilon = \inf\{t \in (0,T) : |W_t - z_t| \leq \varepsilon\}$. Note that this allows for $T = \infty$. We call $\gamma \in \mathcal{X}^C_{z_0}$ a fully grown curve with respect to ∞ and z_0.

The following proposition gives upper and lower bounds of the ρ-Loewner energy in terms of the first two terms of the right-hand side of (18).

Proposition 13. *Suppose $\rho < -4$ and suppose $\gamma : (0,T] \to \mathbb{H} \setminus \{z_0\}$ is a simple curve with $\gamma(0) = 0$. Then*

$$I^C_{\rho,z_0}(\gamma) \leq \max(1, -\tfrac{4+\rho}{4})\left(\max(1, -\tfrac{4+\rho}{4}) I^C(\gamma) + \rho \log \frac{\sin\theta_T}{\sin\theta_0} \right),$$
$$I^C_{\rho,z_0}(\gamma) \geq \min(1, -\tfrac{4+\rho}{4})\left(\min(1, -\tfrac{4+\rho}{4}) I^C(\gamma) + \rho \log \frac{\sin\theta_T}{\sin\theta_0} \right).$$

Proof. We estimate $|g_t'(z_0)|$ in terms of the chordal Loewner energy as in [13]. That is, we have

$$\begin{aligned}\partial_t \log |g_t'(z_0)| y_t &= -4\frac{(W_t - x_t)^2}{|W_t - z_t|^4} = 2\partial_t \log\sin\theta_t + 4\frac{(W_t - x_t)^2}{|W_t - z_t|^4} + 2\dot{W}_t \frac{W_t - x_t}{|W_t - z_t|^2} \\ &\geq 2\partial_t \log \sin\theta_t - \frac{1}{4}\dot{W}_t^2,\end{aligned}$$

where the equalities follow from the chordal Loewner equation. This shows

$$2\log\frac{\sin\theta_T}{\sin\theta_0} - \frac{1}{2}I^C(\gamma) \leq \log\frac{|g_T'(z_0)|y_t}{y_0} \leq 0.$$

Plugging this into (18) one obtains the desired bounds. □

Corollary 4. *Let $\gamma \in \mathcal{X}^C_{z_0}$ and let $\rho < -4$. Then,*

(a) If $I^C(\gamma) = \infty$, then $I^C_{\rho,z_0}(\gamma) = \infty$.

(b) If γ is unbounded, then $I^C_{\rho,z_0}(\gamma) = \infty$.

Proof. Note that $\min(1, -\frac{4+\rho}{4}) > 0$ and recall that $I^C_{\rho,z_0}(\gamma_t)$ is increasing in t. Statement (a) holds since $\rho \log \sin\theta_t$ is non-negative. Thus, if $I^C(\gamma_t)$ diverges, then $I^C_{\rho,z_0}(\gamma_t)$ must also do so. Consider statement (b). We claim that if γ is unbounded, then for every $\varepsilon > 0$ there is a

$t \in (0, T)$ such that $\sin\theta_t < \varepsilon$. If so, then (b) follows, since $I^C(\gamma_t)$ is non-negative. We now prove the claim. Observe that

$$\frac{\theta_t}{\pi} = \omega(z_t, (-\infty, W_t], \mathbb{H}) = \omega(z_0, \mathbb{R}^- \cup \gamma_t^-, \mathbb{H} \setminus \gamma_t),$$

$$\frac{\pi - \theta_t}{\pi} = \omega(z_t, [W_t, \infty), \mathbb{H}) = \omega(z_0, \mathbb{R}^+ \cup \gamma_t^+, \mathbb{H} \setminus \gamma_t).$$

For $R > |z_0|$, let $T_R = \inf\{t : |\gamma(t) - x_0| \geq R\}$ and $D_R = \{z \in \mathbb{H} : |z - x_0| < R\}$. Note that $D_R \setminus \gamma_{T_R}$ consists of two components, D_R^- to the left of γ and D_R^+ to the right of γ. Suppose $z_0 \in D_R^+$. Then, by monotonicity of harmonic measure

$$\omega(z_0, \mathbb{R}^- \cup \gamma_t^-, \mathbb{H} \setminus \gamma_t) \leq \omega(z_0, \partial D_R \cap \mathbb{H}, D_R^+) \leq \omega(z_0, \partial D_R \cap \mathbb{H}, D_R).$$

By symmetry, we have $\omega(z_0, \mathbb{R}^+ \cup \gamma_t^+, \mathbb{H} \setminus \gamma_t) \leq \omega(z_0, \partial D_R \cap \mathbb{H}, D_R)$ if $z_0 \in D_R^-$. Since

$$\omega(z_0, \partial D_R \cap \mathbb{H}, D_R) \to 0 \quad \text{as } R \to \infty$$

(this can be seen by an explicit computation) it follows that $\sin\theta_{T_R} \to 0$ as $R \to \infty$. □

Suppose $\gamma \in \mathcal{X}_{z_0}^C$ has finite ρ-Loewner energy. Then $T < \infty$ since $T = \infty$ implies that γ is unbounded. We can conclude that $\gamma \in \mathcal{X}_{z_0}^C$ can have finite ρ-Loewner energy only if $T < \infty$ and γ can be continuously extended by $\gamma(T) = z_0$: we can assume that the driving function W can be continuously extended to T, for otherwise $I^C(\gamma) = \infty$ which implies $I_{\rho,z_0}^C(\gamma) = \infty$. This also means that γ can be extended continuously to T and must be simple on $[0, T]$ (otherwise it could not have finite chordal Loewner energy). By the definition of $T = \tau_{0+}$ and Loewner's theorem, we must have $\gamma(T) = z_0$.

Proposition 14. *Fix $\rho < -4$. Let $\gamma : (0, T) \to \mathbb{H} \setminus \{z_0\}$ be a simple curve with $\gamma(0+) = 0$ and with $\gamma(T-) = z_0$. Then $I_{\rho,z_0}^C(\gamma)$ is finite if and only if $I^C(\gamma)$ is finite. Moreover, if they are finite then $\sin\theta_t \to 0$ as $t \to T-$.*

Proof. From Corollary 4 we already have that $I^C(\gamma)$ is finite if $I_{\rho,z_0}^C(\gamma)$ is finite. To show the other direction, assume $I^C(\gamma)$ is finite. Note that $g_t(\gamma_{[t,T]}) - W_t$ is a curve from 0 to $z_t - W_t$, with $\arg(z_t - W_t) = \theta_t$. Therefore, (15) implies $I^C(g_t(\gamma_{[t,T]}) - W_t) \geq -8 \log \sin\theta_t$. The assumption $I^C(\gamma) < \infty$ gives

$$I^C(g_t(\gamma_{[t,T]}) - W_t) = \frac{1}{2} \int_t^T \dot{W}_t^2 dt \to 0 \text{ as } t \to T - .$$

It follows that $-\log \sin\theta_t \to 0$ as $t \to T-$ and by Proposition 13, it follows that $I_{\rho,z_0}^C(\gamma)$ is finite. □

Remark 17. The statement that $I^C(\gamma) < \infty$ implies that $\sin\theta_t \to 0$ also appears in [24, Lemma 3.2].

Corollary 5. *Let γ be a simple curve from 1 to 0 in $\mathbb{D}$. Then*

$$\frac{1}{4} I^{(\mathbb{D};1,-1)}(\gamma) \leq I^R(\gamma) \leq I^{(\mathbb{D};1,-1)}(\gamma).$$

Proof. The corollary follows by changing coordinates to $\mathbb{H}$, applying Proposition 13 with $\rho = -6$, and $\theta_0 = \pi/2$, and finally applying Proposition 14. □

6.1.1 Proof of Theorem 2

Recall that $\Sigma = \mathbb{C} \setminus \mathbb{R}^+$. Fix $z_0 \in \mathbb{H}$ and $\rho > -4$. Let $\gamma : (0, T) \to \mathbb{H} \setminus \{z_0\}$ be a simple curve with $\gamma(0+) = 0$, $\gamma(T-) = z_0$. Let $\tilde{\gamma}(t) = (\gamma(t))^2$ and for each $t \in [0, T]$, and let $h_t : \Sigma \setminus \tilde{\gamma}_t \to \Sigma$ defined by $h_t(z) = (g_t(\sqrt{z}) - W_t)^2$. It follows from Proposition 14 and (7) that

$$I_{-\rho-6,\infty}^{(\Sigma;0,z_0^2)}(\tilde{\gamma}) < \infty \iff I^{(\Sigma;0,\infty)}(\tilde{\gamma}) < \infty \iff \mathcal{D}(h) < \infty.$$

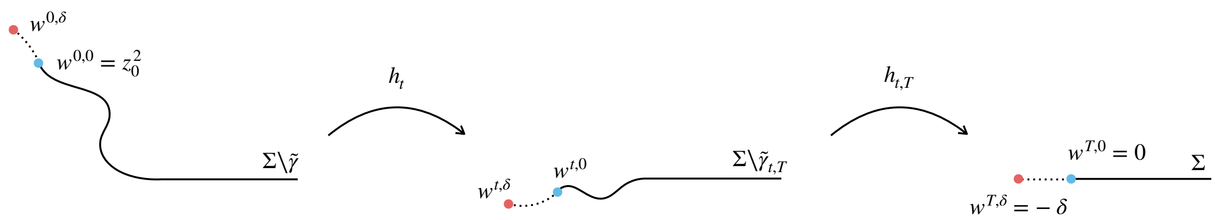

Figure 3: This figure illustrates the set-up for the proof of Lemma 10.

We now aim to show (9). Assume, that $I^C_{\rho,z_0}(\gamma) < \infty$. Using the chain rule we obtain

$$\frac{|g_t'(z_0)|y_t}{y_0} = \frac{|h_t'(z_0^2)|\sin\theta_t}{\sin\theta_0},$$

so that (18) yields

$$I^C_{\rho,z_0}(\gamma) = \lim_{t\to T-}\left(I^C(\gamma_t) - \frac{\rho^2}{8}\log\frac{\sin\theta_t}{\sin\theta_0} - \frac{\rho(8+\rho)}{8}\log|h_t'(z_0^2)|\right). \tag{44}$$

Proposition 14 gives $I^C(\gamma) < \infty$ and $\lim_{t\to T-}\log\sin\theta_t = 0$. Thus $\lim_{t\to T-}\log|h_t'(z_0^2)|$ must also exist and

$$I^C_{\rho,z_0}(\gamma) = I^C(\gamma) + \frac{\rho^2}{8}\log\sin\theta_0 - \frac{\rho(8+\rho)}{8}\lim_{t\to T-}\log|h_t'(z_0^2)|$$

We define for $0 \le s \le t \le T$, $h_{s,t} := h_t \circ h_s^{-1}$, and $w_t = (z_t - W_t)^2 = h_t(z_0^2)$.

Lemma 10. *For a curve γ as above, we have that $\lim_{\delta\to 0+}|(h_T^{-1})'(-\delta)|$ exists and*

$$\lim_{\delta\to 0+}|(h_T^{-1})'(-\delta)| = \lim_{t\to T-}|(h_t^{-1})'(w_t)|. \tag{45}$$

where $w_t = (z_t - W_t)^2$.

Proof. For every $t \in [0,T]$, $\delta \ge 0$, let $w^{t,\delta} = h_{t,T}^{-1}(-\delta)$, $g_{t,T} = g_T \circ g_t^{-1}$, and $z^{t,\delta} = g_{t,T}^{-1}(i\sqrt{\delta} + W_T)$ (see Figure 3). By the same estimate as in the proof of Proposition 13 we have

$$\left(\frac{\sin\pi/2}{\sin\arg(z^{t,\delta} - W_t)}\right)^2 e^{-\frac{1}{2}I^C(\gamma_{t,T})} \le |g_{t,T}'(z^{t,\delta})|\frac{\sqrt{\delta}}{\operatorname{Im} z^{t,\delta}} \le 1, \ \forall t \in [0,T], \ \delta > 0.$$

Therefore,

$$e^{-\frac{1}{2}I^C(\gamma_{t,T})} \le |h_{t,T}'(w^{t,\delta})| \le 1, \ \forall t \in [0,T], \ \delta > 0,$$

and hence, $e^{-\frac{1}{2}I^C(\gamma_{t_0,T})} \le |h_{t,T}'(w^{t,\delta})| \le 1$, for all $t \in (t_0, T)$ and $\delta > 0$. By the chain rule $|h_T'(w^{0,\delta})| = |h_t'(w^{0,\delta})||h_{t,T}'(w^{t,\delta})|$, and thus

$$e^{-\frac{1}{2}I^C(\gamma_{t_0,T})}|h_t'(w^{0,\delta})| \le |h_T'(w^{0,\delta})| \le |h_t'(w^{0,\delta})|,$$

for all $t \in (t_0, T)$ and $\delta > 0$. Taking the limit $\delta \to 0+$ we obtain

$$e^{-\frac{1}{2}I^C(\gamma_{t_0,T})}|h_t'(z_0^2)| \le \liminf_{\delta\to 0+}|h_T'(w^{0,\delta})| \le \limsup_{\delta\to 0+}|h_T'(w^{0,\delta})| \le |h_t'(z_0^2)|,$$

since for each $t < T$, $z_0^2 \in \Sigma\setminus\tilde{\gamma}_t$ so that h_t' is continuous at z_0^2. Taking the limit $t \to T-$ followed by $t_0 \to T-$ yields

$$\lim_{t\to T-}|h_t'(z_0^2)| \le \liminf_{\delta\to 0+}|h_T'(w^{0,\delta})| \le \limsup_{\delta\to 0+}|h_T'(w^{0,\delta})| \le \lim_{t\to T-}|h_t'(z_0^2)|.$$

Recall from the discussion above that the limit on the left- (and right-)hand side exists. This completes the proof. ∎

Let $\eta(s) := h_T^{-1}(-s)$, $s \in [0,\infty)$, be the hyperbolic geodesic from z_0^2 to ∞ in $\Sigma \setminus \tilde\gamma$. Since

$$I^{(\Sigma;0,\infty)}(\tilde\gamma \cup \eta) = I^C(\gamma) < \infty,$$

$\tilde\gamma \cup \eta$ is asymptotically smooth (see [46, Theorem 2.18]). In particular, this implies that $\lim_{s\to 0} \frac{|\eta(s)-z_0^2|}{\ell(\eta_s)} = 1$, where $\ell(\eta_s)$ denotes the length of η_s, and as a consequence

$$\begin{aligned}|(h_T^{-1})'(0)|_{\mathbb{R}^-} &:= \lim_{\delta\to 0+} \frac{|h_T^{-1}(-\delta) - h_T^{-1}(0)|}{\delta} = \lim_{\delta\to 0+} \frac{|\eta(\delta) - z_0^2|}{\delta} = \lim_{\delta\to 0+} \frac{\ell(\eta_\delta)}{\delta} \\ &= \lim_{\delta\to 0+} \frac{1}{\delta}\int_0^\delta |(h_T^{-1})'(-s)|ds = \lim_{\delta\to 0+} |(h_T^{-1})'(-\delta)|,\end{aligned}$$

so $|(h_T^{-1})'(0)|_{\mathbb{R}^-}$ as well as $|h_T'(z_0^2)|_\eta := \lim_{\delta\to 0+} |h_T(\eta(\delta))|/|\eta(\delta) - \eta(0)|$ exist and we finally deduce

$$I^{(\Sigma;0,\infty)}_{\rho,z_0^2}(\tilde\gamma) = I^C_{\rho,z_0}(\gamma) = I^C(\gamma) + \frac{\rho^2}{8}\log\sin\theta_0 - \frac{\rho(8+\rho)}{8}\log|h_T'(z_0^2)|_\eta. \tag{46}$$

In particular this means that, if $\tilde\gamma^0$ denotes the $\mathrm{SLE}_0(\rho)$ in Σ from 0, with reference point ∞ and force point z_0^2, then

$$I^{(\Sigma;0,\infty)}_{\rho,z_0}(\tilde\gamma) = I^{(\Sigma;0,\infty)}(\tilde\gamma) - I^{(\Sigma,0,\infty)}(\tilde\gamma^0) - \frac{\rho(8+\rho)}{8}\log|H'(z_0^2)|_\eta,$$

where $H : \Sigma \setminus \tilde\gamma \to \Sigma \setminus \tilde\gamma^0$ is the conformal map with $H(z_0^2) = z_0^2$, $H(\infty) = \infty$ and $|H'(\infty)| = 1$. Using (7) and recalling that

$$I^{(\Sigma;0,z_0^2)}_{-6-\rho,\infty}(\tilde\gamma) = I^{(\Sigma;0,\infty)}_{\rho,z_0^2}(\tilde\gamma)$$

we obtain (9). This finishes the proof of Theorem 2.

6.2 Chordal setting

Throughout this section $\rho > -2$ and the force point $x_0 > 0$. Since $I^C_{\rho,x_0}(\gamma) = I^C_{\rho,-x_0}(-\overline{\gamma})$ where $-\overline{\gamma}$ denotes the reflection of γ in the imaginary axis, the assumption on x_0 can be imposed without loss of generality. Consider the class $\mathcal{X}^C_{x_0}$, of simple curves $\gamma : (0,T) \to \mathbb{H}$, with $\gamma(0+) = 0$, and maximal in the sense that $T = \tau_{0+} := \lim_{\varepsilon\to 0+}\tau_\varepsilon$ where $\tau_\varepsilon = \inf\{t \in (0,T) : |W_t - x_t| \le \varepsilon\}$. We call $\gamma \in \mathcal{X}^C_{x_0}$ a fully grown curve with respect to x_0 and ∞.

We have the following analog of Proposition 13. As the proof is very similar we omit it.

Proposition 15. *Suppose $\rho \in (-2,\infty)$ and suppose $\gamma : (0,T] \to \mathbb{H}$ is a simple curve with $\gamma(0) = 0$. Then*

$$\begin{aligned}I^C_{\rho,x_0}(\gamma) &\le \max(\tfrac{\rho+2}{2},1)\left(\max(\tfrac{\rho+2}{2},1)I^C(\gamma) + |\rho|\log\frac{|W_t - x_t|}{|x_0|}\right),\\ I^C_{\rho,x_0}(\gamma) &\ge \min(\tfrac{\rho+2}{2},1)\left(\min(\tfrac{\rho+2}{2},1)I^C(\gamma) - |\rho|\log\frac{|W_t - x_t|}{|x_0|}\right).\end{aligned}$$

This immediately shows that $\tau_{0+} < \infty$ implies that $I^C_{\rho,z_0}(\gamma) = \infty$. So, if $I^C_{\rho,z_0}(\gamma) < \infty$ for a $\gamma \in \mathcal{X}^C_{x_0}$, then γ is unbounded. We saw in Corollary 2 that the $\mathrm{SLE}_0(\rho)$ curve approaches ∞ with an angle $\alpha\pi$ where $\alpha = \alpha(\rho) = \frac{2+\rho}{4+\rho}$. For the remainder of this section α refers to $\alpha(\rho)$.

Proposition 16. *Fix $\rho > -2$ and $x_0 > 0$. Let $\gamma \in \mathcal{X}^C_{x_0}$ and suppose that $I^C_{\rho,x_0}(\gamma) < \infty$. Then for each $0 < \alpha_- < \alpha(\rho) < \alpha_+ < 1$ there is an $R > 0$ such that $\gamma \setminus B(0,R) \subset C(\alpha_-,\alpha_+)$ where*

$$C(\alpha_-,\alpha_+) = \{re^{i\theta} : r > 0,\ \theta \in (\alpha_-\pi, \alpha_+\pi)\}.$$

Remark 18. As a consequence of Proposition 17, curves of finite ρ-Loewner energy are continuous at the end point. This means that we may strengthen the above: for each γ, α_-, and α_+ as above there exists a $T > 0$ such that $\gamma_{[T,\infty)} \subset C(\alpha_-,\alpha_+)$.

Lemma 11. *Fix $\rho > -2$ and $x_0 > 0$, and let $y_0 = \frac{-2x_0}{2+\rho}$. Suppose $\gamma : (0, T] \to \mathbb{H}$ driven by W_t and define $r_t := \frac{W_t - y_t}{x_t - y_t} \in (0, 1)$ where $x_t = g_t(x_0)$ and $y_t = g_t(y_0)$. Then,*

$$I^C_{\rho,x_0}(\gamma) \geq -(2+\rho)\log\frac{1-r_T}{1-r_0} - 2\log\frac{r_T}{r_0}, \tag{47}$$

where the right hand side is positive whenever $r_T \neq r_0 = 1 - \alpha$.

Remark 19. The role of the point y_0 in Lemma 11 might seem artificial, but by viewing γ as the (mapped out) continuation of an $\mathrm{SLE}_0(\rho)$ with force point at $0+$ we see that is is quite natural: Recall from Remark 13 that $\{re^{i\pi\alpha} : r \in [0, R]\}$ is an initial part of the $\mathrm{SLE}_0(\rho)$ with force point at $0+$. There exists, for each $x_0 > 0$ a unique $R > 0$ such that there is a (unique) conformal map

$$\varphi_{x_0} : \mathbb{H} \setminus \{re^{i\pi\alpha} : r \in [0, R]\} \to \mathbb{H}$$

satisfying $\varphi_{x_0}(Re^{i\pi\alpha}) = 0$, $\varphi_{x_0}(\infty) = \infty$, $\varphi'_{x_0}(\infty) = 1$, and $\varphi_{x_0}(0+) = x_0$. Then $\varphi_{x_0}(0-) = y_0$ (this can be checked by an explicit computation).

Remark 20. With Remark 19 in mind, Lemma 11 is reminiscent of [24, Theorem 4.4] where Mesikepp answers the question: For $X, Y > 0$ fixed, what is the minimal chordal Loewner energy required for a curve to have $g_T(0+) - W_T = X$, $W_T - g_T(0-) = Y$? Mesikepp also finds that the optimal energy is attained for a unique curve corresponding to an $\mathrm{SLE}_0(-4, -4)$ with force points at $(0-, 0+)$. To fully see the analogy with the Lemma above we would have to define, in the natural way, the ρ-Loewner energy when the force point $x_0 = 0+$. If one does this, the above gives a lower bound on the minimal ρ-Loewner energy, with respect to $x_0 = 0+$, required to obtain $g_T(0+) - W_T = X$, $W_t - g_T(0-) = Y$. If one works a little more (e.g., by following the proof of [24, Theorem 4.4(i)]), one finds that the bound above is optimal and that the unique curve of optimal energy is an $\mathrm{SLE}_0(-4, -4-\rho)$, again with force points $(0-, 0+)$.

Proof. We may suppose that W_t is absolutely continuous, for otherwise (47) is trivial. We may also assume that $T = \inf\{t \in [0, T] : r_t = r_T\}$. Let $X_t = x_t - W_t$, $Y_t = W_t - y_t$, and note that they, as well as r_t, are absolutely continuous with

$$\dot{X}_t = \frac{2}{X_t} - \dot{W}_t, \qquad \dot{Y}_t = \frac{2}{Y_t} + \dot{W}_t, \qquad \dot{r}_t = \frac{\dot{W}_t + \frac{2}{Y_t} - \frac{2}{X_t}}{X_t + Y_t}, \qquad \text{a.e.}$$

Assume $r_T > r_0$. In this case, we may assume that $r_t \geq r_0$ for all $t \in [0, T]$. Let

$$E = \{t \in [0, T] : r_t = \sup_{s<t} r_s\}.$$

Note that $\dot{r}_t \geq 0$ a.e. on E, that E is closed, and that $0, T \in E$. Therefore, $[0, T] \setminus E$ consists of countably many disjoint open intervals $I_n = (\alpha_n, \beta_n)$ for which $r_{\alpha_n} = r_{\beta_n}$. We have

$$I^C_{\rho,x_0}(\gamma) \geq \frac{1}{2}\int_E \left(\dot{W}_t + \frac{\rho}{X_t}\right)^2 \mathbb{1}_{\{\dot{r}_t > 0\}} dt = \frac{1}{2}\int_E \frac{(\dot{W}_t + \frac{\rho}{X_t})^2}{\dot{r}_t} \dot{r}_t \mathbb{1}_{\{\dot{r}_t > 0\}} dt.$$

Under the assumption $\dot{r}_t > 0$ (and $r_t > r_0$), a computation shows that,

$$\frac{(\dot{W}_t + \frac{\rho}{X_t})^2}{\dot{r}_t} \geq 4\left(\frac{2+\rho}{1-r_t} - \frac{2}{r_t}\right)$$

where equality is attained when $\dot{W}_t + \frac{\rho}{X_t} = 2(\frac{2+\rho}{X_t} - \frac{2}{Y_t})$. Define $f(t) = -2(2+\rho)\log(1-r_t) - 4\log r_t$ and observe that f is absolutely continuous with $f'(t) = (2\frac{2+\rho}{1-r_t} - \frac{4}{r_t})\dot{r}_t$ a.e. Thus,

$$I^C_{\rho,x_0}(\gamma) \geq \int_E f'(t)dt = \int_0^T f'(t)dt$$

since $f(\beta_n) - f(\alpha_n) = \int_{I_n} f'(t) = 0$ and $r_{\beta_n} = r_{\alpha_n}$ for all n. This shows (47) in the case $r_T > r_0$. The case $r_T < r_0$ is shown in the same way by replacing r_t with $1 - r_t$. ☐

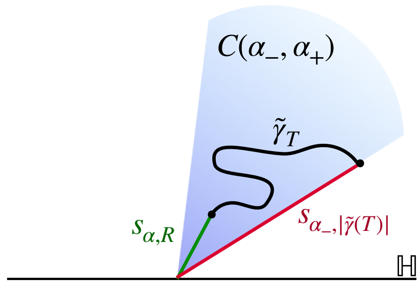

Figure 4: Illustration of the set-up in Lemma 12.

Lemma 12. *Let $R > 0$ and φ_{x_0} be as in Remark 19. For each $0 < \alpha_- < \alpha < \alpha_+ < 1$ there is an $M > 0$, such that for any unbounded simple curve γ in $\mathbb{H}$ starting at 0*

$$\varphi_{x_0}^{-1}(\gamma) \not\subset C(\alpha_-, \alpha_+) \implies I^C_{\rho,z_0}(\gamma) \geq M.$$

Proof. For $\beta \in (0,1)$ and $\tilde R > 0$ and let $s_{\beta,\tilde R} = \{re^{i\pi\beta} : r \in [0,\tilde R]\}$. Then

$$\frac{\omega_\infty(s^-_{\beta,\tilde R}, \mathbb{H} \setminus s_{\beta,\tilde R})}{\omega_\infty(s_{\beta,\tilde R}, \mathbb{H} \setminus s_{\beta,\tilde R})} = 1 - \beta. \tag{48}$$

This is easily checked using an (inverse) Schwarz-Christoffel map to map out the slit. Suppose γ is an unbounded simple curve in $\mathbb{H}$ staring at 0 and denote $\tilde\gamma := \varphi_{x_0}^{-1}(\gamma)$. Suppose further that $\tilde\gamma \not\subset C(\alpha_-,\alpha_+)$ and let $T = \inf\{t : \tilde\gamma(t) \notin C(\alpha_-,\alpha_+)\}$. Assume that $\arg(\tilde\gamma(T)) = \pi\alpha_-$. We claim that this implies, in the notation of Lemma 11, that γ has $r_T \geq 1-\alpha_-$. To prove this claim, it suffices to show that $\frac{1}{r_T} - 1 \leq \frac{\alpha_-}{1-\alpha_-}$. First note that conformal covariance of ω_∞ gives

$$\frac{1}{r_T} - 1 = \frac{x_T - W_T}{W_T - y_T} = \frac{\omega_\infty(\varphi_{x_0}^{-1}([W_T, x_T]), \varphi_{x_0}^{-1}(\mathbb{H}))}{\omega_\infty(\varphi_{x_0}^{-1}([y_t, W_T]), \varphi_{x_0}^{-1}(\mathbb{H}))} = \frac{\omega_\infty(s^+_{\alpha,R} \cup \tilde\gamma^+_T, \mathbb{H} \setminus (s_{\alpha,R} \cup \tilde\gamma_T))}{\omega_\infty(s^-_{\alpha,R} \cup \tilde\gamma^-_T, \mathbb{H} \setminus (s_{\alpha,R} \cup \tilde\gamma_T))}. \tag{49}$$

Observe that $s_{\alpha,R} \cup \tilde\gamma_T \cup s_{\alpha_-,\tilde\gamma(T)}$ is a Jordan curve, see Figure 4. Monotonicity of harmonic measure then implies

$$\omega_\infty(s^+_{\alpha,R} \cup \tilde\gamma^+_T, \mathbb{H} \setminus (s_{\alpha,R} \cup \tilde\gamma_T)) \leq \omega_\infty(s^+_{\alpha_-,\tilde\gamma(T)}, \mathbb{H} \setminus s_{\alpha_-,\tilde\gamma(T)}),$$
$$\omega_\infty(s^-_{\alpha,R} \cup \tilde\gamma^-_T, \mathbb{H} \setminus (s_{\alpha,R} \cup \tilde\gamma_T)) \geq \omega_\infty(s^-_{\alpha_-,\tilde\gamma(T)}, \mathbb{H} \setminus s_{\alpha_-,\tilde\gamma(T)}).$$

Inserting this into (49) and using (48) yields $\frac{1}{r_T} - 1 \leq \frac{\alpha_-}{1-\alpha_-}$ as desired. By a similar argument, one finds that $\arg(\tilde\gamma(T)) = \pi\alpha_+$ implies $r_T \leq 1 - \alpha_+$. Applying Lemma 11 shows the desired implication with

$$M = \min\Big\{ -(2+\rho)\log\frac{\alpha_-}{\alpha} - 2\log\frac{1-\alpha_-}{1-\alpha}, -(2+\rho)\log\frac{\alpha_+}{\alpha} - 2\log\frac{1-\alpha_+}{1-\alpha}\Big\} > 0.$$

□

Proof of Proposition 16. Suppose $\gamma \in \mathcal{X}^C_{x_0}$ and that $I^C_{\rho,x_0}(\gamma) < \infty$. Suppose also, for contradiction, that there exists α_- and α_+ for which the statement fails. Take $\varepsilon > 0$ sufficiently small so that $\tilde\alpha_- := \alpha_- + \varepsilon > 0$, $\tilde\alpha_+ := \alpha_+ - \varepsilon < 1$. Now, let $M = M(\tilde\alpha_-, \tilde\alpha_+)$ as in Lemma 12. Since $I^C_{\rho,x_0}(\gamma) < \infty$ there exists a $T > 0$ such that $I^C_{\rho,x_T - W_T}(g_T(\gamma_{[T,\infty)}) - W_T) < M$. Therefore,

$$\tilde\gamma := \varphi_{x_T - W_T}^{-1}(g_T(\gamma_{[T,\infty)}) - W_T) \subset C(\tilde\alpha_-, \tilde\alpha_+)$$

which implies that

$$\gamma_{[T,\infty)} \subset g_T^{-1}(\varphi_{x_T - W_T}(C(\tilde\alpha_-, \tilde\alpha_+)) + W_T).$$

Since both $\varphi_{x_T-W_T}$ and g_T and are analytic at ∞ the rays $\{re^{i\pi\tilde{\alpha}_-}, r>0\}$ and $\{re^{i\pi\tilde{\alpha}_+}, r>0\}$ are mapped by $g_T^{-1}(\varphi_{x_T-W_T} - W_T)$ to smooth curves approaching infinity at the same angle. Hence, there exists an $R>0$ such that

$$g_T^{-1}(\varphi_{x_T-W_T}(C(\tilde{\alpha}_-, \tilde{\alpha}_+)) + W_T) \setminus B(0,R) \subset C(\alpha_-, \alpha_+).$$

Since, by choosing R larger, we can ensure that $\gamma_T \subset B(0,R)$ this finishes the proof. □

6.2.1 Proof of Theorem 3

Let $\gamma : (0,\infty) \to \Sigma$ be a simple curve from 0 to ∞. Let $h : \Sigma \setminus \gamma \to \Sigma \setminus \mathbb{R}^-$ be a map which is conformal on each component of $\Sigma \setminus \gamma$ mapping the upper (lower) component to $\mathbb{H}$ ($\mathbb{H}^*$) with $\infty \mapsto \infty$. For $\beta \in (0,1)$, we define

$$\mathcal{D}_\beta(\gamma) := \mathcal{D}_\beta(h)$$

whenever the right-hand side (defined in (11)) exists. Since $|\nabla \log |h'(z)|| = |h''(z)/h'(z)|$, and h is unique up to post-composition by translation and scaling on each component, $\mathcal{D}_\beta(\gamma)$ is well-defined. We will write $\sigma_h := \log |h'|$ for conformal maps h.

Lemma 13. *Fix $\rho > -2$ and $x_0 > 0$ and let γ^0 be the SLE$_0(\rho)$ from 0 to ∞ in Σ with force point x_0^+. Then, $\mathcal{D}_\alpha(\gamma^{\rho,x_0})$ exists and is finite.*

Proof. By Remark 13, there exists a $\tilde{R} > 0$ and a conformal map $\tilde{h} : \Sigma \setminus \{re^{i2\alpha\pi} : r \in [0,\tilde{R}]\} \to \Sigma$ satisfying

$$\tilde{h}(\{re^{i2\alpha\pi} : r \in [\tilde{R},\infty)\}) = \gamma^0 \quad \text{and} \quad \sqrt{\tilde{h}(z^2)} = z + O(1) \text{ analytic at } \infty.$$

The second property gives

$$|\tilde{h}(z)| = |z| + O(|z|^{1/2}), \tag{50}$$
$$\log |\tilde{h}'(Re^{i\theta})| = O(|z|^{-1/2}), \tag{51}$$
$$|\nabla \sigma_{\tilde{h}}(z)| = O(|z|^{-3/2}). \tag{52}$$

(For the same reason (50)-(52) also hold for $\tilde{h}^{-1}$.) Further, let

$$\hat{h} : \Sigma \setminus \{re^{i2\alpha\pi} : r \in \mathbb{R}^+\} \to \Sigma \setminus \mathbb{R}^-$$

$$\hat{h}(re^{i\theta}) = \begin{cases} r^{1/(2\alpha)} e^{i\theta/(2\alpha)}, & \theta \in (0, 2\alpha\pi), \\ -r^{1/(2-2\alpha)} e^{i(\theta-2\alpha\pi)/(2-2\alpha)}, & \theta \in (2\alpha\pi, 2\pi), \end{cases}$$

and set $h = \hat{h} \circ \tilde{h}^{-1}$. Then $\mathcal{D}_\alpha(\gamma^0) = \mathcal{D}_\alpha(h)$. Since $\gamma^0_{[0,T]}$ has finite chordal Loewner energy for every $T < \infty$, we have

$$\int_{B(0,r)\setminus(\gamma^0\cup\mathbb{R}^+)} |\nabla \sigma_h(z)|^2 dz^2 < \infty$$

for every $r > 0$. Hence, it suffices to show that

$$\lim_{R\to\infty} \left(\frac{1}{\pi} \int_{A(r,R)\setminus(\gamma^0\cup\mathbb{R}^+)} |\nabla \sigma_h(z)|^2 dz^2 - c_\alpha \log R \right) \tag{53}$$

exists for some r, where $A(r,R) = \{r < |z| < R\}$. We write

$$|\nabla \sigma_h(z)|^2 = |\nabla \sigma_{\tilde{h}^{-1}}(z)|^2 + 2\nabla \sigma_{\tilde{h}^{-1}}(z) \cdot \nabla \sigma_{\hat{h}}(\tilde{h}^{-1}(z)) + |\nabla \sigma_{\hat{h}}(\tilde{h}^{-1}(z))|^2,$$

and study the integral over these terms separately. First of all,

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{A(r,R)\setminus(\gamma^0\cup\mathbb{R}^+)} |\nabla \sigma_{\tilde{h}^{-1}}(z)|^2 dz^2$$

exists and is finite for sufficiently large $r > 0$ by (52). Second, by conformal invariance of the Dirichlet inner product and Stokes' theorem, we have

$$\int_{A(r,R)\setminus(\gamma^0\cup\mathbb{R}^+)} \nabla\sigma_{\tilde{h}^{-1}}(z)\cdot\nabla\sigma_{\hat{h}}(\tilde{h}^{-1}(z))dz^2 = -\int_{\tilde{h}^{-1}(A(r,R))\setminus(\{re^{i2\alpha}\}\cup\mathbb{R}^+)} \nabla\sigma_{\tilde{h}}(z)\cdot\nabla\sigma_{\hat{h}}(z)dz^2$$
$$= -\int_{\Gamma_R^+\cup\Gamma_R^-} \sigma_{\tilde{h}}\partial_n\sigma_{\hat{h}}(z)d\ell,$$

where Γ_R^+ and Γ_R^- are the boundaries of the upper and lower connected components of $\tilde{h}^{-1}(A(r,R))\setminus(\{re^{i2\alpha}\}\cup\mathbb{R}^+)$ respectively. Note that r can be chosen sufficiently large so that $\tilde{h}^{-1}$ is conformal on $\mathbb{C}\setminus(\mathbb{R}^+\cup B(0,r))$, and can be analytically extended across $\mathbb{R}^+$ (from both sides separately). This ensures that $\Gamma_R^\pm$ is piece-wise smooth, and that $\sigma_{\tilde{h}}$ and $\sigma_{\hat{h}}$ are smooth up to (and including) $\Gamma_R^\pm$, so Stokes' theorem may be used. Note that $\partial_n\sigma_{\hat{h}}$ vanishes along both sides of $\mathbb{R}^+$ and $\{re^{i2\alpha\pi} : r\in\mathbb{R}^+\}$. Thus, we are left with integrals along $\tilde{h}^{-1}(\partial B(0,r))$ and $\tilde{h}^{-1}(\partial B(0,R))$. We have

$$|\nabla\sigma_{\hat{h}}(Re^{i\theta})| = \begin{cases} |\frac{1}{2\alpha}-1|\frac{1}{R}, & \theta\in(0,2\alpha\pi),\\ |\frac{1}{2-2\alpha}-1|\frac{1}{R}, & \theta\in(2\alpha\pi,2\pi),\end{cases}$$

and the length of $\tilde{h}^{-1}(\partial B(0,R))$ is $O(R)$. From this and (51), we conclude that

$$\lim_{R\to\infty}\int_{A(r,R)\setminus(\gamma^0\cup\mathbb{R}^+)} \nabla\sigma_{\tilde{h}^{-1}}(z)\cdot\nabla\sigma_{\hat{h}}(\tilde{h}^{-1}(z))dz^2 = \int_{\tilde{h}^{-1}(\partial B(0,r))} \sigma_{\tilde{h}}(z)\partial_n\sigma_{\hat{h}}(z)d\ell.$$

Finally, using the computation of $|\nabla\sigma_{\hat{h}}|$ and (50) we obtain

$$\left(\int_{\tilde{h}^{-1}(A(r,R))\setminus(\{re^{i2\alpha}\}\cup\mathbb{R}^+)} - \int_{A(r,R)\setminus(\{re^{i2\alpha}\}\cup\mathbb{R}^+)}\right)|\nabla\sigma_{\hat{h}}(z)|^2dz^2 = O(1), \quad \text{as } R\to\infty$$

since the symmetric difference of the sets to be integrated over consists of a bounded part and a part contained in an annulus of radius R and thickness $O(\sqrt{R})$. Moreover, the second term is easily computed and equals $\pi c_\alpha\log\frac{R}{r}$. We conclude that (53) exists. □

We are now ready to prove Theorem 3. Let $\gamma\subset\Sigma$ be a simple curve from 0 to ∞ such that $\gamma_{[T,\infty)}$ is the ρ-Loewner energy optimal continuation of γ_T. Let $h_T:\Sigma\setminus\gamma_T\to\Sigma$ be the conformal map with $h_T(\infty)=\infty$, $h_T(\gamma(T))=0$ and $|h_T'(\infty)|=1$. We may assume that $h_T(x_0^+)\geq x_0$ (for otherwise we may achieve this by increasing T). Under this assumption, there exists a $\tilde{T}\geq 0$ such that

$$\tilde{h}_{\tilde{T}}(x_0^+) = h_T(x_0^+) =: x_T,$$

where $\tilde{h}_{\tilde{T}}:\Sigma\setminus\gamma^0_{\tilde{T}}\to\Sigma$ is the conformal map with the same normalization as h_T. Note that $H=\tilde{h}_{\tilde{T}}^{-1}\circ h_T$ and that (19) gives

$$I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\gamma) = I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\gamma_T) - I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\gamma^0_{\tilde{T}}) = I^{(\Sigma;0,\infty)}(\gamma_T) - I^{(\Sigma;0,\infty)}(\gamma^0_{\tilde{T}}) - \frac{\rho(4+\rho)}{4}\log|H'(x_0)|.$$

So, it only remains to show that $I^{\Sigma,0,\infty}(\gamma_T) - I^{\Sigma,0,\infty}(\gamma^0_{\tilde{T}}) = \mathcal{D}_\alpha(\gamma)-\mathcal{D}_\alpha(\gamma^0)$ and from (7) we already know that

$$I^{(\Sigma;0,\infty)}(\gamma_T) - I^{(\Sigma;0,\infty)}(\gamma^0_{\tilde{T}}) = \frac{1}{\pi}\int_{\Sigma\setminus\gamma}|\nabla\sigma_{h_T}(z)|^2dz^2 - \frac{1}{\pi}\int_{\Sigma\setminus\gamma^0}|\nabla\sigma_{\tilde{h}_{\tilde{T}}}(z)|^2dz^2.$$

Let $\tilde{\gamma}^0 := \tilde{h}_{\tilde{T}}(\gamma^0_{[\tilde{T},\infty)}) = h_T(\gamma_{[T,\infty)})$ and let $h_{\tilde{\gamma}^0}:\Sigma\setminus\tilde{\gamma}^0\to\Sigma\setminus\mathbb{R}^-$ be a conformal map, fixing ∞ and mapping the components appropriately. See, Figure 5. Define

$$h_\gamma := h_{\tilde{\gamma}^0}\circ h_T \quad\text{and}\quad h_{\gamma^0} := h_{\tilde{\gamma}^0}\circ\tilde{h}_{\tilde{T}}$$

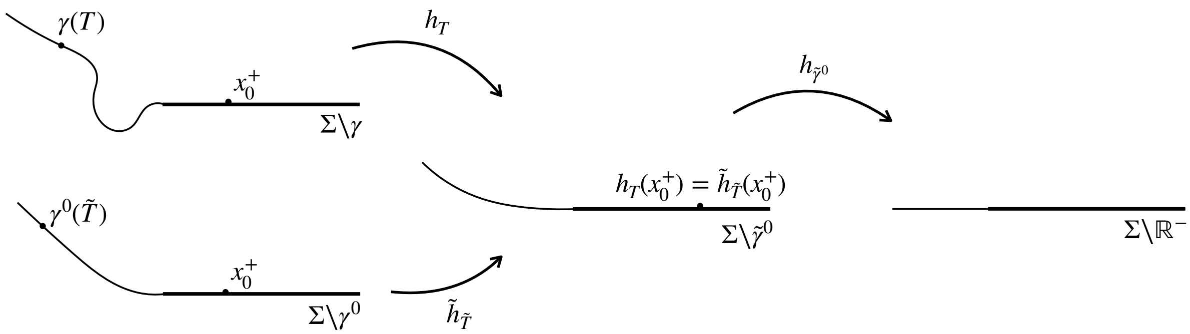

Figure 5: Illustration of the conformal maps h_T, $\tilde{h}_{\tilde{T}}$, and $h_{\tilde{\gamma}^0}$ from the proof of Theorem 3.

and note that $\mathcal{D}_\alpha(\gamma) = \mathcal{D}_\alpha(h_\gamma)$ and $\mathcal{D}_\alpha(\gamma^0) = \mathcal{D}_\alpha(h_{\gamma^0})$. If $I^{(\Sigma,0,\infty)}(\gamma) = \infty$ we have that $\mathcal{D}_\alpha(\gamma) = \infty$, since for each sufficiently large R we have that

$$\int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} |\nabla\sigma_{h_T}(z)|^2 dz^2 = \infty, \qquad \int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} |\nabla\sigma_{h_{\tilde{\gamma}^0}}(h_T(z))|^2 dz^2 < \infty.$$

Hence, we may assume that $I^{(\Sigma,0,\infty)}(\gamma) < \infty$. As we already know that $\mathcal{D}_\alpha(\gamma^0)$ exists and is finite (by Lemma 13), we need only to check that

$$\lim_{R\to\infty} \int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} \nabla\sigma_{h_{\tilde{\gamma}^0}}(h_T)\cdot\nabla\sigma_{h_T} dz^2 = 0 \tag{54}$$

$$\lim_{R\to\infty} \int_{B(0,R)\setminus(\gamma^0\cup\mathbb{R}^+)} \nabla\sigma_{h_{\tilde{\gamma}^0}}(\tilde{h}_{\tilde{T}})\cdot\nabla\sigma_{\tilde{h}_{\tilde{T}}} dz^2 = 0 \tag{55}$$

$$\lim_{R\to\infty} \left(\int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} |\nabla\sigma_{h_{\tilde{\gamma}^0}}(h_T)|^2 dz^2 - \int_{B(0,R)\setminus(\gamma^0\cup\mathbb{R}^+)} |\nabla\sigma_{h_{\tilde{\gamma}^0}}(\tilde{h}_{\tilde{T}})|^2 dz^2 \right) = 0. \tag{56}$$

We start by showing (54). Let $\phi \in C_c^\infty(\mathbb{C})$ is some function with $\phi|_{h_T(B(0,R))} \equiv 1$. It then follows from [45, Lemma 5.3] and Stokes' theorem that

$$\begin{aligned}\int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} \nabla\sigma_{h_{\tilde{\gamma}^0}}(h_T)\cdot\nabla\sigma_{h_T} dz^2 &= -\int_{\mathbb{C}\setminus(B(0,R)\cup\gamma\cup\mathbb{R}^+)} \nabla(\phi\sigma_{h_{\tilde{\gamma}^0}})(h_T)\cdot\nabla\sigma_{h_T} dz^2 \\ &= \int_{\partial B(0,R)} \sigma_{h_T}\partial_n\sigma_{h_{\tilde{\gamma}^0}}(h_T) d\ell.\end{aligned}$$

Here we are using that σ_{h_T} and $\sigma_{\tilde{\gamma}^0}(h_T)$ are smooth on $\mathbb{C}\setminus(B(0,R)\cup\mathbb{R}^+)$ and that $\partial_n\sigma_{h_{\tilde{\gamma}^0}}$ vanishes along both sides of $\mathbb{R}^+$ since

$$\partial_n\sigma_{h_{\tilde{\gamma}^0}}(re^{i\theta})|_{\theta=0} = \partial_r \arg(h'_{\tilde{\gamma}^0}(r)) = \partial_r(0) = 0.$$

Further, since $\sqrt{h_T(z^2)} = z + O(1)$ is analytic at ∞

$$|h_T(Re^{i\theta})| = R + O(1/\sqrt{R}), \qquad |h'_T(Re^{i\theta})| = 1 + O(1/\sqrt{R}),$$

as $R\to\infty$ and by the proof of Lemma 13 we have $|\nabla\sigma_{h_{\tilde{\gamma}^0}}(Re^{i\theta})| = O(1/R)$, as $R\to\infty$. Applying these estimates to the right-hand side above shows (54). Since (55) is a special case of (54), we have also shown (55). We move on to (56). Using conformal invariance of the Dirichlet inner product we get

$$\int_{B(0,R)\setminus(\gamma\cup\mathbb{R}^+)} |\nabla\sigma_{h_{\tilde\gamma^0}}(h_T(z))|^2 dz^2 - \int_{B(0,R)\setminus(\gamma^0\cup\mathbb{R}^+))} |\nabla\sigma_{h_{\tilde\gamma^0}}(\tilde h_{\tilde T}(z))|^2 dz^2$$
$$= \Big(\int_{h_T(B(0,R))\setminus(\tilde\gamma^0\cup\mathbb{R}^+)} - \int_{\tilde h_{\tilde T}(B(0,R))\setminus(\tilde\gamma^0\cup\mathbb{R}^+)} \Big) |\nabla\sigma_{h_{\tilde\gamma^0}}(z)|^2 dz^2.$$

Since, $|h_T(Re^{i\theta})| = R + O(1/\sqrt{R})$ and $|\tilde h_{\tilde T}(Re^{i\theta})| = R + O(1/\sqrt{R})$, the symmetric difference $h_T(B(0,R))\triangle \tilde h_{\tilde T}(B(0,R))$ is contained in an annulus of thickness $O(1/\sqrt{R})$ and radius R. Combining this with the estimate $|\nabla\sigma_{h_{\tilde\gamma^0}}(Re^{i\theta})| = O(1/R)$, we conclude that (56) holds. This finishes the proof.

7 ζ-regularized determinants of Laplacians

In this section, we prove Proposition 1. We will give a detailed proof of the chordal case and an outline of the proof of the radial case (the details in the radial case are almost identical to those in the chordal case). Recall from Section 1.2, the construction of the smooth slit structures $(\overline{\mathbb{D}},\varphi)$. We say that a Riemannian metric g_0 on $\mathbb{D}$ is a $(\overline{\mathbb{D}},\varphi)$-smooth conformal metric if the following holds:

- In the z-coordinate $g_0(z) = e^{2\sigma_0(z)}dz^2$, where $\sigma_0 \in C^\infty(\overline{\mathbb{D}}\setminus\{a\})$.
- In the $w = \varphi(z)$-coordinate $g_0(\varphi^{-1}(w)) = e^{2\hat\sigma_0(w)}dw^2$, where $\hat\sigma_0 \in C^\infty(\mathbb{D}\setminus\Gamma)$ and $\hat\sigma_0$ extends smoothly to both sides of Γ separately.

Before presenting the proof of Proposition 1 we prove two lemmas.

Lemma 14. *Let a, b, c, Γ, φ be as in Proposition 1. Then, for each $(\overline{\mathbb{D}},\varphi)$-smooth conformal metric g_0, $(\overline{\mathbb{D}}, g_0, (a), (2))$ is a curvilinear polygonal domain. Moreover, if $\gamma \in \hat{\mathcal{X}}$ then (the closure of) each of the components of $\mathbb{D}\setminus\gamma$ and $\mathbb{D}\setminus(\gamma\cup\eta)$, endowed with g_0 along with the appropriate n-tuples (p_j) and (β_j), are curvilinear polygonal domains.*

Proof. It is immediate from the construction of the smooth slit structure $(\overline{\mathbb{D}},\varphi)$ that g_0 is a smooth Riemannian metric on $\overline{\mathbb{D}}\setminus\{a\}$. For the corner point $p_1 = a$ we choose $\varphi_1 = \varphi$. We check that (i-iv) of Definition 5 hold. By construction of φ (i) holds with $\beta_1 = 2$. As g_0 is a $(\overline{\mathbb{D}},\varphi)$-smooth conformal metric (ii) is also immediate. Moreover, let Γ_1 and Γ_2 be smooth Jordan curves as in (iii) and (iv). Let $g = e^{2\hat\sigma(w)}dw^2$. We may assume that $\Gamma_1,\Gamma_2 \subset \mathbb{D}$ and thus $\sigma_1 = \hat\sigma$ is already defined on $V_{1,1}^\circ$ and $V_{1,2}^\circ$. Since g_0 is $(\overline{\mathbb{D}},\varphi)$-smooth it follows that $\hat\sigma = \sigma_1 \in C^\infty(V_{1,1})$ and $\hat\sigma = \sigma_1 \in C^\infty(V_{1,2})$. Thus $(\overline{\mathbb{D}}, g_0, (a), (2))$ is a curvilinear polygonal domain.

Let $\gamma \in \hat{\mathcal{X}}$. Then, γ is $(\overline{\mathbb{D}},\varphi)$-smooth. Moreover, as η is a hyperbolic geodesic it is smooth, at least away from its endpoints. If $b \in \mathbb{D}$ smoothness of η at b is forced by the definition of $\hat{\mathcal{X}}$ and smoothness at c follows from the fact that a conformal map $\psi : \mathbb{D}\setminus\gamma \to \mathbb{D}$, with $\psi(b) = 1$ and $\psi(c) = -1$, maps η onto $[-1,1]$ and can be extended conformally to neighbourhood of c using Schwarz reflection. If $b \in \partial\mathbb{D}$, then we may assume, by symmetry that c lies on the counter-clockwise circular arc from a to b. We consider the conformal map

$$\psi : \mathbb{D}\setminus\gamma_{[0,T]} \to \Sigma\setminus\{re^{\alpha\pi i} : r\in[0,1]\}$$

with $\alpha = \alpha(\rho) = \frac{2+\rho}{4+\rho}$, $\psi(\gamma(T)) = e^{2\alpha\pi i}$, $\psi(b) = \infty$ and $\psi(c) = 0+$, where T is chosen so that $\gamma_{[T,\infty)}$ is the ρ-Loewner energy optimal continuation of γ_T. It follows from Remark 13 that

$$\psi(\gamma_{[T,\infty)}) = \{re^{\alpha\pi i} : r\in[1,\infty)\}, \quad \psi(\eta) = \{re^{\alpha\pi i/2} : r\in[0,\infty)\}.$$

Since ψ can be extended to a neighborhood of b (again using Schwarz reflection) we deduce that η is smooth at b. Using the same type of argument as in the radial case one can show that η is smooth at c.

Since $(\mathbb{D}, g, (a), (2))$ is a curvilinear polygonal domain and γ and η are $(\overline{\mathbb{D}}, \varphi)$-smooth and intersect $\partial\mathbb{D}$ non-tangentially it follows that the components of $\mathbb{D}\setminus\gamma$ and $\mathbb{D}\setminus(\gamma\cup\eta)$ are curvilinear polygonal. □

It follows from Lemma 14 that, for each choice of a, b, c, φ, and $(\overline{\mathbb{D}}, \varphi)$-smooth conformal metric g_0, the ρ-Loewner potential, with respect to g_0, is defined for all curves $\gamma \in \hat{\mathcal{X}}$.

Lemma 15. *Let a, b, c, Γ, φ be as in Proposition 1. Let $\gamma : (0, T] \to \mathbb{D}$, with $\gamma(0+) = a$ be a $(\overline{\mathbb{D}}, \varphi)$-smooth slit, smoothly attached at a. Let g_0 be a $(\overline{\mathbb{D}}, \varphi)$-smooth conformal metric. Then $(\overline{\mathbb{D}\setminus\gamma}, g_0, (\gamma(T)), (2))$ is a curvilinear polygonal domain (here $\partial(\mathbb{D}\setminus\gamma)$ refers to the set of prime ends). If $\sigma \in C^\infty(\mathbb{D}\setminus\gamma)$ extends smoothly to both sides of $\gamma((0, T])$ and to $\partial\mathbb{D}\setminus\{a\}$, and $\sigma(\varphi^{-1})$ extends smoothly to both sides of $\varphi(\gamma)\cup\Gamma$ locally at 0, then $\sigma \in C^\infty(\overline{\mathbb{D}\setminus\gamma}, g_0, (\gamma(T)), (2))$.*

Proof. The proof of Lemma 14 shows that $(\overline{\mathbb{D}\setminus\gamma}, g_0, (\gamma(T)), (2))$ is a curvilinear polygonal domain. The z-coordinate is smooth in a neigborhood of $p_1 = \gamma(T)$. Since σ extends smoothly to both sides of γ it is immediate that σ can be smoothly extended to any $V_{1,1}, V_{1,2} \subset \mathbb{D}$ as in Definition 6. Since the boundary of $\mathbb{D}\setminus\gamma$ (treated as prime ends) is $(\overline{\mathbb{D}}, \varphi)$-smooth away from $\gamma(T)$, the inherited smooth structure on $\overline{\mathbb{D}\setminus\gamma}\setminus\{\gamma(T)\}$ is a smooth structure in the classical sense. Since σ extends smoothly to both sides of $\gamma_{(0,T)}$ and to $\partial\mathbb{D}\setminus\{a\}$, and $\sigma(\varphi^{-1})$ extends smoothly (locally at 0) to both sides of $\varphi(\gamma)\cup\Gamma$ we have that $\sigma \in C^\infty(\overline{\mathbb{D}\setminus\gamma}\setminus\{\gamma(T)\})$ with respect to the inherited smooth structure. This shows that $\sigma \in C^\infty(\overline{\mathbb{D}\setminus\gamma}, g_0, (\gamma(T)), (2))$. □

We are now ready to prove Proposition 1.

Proof of Proposition 1, chordal case. Fix $\rho > -2$, a, $b \in \partial\mathbb{D}$, c, Γ, and φ as in Proposition 1. Consider $\gamma_1, \gamma_2 \in \hat{\mathcal{X}}$ and a $(\overline{\mathbb{D}}, \varphi)$-smooth conformal metric $g_0 = e^{2\sigma_0}dz^2$. Since γ_j, $\eta_j = \eta(\gamma_j)$, $j = 1, 2$, are $(\overline{\mathbb{D}}, \varphi)$-smooth the ρ-Loewner potential of γ_j with respect to g_0 is defined (see, Lemma 14).

Let $\psi : \mathbb{D} \to \Sigma$ be a conformal map with $\psi(a) = 0$, $\psi(b) = \infty$. By symmetry, we may assume that $\psi(c) = x_0^+$, where x_0^+ is the upper prime end at $x_0 > 0$. For $j = 1, 2$, write $\tilde{\gamma}_j = \psi(\gamma_j)$ and $\tilde{\eta}_j = \psi(\eta_j)$, $j = 1, 2$, and let $T_j > 0$ be such that $\gamma_j([T_j, \infty))$ is the $I_{\rho,c}^{(\mathbb{D},a,b)}$-optimal continuation of $\gamma_j([0, T_j])$. Next, let

$$h : \Sigma\setminus\tilde{\gamma}_1([0, T_1]) \to \Sigma\setminus\tilde{\gamma}_2([0, T_2])$$

be the conformal map with $h(\tilde{\gamma}_1(T_1)) = h(\tilde{\gamma}_2(T_2))$, $h(\infty) = h(\infty)$, and $h(x_0^+) = x_0^+$. Observe that this has the effect that $\tilde{\gamma}_2([T_2, \infty)) = h(\tilde{\gamma}_1([T_1, \infty)))$ and $\tilde{\eta}_2 = h(\tilde{\eta}_1)$, where $\tilde{\eta}_j = \psi(\eta_j)$ and $\eta_j = \eta(\gamma_j)$. Hence, we have, for each component D of $\mathbb{D}\setminus\gamma_1$ or $\mathbb{D}\setminus(\gamma_1\cup\eta_1)$, that

$$\log\det{}_\zeta \Delta_{((\psi^{-1}\circ h\circ\psi)(D), g_0)} = \log\det{}_\zeta \Delta_{(D, (\psi^{-1}\circ h\circ\psi)^* g_0)}. \tag{57}$$

Write $H = \psi^{-1}\circ h\circ\psi$ and set $g = H^* g_0$. Then $g = e^{2\sigma} g_0$ with $\sigma = \sigma_0(H) + \sigma_H - \sigma_0$. We wish to apply Theorem B on the components of $\mathbb{D}\setminus\gamma_1$ and $\mathbb{D}\setminus(\gamma_1\cup\eta_1)$, with the conformal change of metric $g = e^{2\sigma} g_0$. To do so we must, in light of Lemma 15, show that σ extends smoothly to both sides of $\gamma_1((0, T_1])$ and to $\partial\mathbb{D}\setminus\{a\}$, and that $\sigma(\varphi^{-1})$ extends smoothly, locally at 0, to both sides of $\varphi(\gamma)\cup\Gamma$. We immediately have that σ and extends smoothly to both sides of $\gamma_1((0, T_1])$ and to $\partial\mathbb{D}\setminus\{a\}$ with the possible exception of $\{H^{-1}(a+), H^{-1}(a-)\}$, since σ_0 is smooth on $\overline{\mathbb{D}}\setminus\{a\}$ and H is smooth on $\overline{\mathbb{D}}\setminus\{a+, a-, H^{-1}(a+), H^{-1}(a-))\}$ (here we use Kellogg's theorem, see, e.g., [14, Theorem II.4.3]). Close to $\varphi(a+)$ we have

$$\sigma(\varphi^{-1}(w)) = \sigma_0((H\circ\varphi^{-1})(w)) + \sigma_{H\circ\varphi^{-1}}(w) - (\sigma_0(\varphi^{-1}(w)) + \sigma_{\varphi^{-1}}(w))$$

where $H\circ\varphi^{-1}$ is a smooth conformal map (unless H fixes $a+$) and $\hat{\sigma}_0 = \sigma_0(\varphi^{-1}) + \sigma_{\varphi^{-1}}$ is smooth since g_0 is smooth. Hence we have that σ is $(\overline{\mathbb{D}}, \varphi)$-smooth at $a+$ (the case where H fixes $a+$ is covered in a similar way). The same type of argument shows that σ is $(\overline{\mathbb{D}}, \varphi)$-smooth at $a-$, $H^{-1}(a+)$, and $H^{-1}(a-)$.

Let D_1 denote the component of $\mathbb{D}\setminus\gamma_1$ containing η_1 and let D_2 denote the other component. Observe that the ρ-Loewner potential of γ_1 can be re-written as

$$\mathcal{H}^{(\mathbb{D};a,b)}_{\rho,c}(\gamma_1;g_0)=\mathcal{H}_{(\mathbb{D},g_0)}(\gamma_1)+\frac{\rho(\rho+4)}{12}\mathcal{H}_{(D_1,g_0)}(\eta_1),$$

and similarly for γ_2. We apply Theorem B, which along with (57) yields

$$12\Big(\mathcal{H}_{(D_1,g_0)}(\eta_2)-\mathcal{H}_{(H(D_1),g_0)}(\eta_1)\Big)=-\Big(\frac{1}{\alpha}-\alpha\Big)\sigma(b)+2\Big(\frac{2}{\alpha}-\frac{\alpha}{2}\Big)\sigma(b)+2\Big(2-\frac{1}{2}\Big)\sigma(c) \quad (58)$$

and

$$\begin{aligned}12\Big(&\mathcal{H}_{(\mathbb{D};g_0)}(\gamma_2)-\mathcal{H}_{(\mathbb{D};g_0)}(\gamma_1)\Big)\\ &=\frac{1}{\pi}\int_{D_1\cup D_2}|\nabla_{g_0}\sigma|^2d\operatorname{Vol}_{g_0}+\frac{2}{\pi}\int_{D_1\cup D_2}\sigma K_{g_0}d\operatorname{Vol}_{g_0}+\frac{2}{\pi}\int_{\partial D_1\cup\partial D_2}\sigma k_{g_0}d\ell_{g_0} \quad (59)\\ &\quad+\frac{3}{\pi}\int_{\partial D_1\cup\partial D_2}\partial_{n_{g_0}}\sigma d\ell_{g_0}+\Big(\frac{1}{\alpha}-\alpha\Big)\sigma(b)+\Big(\frac{1}{1-\alpha}-(1-\alpha)\Big)\sigma(b).\end{aligned}$$

Note that

$$\int_{\partial D_1\cup\partial D_2}\partial_{n_{g_0}}\sigma d\ell_{g_0}=\int_{\partial H(D_1)\cup\partial H(D_2)}k_{g_0}d\ell_{g_0}-\int_{\partial D_1\cup\partial D_2}k_{g_0}d\ell_{g_0}=0,$$

since the contributions from integration along the two sides of γ_1 and γ_2 cancel.

Since we aim to relate the ρ-Loewner potential to the ρ-Loewner energy, we consider (59) in ψ-coordinates. Denote by $\tilde{g}_0=(\psi^{-1})^*g_0$, $\tilde{g}=e^{2\sigma(\psi^{-1})}\tilde{g}_0$, $\tilde{D}_1=\psi(D_1)$, and $\tilde{D}_2=\psi(D_2)$. We have $\tilde{g}_0=e^{\tilde{\sigma}_0(z)}dz^2$ where $\tilde{\sigma}_0=\sigma_0(\psi^{-1})+\sigma_{\psi^{-1}}$ and $\sigma(\psi^{-1})=\tilde{\sigma}_0(h)+\sigma_h-\tilde{\sigma}_0$, so that

$$\int_{D_1\cup D_2}|\nabla_{g_0}\sigma|^2d\operatorname{Vol}_{g_0}=\int_{\tilde{D}_1\cup\tilde{D}_2}|\nabla_{\tilde{g}_0}(\tilde{\sigma}_0(h)+\sigma_h-\tilde{\sigma}_0)|^2d\operatorname{Vol}_{\tilde{g}_0}.$$

We wish to compute the right-hand side by expanding the square and studying each term separately. Note that $\tilde{\sigma}_0(h)$, σ_h and $\tilde{\sigma}_0$ are not necessarily smooth at

$$x\in\{0+,0-,h^{-1}(0+),h^{-1}(0-),\infty\}$$

when considered separately. Let $D_\varepsilon=\tilde{D}_1\cup\tilde{D}_2\setminus\cup B_{x,\varepsilon}$ where $B_{x,\varepsilon}=B_{(\tilde{D}_1\cup\tilde{D}_2,\tilde{g}_0)}(x,\varepsilon)$ if $x\in\{0+,0-,\infty\}$ and $B_{h^{-1}(0\pm),\varepsilon}=h^{-1}(B_{0\pm,\varepsilon})$. (Here we assume that $0+$ and $0-$ are not fixed by h. The other case can be treated in a similar way.) By applying Stokes' theorem on D_ε and using (17) we find

$$\begin{aligned}\int_{D_\varepsilon}\nabla_{\tilde{g}_0}\sigma_h\cdot\nabla_{\tilde{g}_0}\tilde{\sigma}_0(h)d\operatorname{Vol}_{\tilde{g}_0}&=\int_{\partial h(D_\varepsilon)}\tilde{\sigma}_0kd\ell-\int_{\partial D_\varepsilon}\tilde{\sigma}_0(h)kd\ell,\\ \int_{D_\varepsilon}\nabla_{\tilde{g}_0}\sigma_h\cdot\nabla_{\tilde{g}_0}\tilde{\sigma}_0d\operatorname{Vol}_{\tilde{g}_0}&=\int_{\partial D_\varepsilon}\sigma_h\partial_n\tilde{\sigma}_0d\ell+\int_{D_\varepsilon}\sigma_hK_{\tilde{g}_0}d\operatorname{Vol}_{\tilde{g}_0},\\ \int_{D_\varepsilon}\nabla_{\tilde{g}_0}\tilde{\sigma}_0(h)\cdot\nabla_{\tilde{g}_0}\tilde{\sigma}_0(h)d\operatorname{Vol}_{\tilde{g}_0}&=\int_{\partial h(D_\varepsilon)}\tilde{\sigma}_0\partial_n\tilde{\sigma}_0d\ell+\int_{h(D_\varepsilon)}\tilde{\sigma}_0K_{\tilde{g}_0}d\operatorname{Vol}_{\tilde{g}_0},\\ \int_{D_\varepsilon}\nabla_{\tilde{g}_0}\tilde{\sigma}_0\cdot\nabla_{\tilde{g}_0}\tilde{\sigma}_0d\operatorname{Vol}_{\tilde{g}_0}&=\int_{\partial D_\varepsilon}\tilde{\sigma}_0\partial_n\tilde{\sigma}_0d\ell+\int_{D_\varepsilon}\tilde{\sigma}_0K_{\tilde{g}_0}d\operatorname{Vol}_{\tilde{g}_0},\\ \int_{D_\varepsilon}\nabla_{\tilde{g}_0}\tilde{\sigma}_0(h)\cdot\nabla_{\tilde{g}_0}\tilde{\sigma}_0d\operatorname{Vol}_{\tilde{g}_0}&=\int_{\partial D_\varepsilon}\tilde{\sigma}_0(h)\partial_n\tilde{\sigma}_0d\ell+\int_{D_\varepsilon}\tilde{\sigma}_0(h)K_{\tilde{g}_0}d\operatorname{Vol}_{\tilde{g}_0},\end{aligned}$$

(here we consider ∂D_ε in terms of prime ends). By combining the above and

$$\int_{\partial D_\varepsilon}\sigma(\psi^{-1})\partial_n\tilde{\sigma}_0d\ell=\int_{\partial D_\varepsilon}\sigma(\psi^{-1})k_{\tilde{g}_0}d\ell_{\tilde{g}_0}-\int_{\partial D_\varepsilon}\sigma kd\ell,$$

we find that

$$\begin{aligned}\int_{D_\varepsilon} |\nabla_{\tilde{g}_0}\sigma(\psi^{-1})|^2 d\operatorname{Vol}_{\tilde{g}_0} = &\int_{D_\varepsilon} |\nabla\sigma_h|^2 dz^2 + 2\int_{\partial D_\varepsilon} \sigma_h k d\ell - 2\int_{D_\varepsilon} \sigma(\psi^{-1}) K_{\tilde{g}_0} d\operatorname{Vol}_{\tilde{g}_0} \\ &- 2\int_{\partial D_\varepsilon} \sigma(\psi^{-1}) k_{\tilde{g}_0} d\ell_{\tilde{g}_0} + \left(\int_{h(D_\varepsilon)} - \int_{D_\varepsilon}\right)\tilde{\sigma}_0 K_{\tilde{g}_0} d\operatorname{Vol}_{\tilde{g}_0} \\ &+ \left(\int_{\partial h(D_\varepsilon)} - \int_{\partial D_\varepsilon}\right)\tilde{\sigma}_0 k_{\tilde{g}_0} d\ell_{\tilde{g}_0} + \left(\int_{\partial h(D_\varepsilon)} - \int_{\partial D_\varepsilon}\right)\tilde{\sigma}_0 k d\ell.\end{aligned} \tag{60}$$

We now study the limit as $\varepsilon \to 0+$. First of all,

$$\begin{aligned}\int_{D_\varepsilon} |\nabla_{\tilde{g}_0}\sigma(\psi^{-1})|^2 d\operatorname{Vol}_{\tilde{g}_0} &\to \int_{\tilde{D}_1\cup\tilde{D}_2} |\nabla_{\tilde{g}_0}\sigma|^2 d\operatorname{Vol}_{\tilde{g}_0}, \\ \int_{D_\varepsilon} |\nabla\sigma_h|^2 dz^2 &\to \int_{\tilde{D}_1\cup\tilde{D}_2} |\nabla\sigma_h|^2 dz^2, \\ \int_{D_\varepsilon} \sigma K_{\tilde{g}_0} d\operatorname{Vol}_{\tilde{g}_0} &\to \int_{\tilde{D}_1\cup\tilde{D}_2} \sigma K_{\tilde{g}_0} d\operatorname{Vol}_{\tilde{g}_0}.\end{aligned}$$

Next, observe that

$$\psi^{-1}(z) = b + Cz^{-1/2} + o(z^{-1/2}), \quad (\psi^{-1})'(z) = -\frac{C}{2}z^{-3/2} + o(z^{-3/2}), \quad \text{ as } z \in \overline{\Sigma} \to \infty \tag{61}$$

for some $C \neq 0$. Similarly, [48, Theorem 3] implies that

$$\varphi \circ \psi^{-1}(z) = Dz + o(z), \quad (\varphi\circ\psi^{-1})'(z) = D + o(1), \quad \text{ as } z \in \overline{\Sigma} \to 0,$$

for some $D \neq 0$. Moreover ψ^{-1} is smooth at $h^{-1}(0+)$, and $h^{-1}(0-)$. Hence,

$$\lim_{\varepsilon\to 0+}\left(\int_{h(D_\varepsilon)} - \int_{D_\varepsilon}\right)\tilde{\sigma}_0 K_{\tilde{g}_0} d\operatorname{Vol}_{\tilde{g}_0} = 0.$$

For each $x \in \{0+, 0-, h^{-1}(0+), h^{-1}(0-), \infty\}$, let $C_{x,\varepsilon} = \partial B_{x,\varepsilon} \cap (\tilde{D}_1 \cup \tilde{D}_2)$ and $L_{x,\varepsilon} = \partial B_{x,\varepsilon} \setminus C_{x,\varepsilon}$. Then

$$\lim_{\varepsilon\to 0+}\left(\int_{\partial D_\varepsilon} - \int_{\partial\tilde{D}_1\cup\partial\tilde{D}_2}\right)\sigma(\psi^{-1}) k_{\tilde{g}_0} d\ell_{\tilde{g}_0} = \lim_{\varepsilon\to 0+}\int_{\cup C_{x,\varepsilon}} \sigma(\psi^{-1}) k_{\tilde{g}_0} d\ell_{\tilde{g}_0},$$

and the same holds when replacing $\sigma(\psi^{-1})k_{\tilde{g}_0} d\ell_{\tilde{g}_0}$ with $\sigma_h k d\ell$. Similarly,

$$\lim_{\varepsilon\to 0+}\left(\int_{\partial h(D_\varepsilon)} - \int_{\partial D_\varepsilon}\right)\tilde{\sigma}_0 k_{\tilde{g}_0} d\ell_{\tilde{g}_0} = \lim_{\varepsilon\to 0+}\left(\int_{\cup h(C_{x,\varepsilon})} - \int_{\cup C_{x,\varepsilon}}\right)\tilde{\sigma}_0 k_{\tilde{g}_0} d\ell_{\tilde{g}_0},$$

which also holds when replacing $k_{\tilde{g}_0} d\ell_{\tilde{g}_0}$ with $kd\ell$. To compute the integrals over $C_{x,\varepsilon}$ we note that $\ell_{\tilde{g}_0}(C_{x,\varepsilon}) \sim \pi\varepsilon$ (since in the limit $C_{x,\varepsilon}$ is a semicircle of $\tilde{g}_0$-radius ε) and $k_{\tilde{g}_0} \sim -\frac{1}{\varepsilon}$ (since the semicircle is traversed clockwise). We therefore obtain

$$\begin{aligned}\frac{1}{\pi}\int_{\cup_x C_{x,\varepsilon}} \sigma(\psi^{-1}) k_{g_0} d\ell_{g_0} \to &-\sigma(\psi^{-1}(\infty)) - \sigma(\psi^{-1}(0+)) - \sigma(\psi^{-1}(0-)) \\ &- \sigma(\psi^{-1}(h^{-1}(0+))) - \sigma(\psi^{-1}(h^{-1}(0-))),\end{aligned}$$

as $\varepsilon \to 0+$. In a similar manner

$$\frac{1}{\pi}\int_{\cup_x C_{x,\varepsilon}} \sigma_h k d\ell \to -2\sigma_h(\infty) - \sigma_h(0+) - \sigma_h(0-) - \sigma_h(h^{-1}(0+)) - \sigma_h(h^{-1}(0-)),$$

$$\frac{1}{\pi}\Big(\int_{\cup h(C_{x,\varepsilon})} - \int_{\cup C_{x,\varepsilon}}\Big)\tilde{\sigma}_0 k_{\tilde{g}_0} d\ell_{\tilde{g}_0} \to -\tfrac{3}{2}\sigma_h(\infty) - \tilde{\sigma}_0(h(0+)) - \tilde{\sigma}_0(h(0-)) + \tilde{\sigma}_0(h^{-1}(0+)) + \tilde{\sigma}_0(h^{-1}(0-)),$$

$$\frac{1}{\pi}\Big(\int_{\cup h(C_{x,\varepsilon})} - \int_{\cup C_{x,\varepsilon}}\Big)\tilde{\sigma}_0 k d\ell \to 3\sigma_h(\infty) - \tilde{\sigma}_0(h(0+)) - \tilde{\sigma}_0(h(0-)) + \tilde{\sigma}_0(h^{-1}(0+)) + \tilde{\sigma}_0(h^{-1}(0-)),$$

where we use the convention $\sigma_h(\infty) = \log|(1/h(1/z))'|_{z=0}$. In the bottom two limits we have used that $h(C_{h^{-1}(0\pm),\varepsilon}) = C_{0\pm,\varepsilon}$ and that $\tilde{\sigma}_0$ is smooth at $h^{-1}(0\pm)$ and $h(0\pm)$. Combining the computed limits with (59) and (60) yields

$$12\Big(\mathcal{H}_{(\mathbb{D};g_0)}(\gamma_2) - \mathcal{H}_{(\mathbb{D};g_0)}(\gamma_1)\Big) = \frac{1}{\pi}\int_{\tilde{D}_1\cup\tilde{D}_2} |\nabla\sigma_h|^2 dz^2 + \frac{2}{\pi}\int_{\partial\tilde{D}_1\cup\partial\tilde{D}_2} \sigma_h k d\ell + \frac{(2\alpha-1)^2}{2\alpha(1-\alpha)}\sigma_h(\infty) \tag{62}$$

where we have used that $\sigma(\psi^{-1}(\infty)) = \sigma_h(\infty)/2$ which is found by using (61). Since $\tilde{\sigma}_0$ is smooth at x_0 we find that $\sigma(c) = \sigma_h(x_0)$. Then, (58) and (62) imply

$$12\Big(\mathcal{H}^{(\mathbb{D};a,b)}_{\rho,c}(\gamma_2;g_0) - \mathcal{H}^{(\mathbb{D};a,b)}_{\rho,c}(\gamma_1;g_0)\Big) = \frac{1}{\pi}\int_{\tilde{D}_1\cup\tilde{D}_2} |\nabla\sigma_h|^2 dz^2 + \frac{2}{\pi}\int_{\partial\tilde{D}_1\cup\partial\tilde{D}_2} \sigma_h k d\ell + \frac{\rho(\rho+4)}{4}\sigma_h(x_0) + \frac{\rho(8+\rho)}{8}\sigma_h(\infty).$$

Now, for the final step, let $h_1 : \Sigma \setminus \tilde{\gamma}_1([0,T_1]) \to \Sigma$ with $h_1(\tilde{\gamma}_1(T_1)) = 0$, $h_1(\infty) = \infty$ and $h_1'(\infty) = 1$. Let $h_2 := h_1 \circ h^{-1} : \Sigma \setminus \tilde{\gamma}([0,T_2]) \to \Sigma$. We may assume that $h_2'(\infty) = 1$ since this can be achieved by increasing T_1 or T_2 by an appropriate amount. Then it follows, from the Dirichlet energy formula (7) and (19), that

$$I^{(\mathbb{D};a,b)}_{\rho,c}(\gamma_2) - I^{(\mathbb{D};a,b)}_{\rho,c}(\gamma_1) = I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\tilde{\gamma}_2) - I^{(\Sigma;0,\infty)}_{\rho,x_0^+}(\tilde{\gamma}_1) = \frac{1}{\pi}\int_\Sigma |\nabla\sigma_{h_2^{-1}}|^2 dz^2 - \frac{1}{\pi}\int_\Sigma |\nabla\sigma_{h_1^{-1}}|^2 dz^2 + \frac{\rho(\rho+4)}{4}\log|h'(x_0)|.$$

Since, $\sigma_h(\infty) = 0$ by assumption, it only remains to show that

$$\int_{\tilde{D}_1\cup\tilde{D}_2} |\nabla\sigma_h|^2 dz^2 + \int_{\partial\tilde{D}_1\cup\partial\tilde{D}_2} \sigma_h k d\ell = \int_\Sigma |\nabla\sigma_{h_2^{-1}}|^2 dz^2 - \int_\Sigma |\nabla\sigma_{h_1^{-1}}|^2 dz^2 \tag{63}$$

Write $\sigma_h = \sigma_{h_1} + \sigma_{h_2^{-1}}(h_1)$. We have,

$$\int_{\tilde{D}_1\cup\tilde{D}_2} |\nabla\sigma_h|^2 dz^2 = \int_{\tilde{D}_1\cup\tilde{D}_2} \Big(|\nabla\sigma_{h_2^{-1}}(h_1)|^2 - |\nabla\sigma_{h_1}|^2 + 2\nabla\sigma_{h_1}\cdot\nabla\sigma_h\Big) dz^2.$$

By conformal invariance of the Dirichlet inner product, the first two terms on the right-hand side equal the right-hand side of (63). We apply Stokes' theorem to the third term of the right-hand side and use the change of variable formula for the geodesic curvature (17), yielding

$$2\int_{\tilde{D}_1\cup\tilde{D}_2} \nabla\sigma_{h_1}\cdot\nabla\sigma_h dz^2 = 2\int_{\partial\tilde{D}_1\cup\partial\tilde{D}_2} \sigma_h \partial_n \sigma_{h_1} d\ell = 2\int_{\partial\Sigma} \sigma_h(h_1^{-1}) k d\ell - 2\int_{\partial\tilde{D}_1\cup\partial\tilde{D}_2} \sigma_h k d\ell.$$

Since $k = 0$ along $\partial\Sigma$, this shows (63) and finishes the proof. □

Proof of Proposition 1, radial case. The proof of the radial case of Proposition 1 is very similar. We therefore give an outline of the proof and refer the reader to the chordal proof for details. Fix $\rho > -2$, a, $b \in \mathbb{D}$, c, Γ, and φ as in Proposition 1. Fix $\gamma_1, \gamma_2 \in \hat{\mathcal{X}}$ and a $(\overline{\mathbb{D}}, \varphi)$-smooth conformal metric $g_0 = e^{2\sigma_0} dz^2$. Since γ_j and $\eta_j = \eta(\gamma_j)$, $j = 1, 2$, are $(\overline{\mathbb{D}}, \varphi)$-smooth the ρ-Loewner potential of γ_j with respect to g_0 is defined.

As in the chordal case we fix a conformal map $\psi : \mathbb{D} \to \Sigma$, $\psi(a) = 0$, $\psi(c) = \infty$ and write $\tilde{\gamma}_j = \psi(\gamma_j)$ and $\tilde{\eta}_j = \psi(\eta_j)$, $j = 1, 2$. Denote by $z_0 = \psi(b)$. We let $h : \Sigma \setminus \tilde{\gamma}_1 \to \Sigma \setminus \tilde{\gamma}_2$ be the conformal map with $h(z_0) = z_0$, $h(\infty) = \infty$, and $h'(\infty) = 1$. Then $\tilde{\eta}_2 = h(\tilde{\eta}_1)$ so that (57) holds if D is $\mathbb{D} \setminus \gamma_1$ or a component of $\mathbb{D} \setminus (\gamma_1 \cup \eta_1)$. We introduce a new metric $g = H^* g_0 = e^{2\sigma} g_0$ with $H = \psi^{-1} \circ h \circ \psi$ and $\sigma = \sigma_0(H) + \sigma_H - \sigma_0$. By the same type of argument as in the chordal case, we find that σ is $(\overline{\mathbb{D}}, \varphi)$-smooth. We have

$$\mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_j; g_0) = \mathcal{H}_{(\mathbb{D};g_0)}(\gamma_j) + \frac{\rho(\rho+4)}{12} \mathcal{H}_{(\mathbb{D} \setminus \gamma_j, g_0)}(\eta_j)$$

for $j = 1, 2$. Theorem B implies

$$12\Big(\mathcal{H}_{(\mathbb{D} \setminus \gamma_2, g_0)}(\eta_2) - \mathcal{H}_{(\mathbb{D} \setminus \gamma_1, g_0)}(\eta_1)\Big) = -\Big(\frac{1}{2} - 2\Big)\sigma(b) + 2\Big(2 - \frac{1}{2}\Big)\sigma(c),$$

and

$$\begin{aligned}12\Big(\mathcal{H}_{(\mathbb{D} \setminus \gamma_2, g_0)}(\eta_2) - \mathcal{H}_{(\mathbb{D} \setminus \gamma_1, g_0)}(\eta_1)\Big) =& \frac{1}{\pi} \int_{\mathbb{D} \setminus \gamma_1} |\nabla_{g_0} \sigma|^2 d\,\mathrm{Vol}_{g_0} + \frac{2}{\pi} \int_{\mathbb{D} \setminus \gamma_1} \sigma K_{g_0} d\,\mathrm{Vol}_{g_0} \\ &+ \frac{2}{\pi} \int_{\partial(\mathbb{D} \setminus \gamma_1)} \sigma k_{g_0} d\ell_{g_0} + \frac{3}{\pi} \int_{\partial(\mathbb{D} \setminus \gamma_1)} \partial_{n_{g_0}} \sigma d\ell_{g_0} - \frac{3}{2}\sigma(b).\end{aligned}$$

By carrying out the same type of computation as in the chordal case (the computation is almost identical) one finds

$$12\Big(\mathcal{H}_{(\mathbb{D} \setminus \gamma_2, g_0)}(\eta_2) - \mathcal{H}_{(\mathbb{D} \setminus \gamma_1, g_0)}(\eta_1)\Big) = \frac{1}{\pi} \int_{\Sigma \setminus \tilde{\gamma}_1} |\nabla \sigma_h|^2 dz^2 + \frac{2}{\pi} \int_{\partial \Sigma \setminus \tilde{\gamma}_1} \sigma_h k d\ell - \frac{3}{2}\sigma_h(\infty) - \frac{3}{2}\sigma_h(z_0).$$

A computation shows that $\sigma_h(\infty) = \sigma(c)/2$, but by the normalization of h, $\sigma_h(\infty) = 0$. We also have $\sigma_h(z_0) = \sigma(b)$. Combining the above yields,

$$12\Big(\mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2; g_0) - \mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2; g_0)\Big) = \frac{1}{\pi} \int_{\Sigma \setminus \tilde{\gamma}_1} |\nabla \sigma_h|^2 dz^2 + \frac{2}{\pi} \int_{\partial \Sigma \setminus \tilde{\gamma}_1} \sigma_h k d\ell + \frac{(\rho+6)(\rho-2)}{8} \sigma_h(z_0).$$

Finally, by letting $h_1 : \Sigma \setminus \tilde{\gamma}_1 \to \Sigma$ be the conformal map with $h_1(z_0) = 0$, $h_1(\infty) = \infty$, and $h_1'(\infty) = 1$ and setting $h_2 = h_1 \circ h^{-1}$ we find

$$\begin{aligned}&12\Big(\mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2; g_0) - \mathcal{H}_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2; g_0)\Big) \\ &= \frac{1}{\pi} \int_{\Sigma \setminus \tilde{\gamma}_2} |\nabla \sigma_{h_2}|^2 dz^2 - \frac{1}{\pi} \int_{\Sigma \setminus \tilde{\gamma}_1} |\nabla \sigma_{h_1}|^2 dz^2 - \frac{(\rho+6)(\rho-2)}{8} \log \frac{|h_2'(z_0)|}{|h_1'(z_0)|} \\ &= I_{\rho,\infty}^{(\Sigma;0,z_0)}(\tilde{\gamma}_2) - I_{\rho,\infty}^{(\Sigma;0,z_0)}(\tilde{\gamma}_1) = I_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_2) - I_{\rho,c}^{(\mathbb{D};a,b)}(\gamma_1),\end{aligned}$$

where Stokes' theorem is used in the first equality and Theorem 2 is used in the second. □

Remark 21. If $b \in \partial\mathbb{D}$ we can guarantee that the $I_{\rho,c}^{(\mathbb{D};a,b)}$-minimizer, denoted by γ^0, is in $\hat{\mathcal{X}}$ by choosing φ in the following way. We may assume c lies on the counter-clockwise circular arc from a to b (the other case is covered by symmetry). Consider $\Sigma_\alpha = \Sigma \setminus \{re^{2\alpha\pi i} : r \in [0, 1]\}$ and let $\tilde{\varphi} : \mathbb{D} \to \Sigma_\alpha$ be the conformal map with $\tilde{\varphi}(a) = e^{2\alpha\pi i}$, $\tilde{\varphi}(b) = \infty$, and $\tilde{\varphi}(c) = 0+$. Then

$$\tilde{\varphi}(\gamma^0) = \{re^{2\alpha\pi i} : r \in [1, \infty)\}.$$

So if we choose $\varphi(z) = 1 - \tilde{\varphi}(z)e^{-2\alpha\pi i}$, then $\gamma^0 \in \hat{\mathcal{X}}(\varphi)$.

If instead $b = 0 \in \mathbb{D}$ and a and c are antipodal then the $I_{\rho,c}^{(\mathbb{D};a,b)}$-minimizer, γ^0, is the line segment from a to b. If we then choose $\varphi : \mathbb{D} \to \Sigma$ with $\varphi(a) = 0$, $\varphi(c) = \infty$, and $\varphi(b) = -1$ then $\varphi(\gamma^0) = [-1, 0]$ and $\varphi(\eta(\gamma^0)) = (-\infty, -1]$. Hence, $\gamma^0 \in \hat{\mathcal{X}}(\varphi)$.

A Return estimates

In this section, we will study the energy return estimates and return probability estimates which are used to show the large deviation principle on the infinite time $\mathrm{SLE}_\kappa(\rho)$ curves. This is done by combining some ideas from [12] and [33]. In particular, the following proposition and lemmas will be useful.

Proposition A ([33, Proposition A.3])**.** *Let $\kappa \in (0,4]$. There exists constants, $c_\kappa > 0$ such that $\lim_{\kappa\to 0+} \kappa \log c_\kappa = C \in (-\infty, \infty)$, and the following holds. Let $K \subset \overline{\mathbb{H}}$ be a hull such that*

$$K \cap (\mathbb{H} \setminus \mathbb{D}) = \{z\},$$

and let γ^κ be an SLE_κ from z to ∞ in $\mathbb{H} \setminus K$, then, for any $r \in (0, 1/3)$ we have

$$\mathbb{P}[\gamma^\kappa \cap S_r \neq \varnothing] \leq c_\kappa r^{8/\kappa - 1}.$$

For domains D with smooth boundary and two disjoint subsets $A_1, A_2 \subset \partial D$ the Brownian excursion measure is defined by

$$\mathcal{E}_D(A_1, A_2) = \int_{A_1} \omega_x(A_2, D)|dx|.$$

It can be shown that $\mathcal{E}_D(A_1, A_2)$ is conformally invariant, and therefore the definition may be extended to domains D with non-smooth boundary. We extend the definition in two ways (as in [12]). Firstly, if D is not connected, and $A_1 = \cup_i \eta_{1,i}$ and $A_2 = \cup_i \eta_{2,i}$ are disjoint unions of boundary arcs, then

$$\mathcal{E}_D(A_1, A_2) = \sum_i \sum_j \int_{\eta_{1,i}} \omega_x(\eta_{2,j}, D_{i,j})|dx|,$$

where $D_{i,j}$ is the unique connected component of D where both $\eta_{1,i}$ and $\eta_{2,j}$ are accessible (here we must treat ∂D in terms of prime ends). Finally, if $A_1, A_2 \subset \mathbb{C}$ are not contained in ∂D, but are contained in $\partial(D \setminus (A_1 \cup A_2))$ then we set

$$\mathcal{E}_D(A_1, A_2) = \mathcal{E}_{D\setminus(A_1\cup A_2)}(A_1, A_2).$$

Lemma A ([33, Lemma A.2])**.** *Let $\kappa \in (0,4]$. There exists constants $c'_\kappa \in (0,\infty)$, such that $\lim_{\kappa\to 0+} \kappa \log c'_\kappa = C' \in (-\infty,\infty)$, and the following holds. Let D be a simply connected domain and $x, y \in \partial D$ two distinct boundary points. Let γ' be a chord from x to y in D, and let η be a chord (with arbitrary endpoints in D) disjoint from γ'. Finally, let γ^κ be a chordal SLE_κ in $(D; x, y)$. Then, we have*

$$\mathbb{P}[\gamma^\kappa \cap \eta \neq \varnothing] \leq c_\kappa \mathcal{E}_D(\eta, \gamma')^{8/\kappa - 1}.$$

Lemma B ([12, Lemma 3.3])**.** *Let D be a simply connected domain and $\mathcal{S}$ the set of crosscuts of D that are subsets of the circle $\partial(r\mathbb{D})$. Then*

$$\sum_{\eta\in\mathcal{S}} \mathcal{E}_D(\partial(R\mathbb{D}), \eta) \leq 2\mathcal{E}_D(\partial(R\mathbb{D}), \partial(r\mathbb{D})).$$

A.1 Chordal case

Recall that $\mathcal{X}^C$ denotes the family of all simple curves in $\mathbb{H}$ starting at 0 and ending at ∞. Let $D_R = R\mathbb{D}$ and $S_R = \partial D_R \cap \mathbb{H}$. For a simple curve γ starting at 0, let

$$\tau_R = \inf\{t : |\gamma(t)| = R\}.$$

Note that this differs from the definition of τ_R in Section 4.3. The goal of this section is to prove the following proposition.

Proposition 17. *For all $r > 0$ and every $M \in [0, \infty)$ there is an $R > r$ s.t.*

(a) $\inf\{I^C_{\rho,v_0}(\gamma) | \gamma \in \mathcal{X}^C, \gamma_{[\tau_R,\infty)} \cap S_r \neq \varnothing\} \geq M$

(b) $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma^{\kappa,\rho}_{[\tau_R,\infty)} \cap S_r \neq \varnothing] \leq -M$

(b') $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\kappa+\rho}[\gamma^{\kappa,\kappa+\rho}_{[\tau_R,\infty)} \cap S_r \neq \varnothing] \leq -M$

We prove (a) separately from (b) and (b'). We will assume, without loss of generality, that $x_0 > 0$.

Proof of Proposition 17 (a). Let $y_0 = -\frac{2x_0}{\rho+2}$ and fix $r > 0$. Fix $R > \min(x_0, r)$ and denote

$$E = \{\gamma \in \mathcal{X}^C : \gamma_{[\tau_R,\infty)} \cap S_r \neq \varnothing\}.$$

For any $\gamma \in E$, let

$$\tau_{r,R} = \inf\{t > \tau_R : |\gamma(t)| = r\} \quad \text{and} \quad R' = \sup_{t\in[\tau_R,\tau_{r,R}]} |\gamma(t)| \geq R.$$

Let $\theta \in (0, \pi)$ be such that $\gamma(\tau_{R'}) = R'e^{i\theta}$. Using

$$|x_{\tau_{R'}} - W_{\tau_{R'}}| = \pi\omega_\infty([W_{\tau_{R'}}, x_{\tau_{R'}}], \mathbb{H}) = \pi\omega_\infty(\gamma^+_{\tau_{R'}} \cup [0, x_0], \mathbb{H} \setminus \gamma_{\tau_{R'}}),$$

and monotonicity of harmonic measure we find

$$|x_{\tau_{R'}} - W_{\tau_{R'}}| \leq \pi\omega_\infty(a^+_\theta, \mathbb{H} \setminus a_\theta) = 2\sin(\theta/2)(1 + \sin(\theta/2))R', \tag{64}$$

where $a_\theta = \{R'e^{i\phi} : \phi \in [0, \theta]\}$. Similarly,

$$|x_{\tau_{R'}} - y_{\tau_{R'}}| \geq |W_{\tau_{R'}} - g_{\tau_{R'}}(0-)| = \pi\omega_\infty(\gamma^-_{\tau_{R'}}, \mathbb{H} \setminus \gamma_{\tau_{R'}}) \geq \omega_\infty(a^-_\theta \cup [0, R'], \mathbb{H} \setminus a_\theta) = \cos^2(\theta/2).$$

We now consider two cases. Let $\varepsilon \in (0, \pi/2)$ and suppose that $\theta \leq \varepsilon$. Then

$$\frac{x_{\tau_{R'}} - W_{\tau_{R'}}}{x_{\tau_{R'}} - y_{\tau_{R'}}} \leq \frac{2\sin(\varepsilon/2)(1 + \sin(\varepsilon/2))}{\cos^2(\varepsilon/2)}.$$

Since the right hand side approaches 0 as $\varepsilon \to 0+$ there is an $\varepsilon_0 \in (0, \pi/2)$ so that Lemma 11 gives

$$\theta \leq \varepsilon_0 \implies I^C_{\rho,x_0}(\gamma) \geq M.$$

Fix such an ε_0. It remains to show that, if R is sufficiently large, any $\gamma \in E$ with $\theta > \varepsilon_0$ has $I^C_{\rho,x_0}(\gamma) \geq M$. Let $\hat\gamma = g_{\tau_{R'}}(\gamma_{[\tau_{R'},\tau_{r,R}]}) - W_{\tau_{R'}}$. If R/r is sufficiently large (e.g., $R/r \geq \sqrt{2}$) a harmonic measure estimate shows that, for any $z \in S_r \setminus \gamma_{\tau_{R'}}$,

$$\sin(\pi\omega(z, \gamma^+_{\tau_{R'}} \cup R^+, \mathbb{H} \setminus \gamma_{\tau_{R'}})) \leq 8\frac{r}{R}.$$

Hence, $I^C(\hat\gamma) \geq -8\log 8r/R$. Moreover, $|x_{\tau_{R'}} - W_{\tau_{R'}}| \geq \sin^2(\varepsilon_0/2)R$, and $|x_{\tau_{r,R}} - W_{\tau_{r,R}}| \leq 4R$, by a arguments similar to those above. So, by Proposition 15, we have

$$I^C_{\rho,x_0}(\gamma) \geq I^C_{\rho,x_{\tau_{R'}}-W_{\tau_{R'}}}(\hat\gamma) \geq \min(\tfrac{\rho+2}{2}, 1)\left(\min(\tfrac{\rho+2}{2}, 1)8\log\frac{R}{8r} - |\rho|\log\frac{4}{\sin^2(\varepsilon_0/2)}\right).$$

Since the second term does not depend on R we see that we can choose R sufficiently large so that the right hand side is larger than M. This finishes the proof. ☐

We now move toward the proof of parts (b) and (b'). The main idea is (loosely) to partition the event $E = \{\gamma \in \mathcal{X}^C : \gamma_{[\tau_R,\infty)} \cap S_r \neq \varnothing\}$ into two sub-events, one where we have uniform control on the Radon-Nikodym derivative of $\text{SLE}_\kappa(\rho)$ with respect to SLE_κ, and another where we do not. The probability of the first sub-event will then be controlled using Lemma A, and the probability of the other sub-event will be controlled using Lemma 16 and 17.

For a simple curve γ and a force point $x_0 > 0$ we let $\hat{\tau}_R := \inf\{t : |x_t - O_t^-| \geq R\}$, where $O_t^- = g_t(0-)$. Note that $R \mapsto \tau_R$ continuous and strictly increasing on $[x_0, \infty)$ since

$$d(x_t - O_t^-) = \frac{2}{x_t - W_t} dt + \frac{2}{W_t - O_t^-} dt > 0$$

by the Loewner equation.

Lemma 16. *Fix $x_0 > 0$ and $\rho > -2$. For any $R > |x_0|$ and any $\varepsilon \in (0,1)$ we have that*

$$\mathbb{P}^{\kappa,\rho}[|x_{\hat{\tau}_R} - W_{\hat{\tau}_R}| < \varepsilon R] \leq C_1(\kappa,\rho)\varepsilon^{2\frac{2+\rho}{\kappa}},$$

where $C_1(\kappa,\rho)$ is a constant such that

$$\lim_{\kappa\to 0+} \kappa \log C_1(\kappa,\rho) = \lim_{\kappa\to 0+} \kappa \log C_1(\kappa,\kappa+\rho) = C_2(\rho) \in (-\infty,\infty).$$

Proof. Let $x_0 > 0$ and $O_t^- = g_t(0-)$. The process

$$Y_s = \frac{x_{t(s)} - W_{t(s)}}{x_{t(s)} - O_{t(s)}^-} \in [0,1], \quad s(t) = \frac{\kappa}{2} \log \frac{x_t - O_t^-}{x_0},$$

satisfies the SDE

$$dY_s = -\frac{2}{\kappa} Y_s ds + \frac{2+\rho}{\kappa}(1-Y_s)ds + \sqrt{Y_s(1-Y_s)}dB_s, \tag{65}$$

with $Y_0 = 1$, where B_s is a Brownian motion (this follows by a simple computation, but is also a direct consequence [50, Lemma 3.3]). For sufficiently small κ, the SDE (65) has an invariant distribution with density

$$\Psi(y) = \frac{\Gamma(2\frac{4+\rho}{\kappa})}{\Gamma(2\frac{2+\rho}{\kappa})\Gamma(\frac{4}{\kappa})} y^{2\frac{2+\rho}{\kappa}-1}(1-y)^{\frac{4}{\kappa}-1}$$

(see, e.g., [50, Proposition 2.20]). Let $\hat{Y}_s$ be a process, satisfying (65) started at the invariant density. Since $Y_0 = 1$ we have, for all $s > 0$ and $\varepsilon \in (0,1)$, that

$$\mathbb{P}^{\kappa,\rho}[Y_s < \varepsilon] \leq \mathbb{P}[\hat{Y}_s < \varepsilon] \leq \frac{\Gamma(2\frac{4+\rho}{\kappa})}{\Gamma(2\frac{2+\rho}{\kappa})\Gamma(\frac{4}{\kappa})} \frac{\kappa}{2(2+\rho)} \varepsilon^{2\frac{2+\rho}{\kappa}}.$$

By Stirling's formula

$$\lim_{\kappa\to 0+} \kappa \log \frac{\Gamma(2\frac{4+\rho}{\kappa})}{\Gamma(2\frac{2+\rho}{\kappa})\Gamma(\frac{4}{\kappa})} = 2(2+\rho)\log\frac{4+\rho}{2+\rho} + 4\log\frac{4+\rho}{2}.$$

Replacing ρ with $\kappa + \rho$ gives the same limit. Since $s(\tau_R) = \frac{\kappa}{2}\log\frac{x_t - O_t^-}{x_0}$ is a deterministic time, this finishes the proof. □

Lemma 17. *Fix $x_0 > 0$, $\rho > -2$, and an integer $n_0 \geq 3^4$. Let $\varepsilon_n = (n+n_0)^{-1/2}$. For every integer $n \geq 0$ we define the event*

$$E_n = \{\exists t \in [\tau_{2^n R}, \tau_{2^{n+1} R}] \text{ s.t. } |x_t - W_t| \leq \varepsilon_n 2^n R\}.$$

Then there exists a constant $C_3 = C_3(\rho)$ such that,

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\cup_{n=0}^{\infty} E_n] \leq C_3 - \frac{\rho+2}{2}\log n_0, \tag{66}$$

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\kappa+\rho}[\cup_{n=0}^{\infty} E_n] \leq C_3 - \frac{\rho+2}{2}\log n_0. \tag{67}$$

Proof. We begin by noting that for any $r > 0$ we have $\hat{\tau}_{r/2} \leq \tau_r$ (this follows using (64)). Hence, $E_n \subset \hat{E}_n$, where

$$\hat{E}_n = \{\exists t \in [\hat{\tau}_{2^{n-1}R}, \tau_{2^{n+1}R}] \text{ s.t. } |x_t - W_t| \leq \varepsilon_n 2^n R\}.$$

Let

$$\hat{E}_n^1 = \{|x_{\hat{\tau}_{2^{n-1}R}} - W_{\hat{\tau}_{2^{n-1}R}}| \leq \sqrt{\varepsilon_n} 2^n R\}, \quad \hat{E}_n^2 = \hat{E}_n \setminus \hat{E}_n^1,$$

so that

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\cup_{n=0}^\infty E_n] \leq \max \left\{ \limsup_{\kappa \to 0+} \kappa \log \sum_{n=0}^\infty \mathbb{P}^{\kappa,\rho}[\hat{E}_n^1], \limsup_{\kappa \to 0+} \kappa \log \sum_{n=0}^\infty \mathbb{P}^{\kappa,\rho}[\hat{E}_n^2] \right\}.$$

We estimate the two limits separately. Using Lemma 16 we obtain

$$\limsup_{\kappa \to 0+} \kappa \log \sum_{n=0}^\infty \mathbb{P}^{\kappa,\rho}[\hat{E}_n^1] \leq \limsup_{\kappa \to 0+} \kappa \log C_1(\kappa,\rho) 2^{2\frac{2+\rho}{\kappa}} \sum_{n=0}^\infty \varepsilon_n^{\frac{2+\rho}{\kappa}} \leq C_4(\rho) - \frac{2+\rho}{2} \log n_0,$$

where $C_4(\rho)$ is a constant. We now study $\hat{E}_n^2$. For $y > 0$ we let Z_t^y be the solution of

$$dZ_t^y = \frac{\rho+2}{Z_t^y} + \sqrt{\kappa} dB_t, \; Z_0^y = y,$$

i.e., a stochastic process satisfying the same SDE as $x_t - W_t$, started at y. By the strong Markov property of Itô diffusions, the bound $\tau_{2^{n+1}R} \leq (2^{n+1}R)^2/2$, and

$$\mathbb{P}[\exists t \in [0, (2^{n+1}R)^2/2] : Z_t^X < \varepsilon_n 2^n R] \leq \mathbb{P}[\exists t \in [0, (2^{n+1}R)^2/2] : Z_t^y < \varepsilon_n 2^n R]$$

for all $y > \varepsilon_n 2^n R$ and random variables $X \in [y, \infty)$ (since Z_t^y and Z_y^X can be coupled to coincide after their collision), we have

$$\mathbb{P}^{\kappa,\rho}[\hat{E}_n^2] \leq \mathbb{P}[\exists t \in [0, (2^{n+1}R)^2/2] : Z_t^{\sqrt{\varepsilon_n} 2^n R} < \varepsilon_n 2^n R].$$

By using Lemma 3 (and the explicit form of the constant, see (31)) on the right-hand side we obtain

$$\limsup_{\kappa \to 0+} \kappa \log \sum_{n=0}^\infty \mathbb{P}^{\kappa,\rho}[\hat{E}_n^2] \leq \limsup_{\kappa \to 0+} \kappa \log C_5(\kappa,\rho) \sum_{n=n_0}^\infty n^{-\frac{\rho+2}{2\kappa} + \frac{3}{4}} = C_6(\rho) - \frac{\rho+2}{2} \log n_0,$$

whenever $n_0^{1/4} \geq 3$ (this comes from the condition $\varepsilon < \varepsilon_0$ of Lemma 3), and where $\limsup_{\kappa \to 0+} \kappa \log C_5 =: C_6 \in \mathbb{R}$. Thus,

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\cup_{n=0}^\infty E_n] \leq \max\{C_4(\rho), C_6(\rho)\} - \frac{\rho+2}{2} \log(n_0),$$

which shows (66). Following the same steps we see that the same computation with ρ replaced with $\kappa + \rho$ holds, which shows (67). □

Define $M_t^{\kappa,\rho} = |g_t'(x_0)|^{(4-\kappa+\rho)\rho/(4\kappa)} |x_t - W_t|^{\rho/\kappa}$.

Lemma 18. *On the event E_n^c (with E_n as in Lemma 17) we have $M_{\tau_{2^{n+1}R}}^{\kappa,\rho} / M_{\tau_{2^n R}}^{\kappa,\rho} \leq f_n(\kappa,\rho)$ where*

$$f_n(\kappa,\rho) = \begin{cases} \left(\frac{8}{\varepsilon_n}\right)^{\frac{\rho}{\kappa}}, & \rho \in [0,\infty), \\ e^{-(n+n_0)\frac{\rho(4-\kappa+\rho)}{\kappa}} \left(\frac{4}{\varepsilon_n}\right)^{-\frac{\rho}{\kappa}}, & \rho \in (-2,0), \end{cases}$$

for all $\kappa < 2$.

Proof. We start by studying the case $\rho \in [0, \infty)$. In this case the exponents of $M_t^{\kappa,\rho}$, are non-negative (at least when $\kappa < 4$). Firstly, recall that $|g_t'(x_0)|$ is decreasing in t. Moreover, since $x_0 < R$, we have $|x_{\tau_{2^{n+1}R}} - W_{\tau_{2^{n+1}R}}| \le 4 \cdot 2^{n+1}R$. Finally, on the event E_n^c we have that $|x_{\tau_{2^nR}} - W_{\tau_{2^nR}}| \ge \varepsilon_n 2^n R$. Therefore,

$$\frac{M^{\kappa,\rho}_{\tau_{2^{n+1}R}}}{M^{\kappa,\rho}_{\tau_{2^nR}}} \le \left(\frac{8}{\varepsilon_n}\right)^{\frac{\rho}{\kappa}}.$$

If $\rho \in (-2, 0)$, the exponents of $M_t^{\kappa,\rho}$ are negative (at least when $\kappa < 2$). Since

$$\log \frac{|g'_{\tau_{2^{n+1}R}}(x_0)|}{|g'_{\tau_{2^nR}}(x_0)|} = -2\int_{\tau_{2^nR}}^{\tau_{2^{n+1}R}} \frac{1}{(x_s - W_s)^2} ds,$$

we have on the event E_n^c that

$$\log \frac{|g'_{\tau_{2^{n+1}R}}(x_0)|}{|g'_{\tau_{2^nR}}(x_0)|} \ge -2\frac{\tau_{2^{n+1}R} - \tau_{2^nR}}{(\varepsilon_n 2^n R)^2} \ge -2\frac{(2^{n+1}R)^2/2}{(\varepsilon_n 2^n R)^2} = -4(n + n_0).$$

Moreover, $|x_{\tau_{2^nR}} - W_{\tau_{2^nR}}| \le 4 \cdot 2^n R$, and $|x_{\tau_{2^{n+1}R}} - W_{\tau_{2^{n+1}R}}| \ge \varepsilon_n 2^n R$ on E_n^c. This yields,

$$\frac{M^{\kappa,\rho}_{\tau_{2^{n+1}R}}}{M^{\kappa,\rho}_{\tau_{2^nR}}} \le e^{-(n+n_0)\frac{\rho(4-\kappa+\rho)}{\kappa}} \left(\frac{4}{\varepsilon_n}\right)^{-\frac{\rho}{\kappa}}$$

□

Proof of Proposition 17 (b) and (b'). For any $R > x_0$, integer $n_0 \ge 3^4$, and E_n as in Lemma 17, we have

$$\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_R,\infty)} \cap S_r \ne \varnothing] \le \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_R,\infty)} \cap S_r \ne \varnothing, (\cup_{n=0}^{\infty} E_n)^c] + \mathbb{P}^{\kappa,\rho}[\cup_{n=0}^{\infty} E_n].$$

Hence,

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_R,\infty)} \cap S_r \ne \varnothing] \tag{68}$$

is bounded above by the maximum of

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_R,\infty)} \cap S_r \ne \varnothing, (\cup_{n=0}^{\infty} E_n)^c], \tag{69}$$

$$\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\cup_{n=0}^{\infty} E_n] \le C_3 - (\rho+2)\log n_0, \tag{70}$$

where the bound on (70) was shown in Lemma 17. We now fix $n_0 \ge 3^4$ such that the right-hand side of (70) is bounded above by $-M$. It remains to show that, given the choice of n_0, there exists an R such that (69) is bounded above by $-M$. For each non-negative integer n we have, by the strong domain Markov property of $\mathrm{SLE}_\kappa(\rho)$, and Lemma 18, that

$$\begin{aligned}
&\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_{2^nR},\tau_{2^{n+1}R}]} \cap S_r \ne \varnothing, (\cup_{n=0}^{\infty} E_n)^c] \\
&\le \int \mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_{2^nR}}}[\hat\gamma_{\hat\tau_{2^{n+1}R}} \cap S_r \ne \varnothing, \hat M^{\kappa,\rho}_{\hat\tau_{2^{n+1}R}}/\hat M_0^{\kappa,\rho} \le f_n(\kappa,\rho)] d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_{2^nR}}],
\end{aligned}$$

where $\mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_{2^nR}}}$ denotes the law of an $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H} \setminus \gamma_{\tau_{2^nR}}$ from $\gamma(\tau_{2^nR})$ to ∞ with force point x_0, and $\hat{\cdot}$ denotes the corresponding curves, stopping times, etc. We now use the absolute continuity of $\mathrm{SLE}_\kappa(\rho)$ with respect to SLE_κ with Radon-Nikodym derivative

$$\frac{d\mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_{2^nR}}}}{d\mathbb{P}^{\kappa,0}_{\gamma_{\tau_{2^nR}}}}(\hat\gamma_{\hat\tau_{2^{n+1}R}}) = \frac{\hat M_{\hat\tau_{2^nR}}}{\hat M_0},$$

(where $\hat{M}_{\hat{\tau}_{2^{n+1}R}} = M_{\tau_{2^{n+1}R}}$ and $\hat{M}_0 = M_{\tau_{2^nR}}$), assume that $r/R < 1/3$ and apply Lemma A. This gives

$$\begin{aligned}
&\int \mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_{2^nR}}}[\hat{\gamma}_{\hat{\tau}_{2^{n+1}R}} \cap S_r \neq \varnothing, \hat{M}^{\kappa,\rho}_{\hat{\tau}_{2^{n+1}R}}/\hat{M}^{\kappa,\rho}_0 \leq f_n(\kappa,\rho)]d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_{2^nR}}] \\
\leq &\int f_n(\kappa,\rho)\mathbb{P}^{\kappa,0}_{\gamma_{\tau_{2^nR}}}[\hat{\gamma} \cap S_r \neq \varnothing]d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_{2^nR}}] \\
\leq &f_n(\kappa,\rho)c_\kappa\left(\frac{r}{2^nR}\right)^{8/\kappa-1}
\end{aligned}$$

and as a consequence,

$$\begin{aligned}
\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_R,\infty)} \cap S_r \neq \varnothing, (\cup_{n=0}^\infty E_n)^c] \leq &\sum_{n=0}^\infty \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_{2^nR},\tau_{2^{n+1}R}]} \cap S_r \neq \varnothing, E_n^c] \\
\leq &c_\kappa\left(\frac{r}{R}\right)^{8/\kappa-1}\sum_{n=0}^\infty f_n(\kappa,\rho)2^{-n(8-\kappa)/\kappa}.
\end{aligned}$$

In the case $\rho \in (-2,0)$, the right-hand side can be bounded above in the following way. Recall that in this case $f_n(\kappa,\rho) = e^{-(n+n_0)\frac{\rho(4-\kappa+\rho)}{\kappa}}(4(n+n_0)^{1/2})^{-\rho/\kappa}$. Since $-\rho(4-\kappa+\rho) \leq 4$ for all $\rho \in (-2,0), \kappa > 0$ we have

$$e^{-\rho(4-\kappa+\rho)}/2^{8-\kappa} \leq e^4/2^7 \leq 1/2,$$

for all $\kappa < 1$. So for all $\kappa < 1$, by ℓ_p-norm monotonicity,

$$\begin{aligned}
\sum_{n=0}^\infty C_n(\kappa,\rho)2^{-n(8-\kappa)/\kappa} \leq &2^{-\rho/\kappa}e^{-n_0\rho(4-\kappa+\rho)/\kappa}\sum_{n=0}^\infty\left(\frac{(n+n_0)^{-\rho/2}}{2^n}\right)^{1/\kappa} \\
\leq &2^{-\rho/\kappa}e^{-n_0\rho(4-\kappa+\rho)/\kappa}\left(\sum_{n=0}^\infty\frac{(n+n_0)^{-\rho/2}}{2^n}\right)^{1/\kappa}
\end{aligned}$$

where the series on the right-hand side is convergent. It follows that

$$\limsup_{\kappa\to0+}\kappa\log\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_R,\infty)} \cap S_r \neq \varnothing, (\cup_{n=0}^\infty E_n)^c] \leq C_8(n_0,\rho) + 8\log\frac{r}{R}, \tag{71}$$

where $C_8(n_0,\rho) < \infty$ is a constant. By a similar argument we obtain (71) in the case $\rho \in [0,\infty)$. By choosing R sufficiently large with respect to ρ and n_0 the right-hand side of (71) is bounded above by $-M$. It follows that (68) is bounded above by $-M$ as desired. The case where ρ is replaced by $\kappa+\rho$ follows in the same way. □

A.2 Radial case

Recall that $\mathcal{X}^R$ denotes the family of simple curves in $\mathbb{D}$ starting at 1 and ending at 0. Fix $v_0 \in (0,2\pi)$ and $\rho > -2$. Let $D_R = \{z \in \mathbb{D} : |z - z_0| < R\}$ and $S_R = \partial D_R$. For $\gamma \in \mathcal{X}^R$, let $\tau_R = \inf\{t : |\gamma(t)| = R\}$. The goal of this section is to prove the following proposition.

Proposition 18. *For every $R \in (0,1)$ and every $M \in [0,\infty)$ there is an $r \in (0,R)$ s.t.*

(a) $\inf\{I^R_{\rho,e^{iv_0}}(\gamma) : \gamma \in \mathcal{X}^R, \gamma_{[\tau_r,\infty)} \cap S_R \neq \varnothing\} \geq M$

(b) $\limsup_{\kappa\to0+}\kappa\log\mathbb{P}[\gamma^{\kappa,\rho}_{[\tau_r,\infty)} \cap S_R \neq \varnothing] \leq -M$

(b') $\limsup_{\kappa\to0+}\kappa\log\mathbb{P}[\gamma^{\kappa,\kappa+\rho}_{[\tau_r,\infty)} \cap S_R \neq \varnothing] \leq -M$

We first show the following topological lemma which will allow us to, in a helpful way, partition the set $\{\gamma \in \mathcal{X}^R, \gamma_{[\tau_r,\infty)} \cap S_R \neq \varnothing\}$.

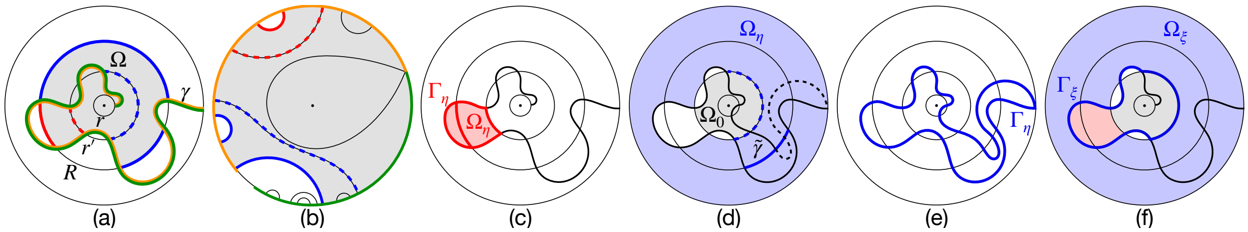

Figure 6: Set-up in Lemma 19 and its proof. In (a) we illustrate the left and right sides of a curve γ in green and orange respectively, and the arcs η in red and blue if they belong to A and B respectively. The dashed arcs correspond to arcs ξ_η. In (b) we show (schematically) how the set-up in (a) looks after mapping out γ. In (c)-(f) we illustrate various parts of the proof. Specifically, in (d) we illustrate an extension $\tilde{\gamma}$ of γ (the solid curve) and a further extension, avoiding Ω_0 (the dashed curve).

Lemma 19. *Let $0 < r < r' < R < 1$ and consider a simple curve $\gamma : (0, T] \to \mathbb{D}$ starting at 1 and with $\tau_r = T$. Let Ω denote the connected component of $D_R \setminus \gamma$ containing 0. The set $S_R \cap \partial\Omega$ consists of (at most) countably many connected arcs η. Let Ω_0 and Ω_η be the connected components of $\mathbb{D} \setminus (\gamma \cup S_{r'})$ containing 0 and η respectively. Then there is a partitioning of the set $\{\eta\}$ into two sets A and B so that the following holds*

(i) If $\eta \in A$, then only one side of γ is accessible within Ω_η.

(ii) If $\eta \in B$ and $\tilde{\gamma} : (0, \tilde{T}] \to \mathbb{D}$, $\tilde{T} > T$, is a simple curve for which $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in (0, T]$, and $\tilde{\gamma}(\tilde{T}) \in \eta$ and $\tilde{T} = \inf\{t > T : |\tilde{\gamma}(t)| = R\}$, then only one side of $\tilde{\gamma}$ is accessible from 0 in $\Omega_0 \setminus \tilde{\gamma}$.

(iii) There exists a curve $\gamma' \subset \mathbb{D} \setminus \gamma$ with endpoints at $\gamma(T)$ and e^{iv_0} which does not intersect $\cup_{\eta \in A} \Omega_\eta$.

Proof. The set $\partial\Omega_0 \cap S_{r'}$ consists of (possibly) countably many connected arcs ξ. Among these arcs there is, for every η, exactly one arc $\xi = \xi_\eta$ separating 0 from η in $\mathbb{D} \setminus \gamma$. We let $\eta \in A$ if the endpoints of ξ_η belong to the same side of γ, and $\eta \in B$ if ξ_η if the endpoints of γ belong to different sides of γ, see Figure 6(a) and (b).

For $\eta \in A$, ξ_η and the subarc of γ connecting the two endpoints of ξ_η form a Jordan curve Γ_η where the bounded component of $\mathbb{C} \setminus \Gamma_\eta$ contains (or equals) Ω_η, and the remaining parts of γ are contained in the unbounded component (see Figure 6(c)). Therefore, $\partial\Omega_\eta$ can only contain points from one side of γ, this shows (i).

If $\eta \in B$, then, by construction, Ω_η has all of $\partial\mathbb{D}$ as accessible points. Thus, any curve $\tilde{\gamma}$ as in (ii) could be continued from $\tilde{\gamma}(\tilde{T})$ to 1 without crossing Ω_0, forming a Jordan curve Γ_η (see Figure 6(d) and (e)). Since only one side of Γ_η is accessible in $\mathbb{C} \setminus \Gamma_\eta$ from the origin, and since the continuation of $\tilde{\gamma}$ does not cross Ω_0, only one side of $\tilde{\gamma}$ can be accessible from the origin in $\Omega_0 \setminus \tilde{\gamma}$. This shows (ii).

Finally, there is exactly one arc $\xi \in \partial\Omega_0 \cap S_{r'}$ separating 0 from 1 in $\mathbb{D} \setminus \gamma$ (namely $\xi = \xi_\eta$ for all $\eta \in B$). Let Ω_ξ be the outer component of $\mathbb{D} \setminus \Gamma_\xi$ where Γ_ξ is the Jordan curve consisting of ξ and the appropriate sub-arc of γ (see Figure 6(f)). Observe that ξ does not separate any of the arcs in A from 0 in $\mathbb{D} \setminus \gamma$, and hence $\cup_{\eta \in A} \Omega_\eta \subset \mathbb{D} \setminus \Omega_\xi$. On the other hand, $\Omega_0 \cup \Omega_\xi$ is an open connected set with $\gamma(T)$ and e^{iv_0} as accessible points. Therefore, there exists a curve γ' in $\Omega_0 \cup \Omega_\xi$ connecting 0 to 1, and any such a curve is disjoint from $\Omega_{\eta \in A} \Omega_\eta$. □

For some $0 < r < r' < R < 1$, and simple curve $\gamma : (0, \tau_r] \to \mathbb{D}$ starting at 1, let $A_{r,r',R,\gamma}$ and $B_{r,r',R,\gamma}$ be the sets of arcs A and B from Lemma 19. We partition the set of simple returning curves

$$E_{r,R} = \{\gamma \in \mathcal{X}^R : \gamma_{[\tau_r,\infty)} \cap S_R \neq \varnothing\} = \tilde{A}_{r,r',R} \cup \tilde{B}_{r,r',R},$$

where

$$\tilde{A}_{r,r',R} := \{\gamma \in E_{r,R} : \gamma(\tau_{r,R}) \in \eta, \eta \in A_{r,r',R,\gamma_{\tau_r}}\},$$

$$\tilde{B}_{r,r',R} := \{\gamma \in E_{r,R} : \gamma(\tau_{r,R}) \in \eta, \eta \in B_{r,r',R,\gamma_{\tau_r}}\},$$

and $\tau_{r,R} = \inf\{t > \tau_r : |\gamma(t)| = R\}$.

Lemma 20. *If $\gamma \in \tilde{B}_{r,r',R}$ and $r/r' < 3 - 2\sqrt{2}$, then* $\sin((v_{\tau_{r,R}} - w_{\tau_{r,R}})/2)) \leq 4\sqrt{r/r'}$.

Proof. By applying Lemma 19(ii) to γ_{τ_r}, we find that only one side of $\gamma_{\tau_{r,R}}$ will be accessible from Ω_0. Without loss of generality, we may assume it to be the right side. Using monotonicity of harmonic measure we find

$$\frac{(v_{\tau_{r,R}} - w_{\tau_{r,R}})}{2\pi} = \omega(0, \gamma^+_{\tau_{r,R}} \cup a, \mathbb{D} \setminus \gamma_{\tau_{r,R}}) \geq \omega(0, \gamma_{\tau_{r,R}}, \Omega_0 \setminus \gamma_{\tau_{r,R}}),$$

where a is the circular arc from 1 to e^{iv_0}. By Beurling's projection theorem (see, e.g., [14, Theorem III.9.3]), and an explicit computation, we have that

$$\omega(0, \gamma_{\tau_{r,R}}, \Omega_0 \setminus \gamma_{\tau_{r,R}}) \geq \omega(0, [r/r', 1], \mathbb{D} \setminus [r/r', 1]) = \frac{2}{\pi} \arcsin \frac{1 - r/r'}{1 + r/r'},$$

and if $r/r' < 3 - 2\sqrt{2}$ then the right-hand side is larger than $\pi/2$. By monotonicity of sin on $[\pi/2, \pi]$, and the explicit computation

$$\sin((v_{\tau_{r,R}} - w_{\tau_{r,R}})/2)) = \sin(\pi\omega(0, [r/r', 1], \mathbb{D} \setminus [r/r', 1])) \leq 4\sqrt{r/r'}.$$

□

Lemma 21. *If $\gamma \in \tilde{A}_{r,r',R}$ and $r'/R < 3 - 2\sqrt{2}$, then* $I^{\mathbb{D},1,e^{iv_0}}(\gamma) \geq -4\log(16r'/R)$.

Proof. Since $\gamma \in \tilde{A}_{r,r',R}$ we have that $\gamma_{\tau_{r,R}} \in \eta$ for some $\eta \in A_{r,r',R,\gamma_{\tau_r}}$, and by Lemma 19(i) only one side of γ_{τ_r} is accessible from Ω_η. Without loss of generality, we may assume that it is the right side. We have, by monotonicity of harmonic measure,

$$\omega(\gamma(\tau_{r,R}), \gamma^+_{\tau_r} \cup a, \mathbb{D} \setminus \gamma_{\tau_r}) \geq \omega(\gamma(\tau_{r,R}), \gamma_{\tau_r}, \Omega_\eta)$$

where a is the circular arc from 1 to e^{iv_0}. One can see, similar to Lemma 19(iii), that there is a curve $\hat{\eta}$ from 0 to 1 in $\mathbb{D} \setminus \gamma_{\tau_r}$, which does not cross Ω_η for any $\eta \in A_{r,r',R,\gamma_{\tau_r}}$, and therefore (again by monotonicity)

$$\omega(\gamma(\tau_{r,R}), \gamma_{\tau_r}, \Omega_\eta) \geq \omega(\gamma(\tau_{r,R}), \hat{\gamma} \cup [1, \infty], \hat{\mathbb{C}} \setminus (\overline{D_{r'}} \cup \hat{\gamma} \cup [1, \infty])).$$

Let $\psi : \hat{\mathbb{C}} \setminus \overline{D_{r'}} \to \mathbb{D}$ be a conformal map with $\psi(\gamma_{\tau_{r,R}}) = 0$. Then, $|\psi(\infty)| = r'/R$ and $\psi(\hat{\gamma} \cup [1, \infty])$ is a path connecting $\partial\mathbb{D}$ and $\psi(\infty)$. Hence, Beurling's projection theorem (see, e.g., [14, Theorem III.9.3]) gives

$$\omega(\gamma(\tau_{r,R}), \hat{\gamma} \cup [1, \infty], \hat{\mathbb{C}} \setminus (\overline{D_{r'}} \cup \hat{\gamma} \cup [1, \infty])) \geq \omega(0, [r'/R, 1], \mathbb{D} \setminus [r'/R, 1]).$$

The same argument as in the proof of Lemma 20 now shows

$$\sin(\pi\omega(\gamma(\tau_{r,R}), \gamma^+_{\tau_r} \cup a(0, v_0), \mathbb{D} \setminus \gamma_{\tau_r})) \leq 4\sqrt{r'/R}, \tag{72}$$

given that $r'/R < 3 - 2\sqrt{2}$, which gives the desired bound. □

Proof of Proposition 18(a). Fix R, ρ, v_0 and M. Set $r' \in (0, R)$ so that $r'/R < 3 - 2\sqrt{2}$ and

$$\min(1, \tfrac{2+\rho}{4})\Big(- 4\min(1, \tfrac{2+\rho}{4}) \log(16r'/R) + (6 + \rho) \log \sin(v_0/2)\Big) \geq M.$$

Then, set $r \in (0, r')$ so that $r/r' < 3 - 2\sqrt{2}$ and

$$\min(1, \tfrac{2+\rho}{4})\Big(- (6 + \rho) \log(4\sqrt{r/r'}) + \log(6 + \rho) \log \sin(v_0/2)\Big) \geq M.$$

Then, any curve $\gamma \in \tilde{A}_{r,r',R} \cup \tilde{B}_{r,r',R}$ has $I^R_{\rho,e^{iv_0}}(\gamma) = I^{\mathbb{D},1,e^{iv_0}}_{-6-\rho,0}(\gamma) \geq M$ by the upper bound from Proposition 13 and the estimates from Lemma 20 and 21. □

Let $0 < \alpha < \beta < 1$ and $0 < r_0 < r_0' < R < 1$ (these constants will be fixed in the proof of Proposition 18(b) and (b'), but for now we consider them as arbitrary). Set $r_n = \alpha^n r_0$ and $r_n' = \beta^n r_0'$. For ease of notation, we denote $\tau_n = \tau_{r_n}$. By Koebe-1/4 we have that $\mathrm{crad}(0, \mathbb{D} \setminus \gamma_{\tau_n}) \in [r_n, 4r_n]$, which gives

$$\tau_n \in [-\log(4\alpha^n r_0), -\log(\alpha^n r_0)], \quad \tau_{n+1} - \tau_n \in [-\log(4\alpha), -\log(\alpha/4)].$$

Lemma 22. *Let* $\varepsilon_n = (n + n_0)^{-1/2}$ *and*

$$n_0 > \max(1/\sin^2(v_0/6), 1/\sin^2((2\pi - v_0)/6), \log(r_0)/\log(\alpha)).$$

Then

$$\limsup_{\kappa\to 0+} \kappa \log\left(\sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\exists t \in [0, \tau_n]:\ \sin((v_t^{\kappa,\rho} - w_t^{\kappa,\rho})/2) < \varepsilon_n]\right) \le C_1(\rho, v_0) - (\rho + 2)\log n_0,$$

where $C_1(\rho, v_0)$ *is a constant. The same holds when* $\mathbb{P}^{\kappa,\rho}$ *is replaced by* $\mathbb{P}^{\kappa,\kappa+\rho}$.

Proof. Using $\varepsilon_n < \varepsilon_0 < \min(\sin(v_0/6), \sin((2\pi - v_0)/6))$, (31) from the proof of Lemma 3, and $\tau_n \le -\log(\alpha^n r_0)$, we find

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\exists t \in [0, \tau_n]:\ \sin((v_t^{\kappa,\rho} - w_t^{\kappa,\rho})/2) < \varepsilon_n] &\le -C_2(\kappa, \rho, v_0) \sum_{n=0}^{\infty} (n \log\alpha + \log r_0)\varepsilon_n^{2\frac{\rho+2}{\kappa} - 1} \\ &\le -C_2(\kappa, \rho, v_0)\log\alpha \sum_{n=0}^{\infty} (n + n_0)^{3/2 - (\rho+2)/\kappa},\end{aligned}$$

where $\limsup_{\kappa\to 0+} \kappa \log C_2(\kappa, \rho, v_0) =: C_1(\rho, v_0) \in \mathbb{R}$, and where the second inequality follows from $n_0 > \log(r_0)/\log(\alpha)$. Hence,

$$\limsup_{\kappa\to 0+} \sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\exists t \in [0, \tau_n]:\ \sin((v_t^{\kappa,\rho} - w_t^{\kappa,\rho})/2) < \varepsilon_n] \le C_1 - (\rho + 2)\log(n_0).$$

The same holds when $\mathbb{P}^{\kappa,\rho}$ is replaced by $\mathbb{P}^{\kappa,\kappa+\rho}$. □

Lemma 23. *If* $4\sqrt{r_0/r_0'} < \min(\sin(v_0/6), \sin((2\pi - v_0)/6))$, *then*

$$\limsup_{\kappa\to 0+} \kappa \log\left(\sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\exists t \in [0, \tau_{r_{n+1}}] : \sin((v_t^{\kappa,\rho} - w_t^{\kappa,\rho})/2) < 4\sqrt{r_n/r_n'}]\right) \le C_3 + (\rho + 2)\log\frac{r_0}{r_0'},$$

where $C_3 = C_3(\rho, v_0)$ *is a constant.*

The proof is almost identical to that of Lemma 22 and is therefore omitted.

Proof of Proposition 18(b) and (b'). Fix $R \in (0, 1)$ and $M > 0$ and consider sequences (r_n), (r_n') as in the set-up above. The constants α and β will depend on ρ but not on M or R (in fact, we will see that fixing $\alpha/\beta \in (0, 1)$ and then choosing $\beta < 1$ so that $\beta \le 4^{-4}(\frac{\alpha}{\beta})^5$ if $\rho \in (-2, 2)$ and $\beta < (\frac{\alpha}{\beta})^{1+\rho/6}$ if $\rho \in [2, \infty)$ will be sufficient for our purposes). Using the lemmas above, we will show that there is a choice of $r = r_0$ such that the statement holds. We start by imposing that $r_0 < r_0'(3 - 2\sqrt{2})$ so that $0 < r_n < r_n'(3 - 2\sqrt{2}) < r_n' < R$ for all $n \ge 0$. Let γ be a radial $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{D}$ from 0 to 1 with force point e^{iv_0}. We have

$$\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_0,\infty)} \cap S_R \ne \varnothing] \le \sum_{n\ge 0} \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap S_R \ne \varnothing],$$

and the n-th term on the right-hand side is bounded above by

$$\mathbb{P}^{\kappa,\rho}[\exists t \in [0, \tau_{n+1}] : \sin(\theta_t) < \tilde{\varepsilon}_n] + \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap S_R \ne \varnothing,\ \sin(\theta_t) \ge \tilde{\varepsilon}_n\ \forall t \in [0, \tau_{n+1}]]$$

where $\tilde{\varepsilon}_n = \max(4\sqrt{r_n/r_n'}, \varepsilon_n)$, $\varepsilon_n = \varepsilon(n_0)$, for some positive integer n_0, is as in Lemma 22, and $\theta_t := (v_t - w_t)/2$. Therefore, $\limsup_{\kappa\to 0+} \kappa \log \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_0,\infty)} \cap S_R \neq \varnothing]$ is bounded above the maximum of

$$\limsup_{\kappa\to 0+} \kappa \log \sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\exists t \in [0,\tau_{n+1}] : \sin(\theta_t) < \tilde{\varepsilon}_n], \tag{73}$$

$$\limsup_{\kappa\to 0+} \kappa \log \sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap S_R \neq \varnothing \text{ and } \sin(\theta_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\tau_{n+1}]]. \tag{74}$$

By choosing n_0 sufficiently large, and the ratio r_0/r_0' sufficiently small, Lemma 22 and 23 show that (73) is bounded above by $-M$. Note that the choices of n_0 and r_0/r_0' depend only on M and ρ, and in particular not on r_0'/R, so if we are able to bound (74) above by $-M$, by choosing r_0'/R appropriately, we are done.

Denote $B_n := B_{r_n,r_n',R,\gamma_{\tau_n}}$. By Lemma 20 and

$$\{\gamma : \gamma_{[\tau_n,\infty)} \cap S_R \neq \varnothing\} = \tilde{A}_{r_n,r_n',R} \cup \tilde{B}_{r_n,r_n',R}$$

we have

$$\begin{aligned}
&\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap S_R \neq \varnothing \text{ and } \sin(\theta_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\tau_{n+1}]] \\
\leq &\mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap (\cup_{\eta\in B_n} \eta) \neq \varnothing \text{ and } \sin(\theta_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\tau_{n+1}]] \\
= &\int_{\{\sin(\theta_t)\geq\tilde{\varepsilon}_n\}} \mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_n}}[\hat{\gamma}_{\hat{\tau}_{n+1}} \cap (\cup_{\eta\in B_n} \eta) \neq \varnothing \text{ and } \sin(\hat{\theta}_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\hat{\tau}_{n+1}]] d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_n}],
\end{aligned}$$

where the strong domain Markov property of $\text{SLE}_\kappa(\rho)$ is used in the last step, and $\mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_n}}$ denotes the law of an $\text{SLE}_\kappa(\rho)$ from $\gamma(\tau_n)$ to 0 in $\mathbb{D} \setminus \gamma_{\tau_n}$ with force point e^{iv_0}, and $\hat{\cdot}$ denotes the corresponding curve, stopping times, etc. Recall that $\text{SLE}_\kappa(\rho)$ in $\mathbb{D}\setminus\gamma^{\kappa,\rho}_{\tau_{r_n}}$ is absolutely continuous with respect to a $\text{SLE}_\kappa(\kappa-6)$ or equivalently a (reparametrized) *chordal* SLE_κ in the same domain from $\gamma(\tau_{r_n})$ to e^{iv_0} for stopping times strictly before the swallowing time of 0 or e^{iv_0}. For the stopping time $\hat{\tau}_n' = \min(\hat{\tau}_{n+1}, \inf\{t : \sin(\hat{\theta}_t) \leq \tilde{\varepsilon}_n\})$, which is such a stopping time, we have

$$\frac{d\mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_n}}}{d\mathbb{P}^{\kappa,\kappa-6}_{\gamma_{\tau_n}}}(\hat{\gamma}_{\hat{\tau}_n'}) = \left(\frac{\hat{g}'_{\hat{\tau}_n'}(0)}{\hat{g}'_0(0)}\right)^{\frac{(4+\rho)\rho-(\kappa-2)(\kappa-6)}{8\kappa}} \left(\frac{|\hat{V}_{\hat{\tau}_n'} - \hat{W}_{\hat{\tau}_n'}|}{|\hat{V}_0 - \hat{W}_0|}\right)^{\frac{\rho-\kappa+6}{\kappa}} \left(\frac{|\hat{g}'_{\hat{\tau}_n'}(V_0)|}{|\hat{g}'_0(V_0)|}\right)^{\frac{(4-\kappa+\rho)\rho+2(\kappa-6)}{4\kappa}}.$$

We claim that, on the set $E_n = \{\hat{\gamma}_{\hat{\tau}_{n+1}} : \sin(\hat{\theta}_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\hat{\tau}_{n+1}]\}$ we have

$$\frac{d\mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_n}}}{d\mathbb{P}^{\kappa,\kappa-6}_{\gamma_{\tau_n}}}(\hat{\gamma}_{\hat{\tau}_n'}) \leq M_{n,\kappa} := f_\kappa \cdot \begin{cases} (\frac{\alpha}{\beta})^{n\frac{\kappa-6-\rho}{2\kappa}} (\frac{\alpha}{4})^{n\frac{(4-\kappa+\rho)\rho+2(\kappa-6)}{8\kappa}}, & \rho \in (-2,2], \\ (\frac{\alpha}{\beta})^{n\frac{\kappa-6-\rho}{2\kappa}}, & \rho \in [2,\infty), \end{cases} \tag{75}$$

where $f_\kappa = f_\kappa(\rho,\alpha,n_0,r_0/r_0')$ (but does not depend on n or r_0'/R), and for every fixed ρ, α, n_0, r_0/r_0' satisfying the constraints above,

$$\limsup_{\kappa\to 0+} f_\kappa(\rho,\alpha,n_0,r_0/r_0') = \limsup_{\kappa\to 0+} f_\kappa(\kappa+\rho,\alpha,n_0,r_0/r_0') = \tilde{C}$$

where $\tilde{C} = \tilde{C}(\rho,\alpha,n_0,r_0/r_0') \in \mathbb{R}$. To show this, first note that

$$\frac{\hat{g}'_{\hat{\tau}_n'}(0)}{\hat{g}'_0(0)} = \frac{e^{-\tau_{n+1}}}{e^{-\tau_n}} \in [\alpha/4, 4\alpha],$$

since $\hat{\tau}_n' = \hat{\tau}_{n+1}$ on E_n. So, the first factor of the Radon-Nikodym derivative can be "swallowed" by f_κ. Secondly, since $|\hat{V}_t - \hat{W}_t| = 2\sin(\hat{\theta}_t)$, and $\rho - \kappa + 6 > 0$ when $\kappa \leq 4$,

$$\left(\frac{|\hat{V}_{\hat{\tau}_n'} - \hat{W}_{\hat{\tau}_n'}|}{|\hat{V}_0 - \hat{W}_0|}\right)^{\frac{\rho-\kappa+6}{\kappa}} \leq \left(4\sqrt{\frac{r_n}{r_n'}}\right)^{\frac{\kappa-6-\rho}{\kappa}} \leq \left(4\sqrt{\frac{r_0}{r_0'}}\right)^{\frac{\kappa-6-\rho}{\kappa}} \left(\frac{\alpha}{\beta}\right)^{n\frac{\kappa-6-\rho}{2\kappa}},$$

where the first factor on the right-hand side is "swallowed" by f_κ. Finally, by the radial Loewner equation

$$\log \frac{|\hat{g}'_{\hat{\tau}'_n}(V_0)|}{|\hat{g}'_0(V_0)|} = -\int_0^{\hat{\tau}'_n} \frac{1}{2\sin^2(\hat{\theta}_t)} dt.$$

This shows that

$$0 \leq -\log \frac{|\hat{g}'_{\hat{\tau}'_n}(V_0)|}{|\hat{g}'_0(V_0)|} \leq \frac{\hat{\tau}'_n}{2\tilde{\varepsilon}_n^2} \leq \frac{(\tau_{n+1}-\tau_n)}{2\varepsilon_n^2} \leq -\log(\alpha/4)\frac{(n+n_0)}{2},$$

and therefore

$$\left(\frac{\alpha}{4}\right)^{\frac{n+n_0}{2}} \leq \frac{|\hat{g}'_{\hat{\tau}'_n}(V_0)|}{|\hat{g}'_0(V_0)|} \leq 1.$$

Since $(4-\kappa+\rho)\rho + 2(\kappa-6)$ is non-negative on $[2,\infty)$ and non-positive on $(-2,2]$, the third factor of the Radon-Nikodym derivative can be bounded above by 1 if $\rho \in [2,\infty)$ and by

$$\left(\frac{\alpha}{4}\right)^{n_0\frac{(4-\kappa+\rho)\rho+2(\kappa-6)}{8\kappa}} \left(\frac{\alpha}{4}\right)^{n\frac{(4-\kappa+\rho)\rho+2(\kappa-6)}{8\kappa}}$$

if $\rho \in (-2,2]$ where again, the first factor is "swallowed" by f_κ. It follows that (75) holds. We see that

$$\begin{aligned}
&\int_{\{\sin(\theta_t)\geq\tilde{\varepsilon}_n\}} \mathbb{P}^{\kappa,\rho}_{\gamma_{\tau_n}}[\hat{\gamma}_{\hat{\tau}_{n+1}} \cap (\cup_{\eta\in B_n}\eta) \neq \varnothing \text{ and } \sin(\hat{\theta}_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\hat{\tau}_{n+1}]] d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_n}] \\
&\leq \int_{\{\sin(\theta_t)\geq\tilde{\varepsilon}_n\}} M_{n,\kappa}\mathbb{P}^{\kappa,\kappa-6}_{\gamma_{\tau_n}}[\hat{\gamma}_{\tau_{n+1}} \cap (\cup_{\eta\in B_n}\eta) \neq \varnothing] d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_n}] \\
&\leq \int_{\{\sin(\theta_t)\geq\tilde{\varepsilon}_n\}} M_{n,\kappa}\tilde{\mathbb{P}}^{\kappa}_{\gamma_{\tau_n}}[\tilde{\gamma} \cap (\cup_{\eta\in B_n}\eta) \neq \varnothing] d\mathbb{P}^{\kappa,\rho}[\gamma_{\tau_n}],
\end{aligned}$$

where $\tilde{\mathbb{P}}^{\kappa}_{\gamma_{\tau_n}}$ denotes the law of a chordal SLE$_\kappa$, $\tilde{\gamma}$, in $\mathbb{D}\setminus\gamma_{\tau_n}$ from $\gamma(\tau_n)$ to e^{iv_0}. Using Lemma A with $D = \mathbb{D}\setminus\gamma_{\tau_n}$, $x = \gamma(\tau_n)$, $y = e^{iv_0}$, $\eta \in B_n$, and γ' as in Lemma 19(iii), we obtain

$$\begin{aligned}
\tilde{\mathbb{P}}^{\kappa}_{\gamma_{\tau_n}}[\tilde{\gamma} \cap (\cup_{\eta\in B}\eta) \neq \varnothing] &\leq \sum_{\eta\in B_n} \tilde{\mathbb{P}}^{\kappa}_{\gamma_{\tau_n}}[\tilde{\gamma}\cap\eta \neq \varnothing] \leq c_\kappa \sum_{\eta\in B_n} \mathcal{E}_{\mathbb{D}\setminus\gamma_{\tau_n}}(\eta,\gamma')^{\frac{8}{\kappa}-1} \\
&\leq c_\kappa \sum_{\eta\in B_n} \mathcal{E}_{\mathbb{D}\setminus\gamma_{\tau_n}}(\eta, S_{r'_n})^{\frac{8}{\kappa}-1} \leq c_\kappa \left(\sum_{\eta\in B_n} \mathcal{E}_{\mathbb{D}\setminus\gamma_{\tau_n}}(\eta, S_{r'_n})\right)^{\frac{8}{\kappa}-1} \\
&\leq c_\kappa \left(2\mathcal{E}_{\mathbb{D}\setminus\gamma_{\tau_n}}(S_R, S_{r'_n})\right)^{\frac{8}{\kappa}-1} \leq c_\kappa\left(\tilde{c}\sqrt{\frac{r'_n}{R}}\right)^{\frac{8}{\kappa}-1} = c_\kappa\left(\tilde{c}\beta^{n/2}\sqrt{\frac{r'_0}{R}}\right)^{\frac{8}{\kappa}-1}
\end{aligned}$$

In the third inequality, we have used that $S_{r'_n}$ separates γ' from every $\eta \in B_n$ (Lemma 19(iii)). In the fourth inequality we use monotonicity of the ℓ^p norm and $8/\kappa - 1 \geq 1$. The fifth and sixth inequalities follow from Lemma B and [12, Equation (2.5)], whenever $r'_0/R \leq 1/2$, where $\tilde{c}$ is a constant. Now,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap S_R \neq \varnothing, \ \sin(\theta_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\tau_{n+1}]] \\
&\leq \sum_{n=0}^{\infty} \mathbb{P}^{\kappa,\rho}[\gamma_{[\tau_n,\tau_{n+1}]} \cap (\cup_{\eta\in B_n}\eta) \neq \varnothing, \ \sin(\theta_t) \geq \tilde{\varepsilon}_n \ \forall t \in [0,\tau_{n+1}]] \\
&\leq \sum_{n=0}^{\infty} M_{n,\kappa} c_\kappa \left(\tilde{c}\beta^{n/2}\sqrt{\frac{r'_0}{R}}\right)^{8/\kappa-1} \\
&\leq f_\kappa c_\kappa \left(\tilde{c}\sqrt{\frac{r'_0}{R}}\right)^{8/\kappa-1} \sum_{n=0}^{\infty} \beta^{n\frac{8-\kappa}{2\kappa}} \cdot \begin{cases} (\frac{\alpha}{\beta})^{n\frac{\kappa-6-\rho}{2\kappa}} (\frac{\alpha}{4})^{n\frac{(4-\kappa+\rho)\rho+2(\kappa-6)}{8\kappa}}, & \rho \in (-2,2], \\ (\frac{\alpha}{\beta})^{n\frac{\kappa-6-\rho}{2\kappa}}, & \rho \in [2,\infty). \end{cases}
\end{aligned}$$

We now set $0 < \alpha < \beta < 1$ in an appropriate way so that the series on the right hand side converges. When $\rho \in [2, \infty)$ we first fix the ratio $\alpha/\beta \in (0,1)$ arbitrarily and set $\beta \le (\frac{\alpha}{\beta})^{1+\frac{\rho}{6}}$. Then

$$\sum_{n=0}^{\infty} \beta^{n\frac{8-\kappa}{2\kappa}} \left(\frac{\alpha}{\beta}\right)^{n\frac{\kappa-6-\rho}{2\kappa}} \le \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^{n\frac{8-\kappa}{2\kappa}(1+\frac{\rho}{6})} \left(\frac{\alpha}{\beta}\right)^{n\frac{\kappa-6-\rho}{2\kappa}} \le \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^{\frac{n}{\kappa}} < \infty.$$

For $\rho \in (-2, 2]$ we again fix $\alpha/\beta \in (0,1)$, but now set $\beta \le 4^{-4}(\frac{\alpha}{\beta})^5$. Then, for $\kappa < 2$

$$\sum_{n=0}^{\infty} \beta^{n\frac{8-\kappa}{2\kappa}} \left(\frac{\alpha}{\beta}\right)^{n\frac{\kappa-6-\rho}{2\kappa}} \left(\frac{\alpha}{4}\right)^{n\frac{(4-\kappa+\rho)\rho+2(\kappa-6)}{8\kappa}} \le \sum_{n=0}^{\infty} \left(\beta^8 \left(\frac{\alpha}{\beta}\right)^{-36} \left(\frac{1}{4}\right)^{-32}\right)^{\frac{n}{8\kappa}} \le \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^{\frac{n}{2\kappa}} < \infty.$$

This shows that (74) is bounded above by $\hat{C}(\rho) + 4\log\frac{r_0'}{R}$, for a constant $\hat{C}(\rho)$, and by choosing the ratio r_0'/R sufficiently small, the upper bound can be made smaller than $-M$. This finishes the proof of (b). The proof of (b') is almost identical. □

References