# Övning 1

**1.1** Låt $z = 1+2i$ och $w = 2-i$. Beräkna följande:

a) $z + 3w = 1 + 2i + 6 - 3i = 7 - i$

b) $\bar{w} - z = \overline{2-i} - (1+2i) = 2+1-1-2i = 1-i$

c) $z^3 = (1+2i)(1+2i)(1+2i) = (1-4+4i)(1+2i) = (-3+4i)(1+2i) = -3-8+(4-6)i = -11-2i$

d) $\mathrm{Re}(w^2+w)$. $w^2 = (2-i)^2 = 4-1-4i = 3-4i$ $\quad \mathrm{Re}(w^2+w) = \mathrm{Re}(w^2) + \mathrm{Re}(w) = 3+2 = 5$

e) $z^2 + \bar{z} + i = \{ z^2 = -3+4i \} = -3+4i+1-2i+i = -2+3i$

**1.2** Bestäm real- och imaginärdelen av följande komplexa tal:

a) För $a \in \mathbb{R}$ $\quad \frac{z-a}{z+a} = \frac{(z-a)(\bar{z}+a)}{|z+a|^2} = \frac{|z|^2 + za - \bar{z}a - a^2}{(x+a)^2+y^2} = \{ z - \bar{z} = 2iy \}$ $\quad \mathrm{Re}\left(\frac{z-a}{z+a}\right) = \frac{x^2+y^2-a^2}{(x+a)^2+y^2}$, $\quad \mathrm{Im}\left(\frac{z-a}{z+a}\right) = \frac{a2iy}{(x+a)^2+y^2}$

b) $\frac{3+5i}{7i+1} = \frac{(3+5i)(1-7i)}{1+49} = \frac{3-21i+5i+35}{50}$ $\quad \mathrm{Re}\left(\frac{3+5i}{7i+1}\right) = \frac{38}{50} = \frac{19}{25}$. $\quad \mathrm{Im}\left(\frac{3+5i}{7i+1}\right) = \frac{-16i}{50} = -\frac{8i}{25}$

c) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3 = \{$ Notera $\frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$ $\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right)$, $-\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$ $\frac{\sqrt{3}}{2} = \sin\left(\frac{2\pi}{3}\right) \}$ $= \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^3 = e^{i\frac{2\pi}{3}\cdot 3} = e^{i2\pi} = 1$

$\mathrm{Re}\left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right) = 1 \quad \mathrm{Im}\left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right) = 0$

d) $i^n = \begin{cases} i & \text{om } n \equiv 1 \mod 4 \\ -1 & \text{om } n \equiv 2 \mod 4 \\ -i & \text{om } n \equiv 3 \mod 4 \\ 1 & \text{om } n \equiv 0 \mod 4 \end{cases}$ $\quad \mathrm{Re}(i^n) = \begin{cases} 0 & \text{om } n \text{ udda} \\ 1 & \text{om } n \text{ delbar med } 4 \\ -1 & \text{annars (om } n \text{ jämn men ej delbar med } 4) \end{cases}$ $\quad \mathrm{Im}(i^n) = \begin{cases} 0 & \text{om } n \text{ jämn} \\ 1 & \text{om } n = 1+4k \\ -1 & \text{om } n = 3+4k \end{cases}$

**1.3** Bestäm absolutbeloppet och konjugatet av de följande komplexa talen:

a) $-2+i$ : $\overline{-2+i} = -2-i \quad |-2+i| = \sqrt{4+1} = \sqrt{5}$

b) $(2+i)(4+3i)$ : $\overline{(2+i)(4+3i)} = (2-i)(4-3i) \quad |(2+i)(4+3i)| = |2+i||4+3i| = \sqrt{(4+1)(16+9)} = \sqrt{5 \cdot 25} = 5\sqrt{5}$

c) $\frac{3-i}{\sqrt{2}+3i}$ : $\overline{\left(\frac{3-i}{\sqrt{2}+3i}\right)} = \frac{\overline{3-i}}{\overline{\sqrt{2}+3i}} = \frac{3+i}{\sqrt{2}-3i}$. $\quad \left|\frac{3-i}{\sqrt{2}+3i}\right| = \sqrt{\frac{9+1}{2+9}} = \sqrt{\frac{10}{11}}$

d) $(1+i)^6$ : $\overline{(1+i)^6} = (\overline{1+i})^6 = (1-i)^6 \quad |(1+i)^6| = |1+i|^6 = |\sqrt{2}|^6 = 2^3 = 8$.

**1.4** Skriv på polär form:

a) $2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2e^{i\pi/2}$

b) $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\pi/4}$

c) $|-3+3i| = \sqrt{12} = 2\sqrt{3}$ $\quad -3+\sqrt{3}i = 2\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\sqrt{3}e^{5\pi i/6}$

d) $-i = 1 \cdot e^{\frac{3\pi}{2}i}$

e) $2-i = \sqrt{5}\left(\frac{2}{\sqrt{5}} - i\frac{1}{\sqrt{5}}\right) = \sqrt{5}e^{-i\arcsin(\frac{1}{\sqrt{5}})}$

f) $|3-4i| = \sqrt{9+16} = 5 = 5e^{i\cdot 0}$

g) $\sqrt{5}-i = \sqrt{5+1}\left(\sqrt{\frac{5}{6}} - i\sqrt{\frac{1}{6}}\right) = \sqrt{6}e^{-i\arcsin(\frac{1}{\sqrt{6}})}$

h) $\left(\frac{1-i}{\sqrt{3}}\right)^4$: $\quad 1-i = \sqrt{2}e^{-i\pi/4}$ $\quad (1-i)^4 = 4\cdot e^{-i\pi} = -4$ $\quad \left(\frac{1-i}{\sqrt{3}}\right)^4 = -\frac{4}{9}$.

## 1.8 Använd kvadratformeln (/pq-formeln) för att lösa:

a) $z^2+25=0$ $\quad z = \pm\sqrt{25} = \pm 5i$

b) $2z^2+2z+5=0$ $\quad z = \frac{-2\pm\sqrt{4-20}}{4} = \frac{-2\pm 4i}{4} = \frac{-1\pm 2i}{2}$

c) $5z^2+4z+1=0$ $\quad z = \frac{-4\pm\sqrt{16-4}}{10} = \frac{-4\pm\sqrt{12}}{10}$

d) $z^2-z=1$ $\quad z = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$

e) $z^2=2z$ $\quad z = \frac{2\pm\sqrt{4}}{2} = \frac{2\pm 2}{2} = 1\pm 1$

## 1.9 Hitta alla lösningar till $z^2+2z+(1-i)=0$

$$z = \frac{-2\pm\sqrt{4-4(1-i)}}{2} = \frac{-2\pm\sqrt{4i}}{2} = \frac{-2\pm 2e^{i\pi/4}}{2} = -1\pm e^{i\pi/4}.$$

## 1.11 Bestäm alla lösningar till

a) $z^6=1$: $\quad z=re^{i\varphi}$ $\quad z^6 = r^6e^{i6\varphi} = 1$ $\Rightarrow r=1$ $\quad 6\varphi = 2\pi k$ $\Rightarrow \varphi = \frac{2\pi k}{6} = \frac{\pi}{3}k$ $\quad k=1,2,\dots$ Lösningar: $1, e^{i\pi/3}, e^{i2\pi/3}, \dots, e^{i5\pi/3}$

b) $z^4=-16$: $\quad z=re^{i\varphi}$ $\quad z^4 = r^4e^{i4\varphi} = 16e^{i\pi}$ $\Rightarrow r=2$ $\quad 4\varphi = \pi+2\pi k$ $\Rightarrow \varphi = \frac{\pi}{4}+\frac{\pi}{2}k$ Lösningar: $2\cdot e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}$.

c) $z^6=-9$: $\quad z^6 = r^6e^{i6\varphi} = 9e^{i\pi}$, $\quad r^3=3$ $\quad r=\sqrt[3]{3}$ $\quad \varphi = \frac{\pi}{6}+\frac{2\pi}{6}k$ Lösningar: $\sqrt[3]{3}e^{i\pi/6}, \sqrt[3]{3}e^{i3\pi/6}, \sqrt[3]{3}e^{i5\pi/6}, \dots, \sqrt[3]{3}e^{i11\pi/6}$

d) $z^6-z^3-2=0$ $\quad w=z^3$ $\quad w^2-w-2=0$ $\quad w = \frac{1}{2}\pm\sqrt{\frac{1}{4}+2} = \frac{1}{2}\pm\sqrt{\frac{9}{4}} = \frac{1}{2}\pm\frac{3}{2}$ $\quad w_1 = \frac{4}{2} = 2$, $w_2 = \frac{-2}{2} = -1$ $\quad z^3 = 2 \Rightarrow z = \underbrace{\sqrt[3]{2}\cdot e^{i2\pi/3}, \sqrt[3]{2}e^{i4\pi/3}, \sqrt[3]{2}}_{\text{Lösn.}}$.

$z^3 = -1 = e^{i\pi}$ $\quad z = \underbrace{e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}}_{\text{Lösn.}}$

## 1.22 Bevisa Prop 1.5 : $\forall z_1, z_2$ (låt $z_j = x_j+iy_j$)

a) $\overline{z_1\pm z_2} = \overline{(x_1+iy_1)\pm(x_2+iy_2)} = \overline{(x_1\pm x_2)+i(y_1\pm y_2)} = (x_1\pm x_2)-i(y_1\pm y_2) = (x_1-iy_1)\pm(x_2-iy_2) = \bar{z}_1\pm\bar{z}_2$

b) $\overline{z_1\cdot z_2} = \bar{z}_1\cdot\bar{z}_2$ (se boken)

c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$. $\quad z_2\neq 0$ Låt $w_2 = \frac{1}{z_2}$ om $\bar{w}_2 = \frac{1}{\bar{z}_2}$ är vi klara eftersom $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{z_1\cdot w_2} \overset{(b)}{=} \bar{z}_1\cdot\bar{w}_2 = \frac{\bar{z}_1}{\bar{z}_2}$

$w_2 = \frac{x_2 - iy_2}{x_2^2 + y_2^2}$ $\quad \overline{w_2} = \frac{x_2 + iy_2}{x_2^2 + y_2^2}$. Men $\frac{1}{\overline{z_2}} = \frac{x_2 + iy_2}{x_2^2 + y_2^2}$.

d) $\overline{\overline{z_1}} = \overline{\overline{x_1 + iy_1}} = \overline{x_1 - iy_1} = x_1 + iy_1 = z_1$

e) $|\overline{z_1}| = |x_1 - iy_1| = \sqrt{x_1^2 + y_1^2} = |z_1|$

f) $|z_1|^2 = x_1^2 + y_1^2 = (x_1 + iy_1)(x_1 - iy_1) = z_1 \overline{z_1}$.

g) $\frac{1}{2}(z_1 + \overline{z_1}) = \frac{1}{2}(x_1 + iy_1 + x_1 - iy_1) = x_1 = \mathrm{Re}(z_1)$

h) $\frac{1}{2}(z_1 - \overline{z_1}) = \mathrm{Im}(z_1)$

i) $\overline{e^{i\varphi}} = \overline{\cos\varphi + i\sin\varphi} = \cos\varphi - i\sin\varphi = \cos(-\varphi) + i\sin(-\varphi) = e^{-i\varphi}$.

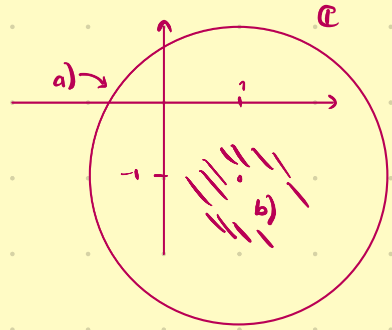
**1.23** Skissa följande delmängder i komplexa talplanet:

a) $\{z \in \mathbb{C} : |z - 1 + i| = 2\} = \{z \in \mathbb{C} : |z - (1 - i)| = 2\}$

b) $\{z \in \mathbb{C} : |z - 1 + i| \leq 2\}$

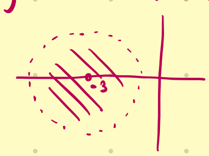
c) $\{z \in \mathbb{C} : \mathrm{Re}(z + 2 - 2i) = 3\} = \{z \in \mathbb{C} : \mathrm{Re}(z) + 2 = 3\}$

$= \{z \in \mathbb{C} : \mathrm{Re}(z) = 1\}$




**1.27** Skissa mängderna definierade genom följande olikheter och avgör om de är öppna, slutna, eller varken eller och om de är begränsade och eller sammanhängande.

a) $|x + 3| < 2$

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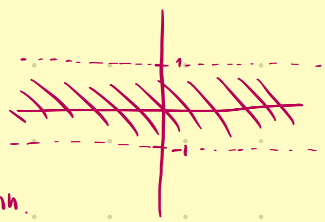
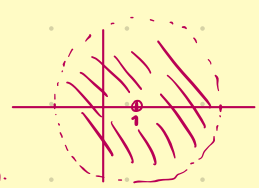
b) $|\mathrm{Im}\, z| < 1$




Öppen, obeg, sammanh.

c) $0 < |z - 1| < 2$

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**1.33** Hitta en parametrisering av kurvorna:

a) Cirkeln $C[1+i, 1]$, moturs : $\gamma(t) = 1 + i + e^{it}$ $\quad 0 \leq t \leq 2\pi$.

b) Linjesegmentet från $-1-i$ till $2i$ : $\gamma(t) = (1-t)(-1-i) + t \cdot (2i)$ $\quad 0 \leq t \leq 1$.