Commutation relations for two-sided radial SLE

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Abstract

We study the commutation relation for 2-radial SLE in the unit disc starting from two boundary points. We follow the framework introduced by Dubédat [9]. Under an additional requirement of the interchangeability of the two curves, we classify all locally commuting 2-radial SLE_κ for $\kappa \in (0,8)$: it is either a two-sided radial SLE_κ with spiral of constant spiraling rate or a chordal SLE_κ weighted by a power of the conformal radius of its complement. Namely, for fixed κ and starting points, we have exactly two one-parameter continuous families of locally commuting 2-radial SLE. Two-sided radial SLE with spiral is a generalization of two-sided radial SLE (without spiral) analyzed in [13, 14, 20, 23] and satisfies the resampling property. We also discuss the semiclassical limit of the commutation relation as $\kappa \to 0$. In particular, we show that the limit for the second family with an appropriately chosen power of conformal radius is a chord that minimizes a modified chordal Loewner energy, which is unique only when the endpoints are not antipodal.

Keywords: Schramm–Loewner evolution, commutation relation, resampling property.
MSC: 60J67.

1 Introduction

1.1 Background on radial SLE

In 1999, O. Schramm [32] introduced the Schramm-Loewner evolution (SLE) as the non-self-crossing random curve driven by a multiple of Brownian motion using Loewner's transform. This definition is motivated by a quest to describe mathematically the random interfaces in two-dimensional critical lattice models, which satisfy the *conformal invariance* and the *domain Markov property.* These properties impose that the chordal Loewner driving function of such a random interface has to be a multiple of Brownian motion, hence justifying the definition of chordal SLE. Indeed, SLEs are proved to be

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the scaling limit of interfaces in many conformally invariant statistical mechanics models, e.g., [6, 7, 22, 33, 34, 37], and play a central role in random conformal geometry.

However, to characterize the other natural variant — radial SLE — one needs an additional condition on the reflection symmetry. As we are mainly concerned with radial SLE in the present article, let us briefly describe its definition and characterization. We will describe the radial Loewner chain in $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ targeting at 0. Radial Loewner chain in other simply connected domain $D \subset \mathbb{C}$ targeting at an interior point $z_0 \in D$ is defined via a conformal map from $\mathbb{D}$ onto D sending 0 to z_0.

Conformal radius and capacity. For any compact subset K (not necessarily connected) of $\overline{\mathbb{D}}$ such that $\mathbb{D} \setminus K$ is simply connected and contains 0, we let g_K be the unique conformal map $\mathbb{D} \setminus K \to \mathbb{D}$ such that $g_K(0) = 0$ and $g_K'(0) > 0$ (called *the radial mapping-out function* of K). The *conformal radius* of $\mathbb{D} \setminus K$ is

$$\mathrm{CR}(\mathbb{D} \setminus K) := (g_K'(0))^{-1}.$$

The *capacity* of K is

$$\mathrm{cap}(K) = \log g_K'(0) = -\log \mathrm{CR}(\mathbb{D} \setminus K).$$

Radial Loewner chain. For $\theta \in [0, 2\pi)$, suppose $\eta : [0, T] \to \overline{\mathbb{D}}$ is a continuous non-self-crossing curve such that $\eta_0 = \mathrm{e}^{\mathrm{i}\theta}$ and $\eta_{(0,T)} \subset \mathbb{D} \setminus \{0\}$. Let D_t be the connected component of $\mathbb{D} \setminus \eta_{[0,t]}$ containing the origin. Let $g_t : D_t \to \mathbb{D}$ be the unique conformal map with $g_t(0) = 0$ and $g_t'(0) > 0$. We say that the curve is parameterized by capacity if $g_t'(0) = \exp(t)$. Then g_t satisfies the radial Loewner equation

$$\partial_t g_t(z) = g_t(z) \frac{\mathrm{e}^{\mathrm{i}\xi_t} + g_t(z)}{\mathrm{e}^{\mathrm{i}\xi_t} - g_t(z)}, \quad g_0(z) = z, \tag{1.1}$$

where $t \mapsto \xi_t \in \mathbb{R}/2\pi\mathbb{Z}$ is continuous and called the *driving function* of η. We note that if $z = \mathrm{e}^{\mathrm{i}V_0} \in \partial\mathbb{D}$, then taking a continuous branch $V_t := \arg g_t(\mathrm{e}^{\mathrm{i}\theta})$, we have

$$\partial_t V_t(z) = \cot((V_t - \xi_t)/2). \tag{1.2}$$

Characterization of radial SLE. Consider a family $(\mathbb{P}_\theta)_{\theta \in \mathbb{R}/2\pi\mathbb{Z}}$ of probability distributions on curves $\eta : [0, \infty) \to \overline{\mathbb{D}}$ with $\eta_0 = \mathrm{e}^{\mathrm{i}\theta}$ and parametrized by capacity satisfies the following properties:

1. (*Conformal invariance*) For all $a \in \mathbb{R}/2\pi\mathbb{Z}$, let $\rho_a(z) = \mathrm{e}^{\mathrm{i}a} z$ be the rotation map $\mathbb{D} \to \mathbb{D}$. For all $a, \theta \in \mathbb{R}/2\pi\mathbb{Z}$, the pullback measure $\rho_a^* \mathbb{P}_\theta = \mathbb{P}_{\theta - a}$. From this, we may extend the definition of $\mathbb{P}_\theta$ to any simply connected domain D with an interior marked point z_0 by pulling back using a uniformizing conformal map $D \to \mathbb{D}$ sending z_0 to 0.
2. (*Domain Markov property*) For any $t > 0$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, let $\eta \sim \mathbb{P}_\theta$. Conditioning on $\eta|_{[0,t]}$, $g_t(\eta_{[t,\infty)}) \sim \mathbb{P}_{\xi_t}$ where $\xi_\cdot$ is the driving function of η. See Figure 1.
3. (*Reflection symmetry*) Let $\iota : z \mapsto \overline{z}$ be the complex conjugation, then $\mathbb{P}_\theta \sim \iota^* \mathbb{P}_{-\theta}$.

Then there exists $\kappa \geq 0$ such that for all θ, the driving function of $\eta \sim \mathbb{P}_\theta$ is $t \mapsto \theta + \sqrt{\kappa} B_t$ modulo $2\pi\mathbb{Z}$, where $B.$ is the standard Brownian motion. This follows from the fact that $(\sqrt{\kappa} B)_{\kappa \geq 0}$ are the only continuous Lévy processes W that have the same law under the transformation $W \mapsto -W$. In this case, as $t \to \infty$, $\eta_t \to 0$ almost surely. The distribution $\mathbb{P}_\theta$ is the law of the *radial* SLE$_\kappa$ in $\mathbb{D}$ starting from $e^{i\theta}$. We also call it radial SLE$_\kappa$ in $(\mathbb{D}; e^{i\theta}; 0)$ for short.

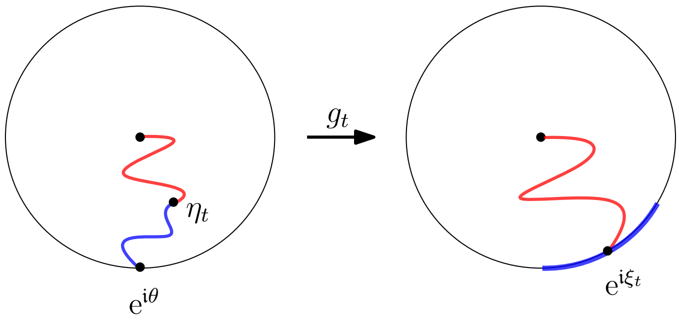

Figure 1: Domain Markov property of radial SLE.

The third assumption on the reflection symmetry is natural for conformally invariant and achiral statistical mechanics models. However, one may also wonder what happens without the reflection symmetry, this means that the law of the driving function is no longer required to be invariant with respect to $W \mapsto -W$. From the classification of continuous Lévy processes, we obtain that there exists $\kappa \geq 0$ and $\mu \in \mathbb{R}$ such that for all θ, the driving function of $\eta \sim \mathbb{P}_\theta$ is

$$t \mapsto \theta + \sqrt{\kappa} B_t + \mu t \qquad (\mathrm{mod})\ 2\pi\mathbb{Z}.$$

The curve generated is called *radial SLE$_\kappa$ with spiraling rate μ* starting from $e^{i\theta}$ and we denote it as radial SLE$_\kappa^\mu$ in $(\mathbb{D}; e^{i\theta}; 0)$. It was shown in [28] that almost surely, $\eta_t \to 0$ as $t \to \infty$ for all $\kappa > 0$ and $\mu \in \mathbb{R}$.

1.2 Locally commuting 2-radial SLE

The random radial curve satisfying conformal invariance and domain Markov property are easily characterized thanks to the fact all simply connected domains (excluding $\mathbb{C}$) with one marked interior point and one marked boundary point are conformally equivalent. If we consider the radial Loewner chain of two curves growing from two distinct boundary points $e^{i\theta_1}, e^{i\theta_2} \in \partial\mathbb{D}$, then we have a one-dimensional moduli space of the boundary data. The goal of this work is to characterize and identify the family of local laws $\mathbb{P}_{(\theta_1,\theta_2)}$ of the pair of continuous non-self-crossing curves $(\eta^{(1)}, \eta^{(2)})$ in $\mathbb{D}$ starting from all possible choices of $e^{i\theta_1}, e^{i\theta_2} \in \partial\mathbb{D}$ under the following axioms.

For this, we first parameterize $\eta^{(1)}, \eta^{(2)}$ by their intrinsic radial capacities and consider the associated Loewner chains $g^{(1)}_.$ and $g^{(2)}_.$ respectively. For all $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_{>0}$, let $g_{\mathbf{t}}$ be the radial mapping-out function of $\eta_{\mathbf{t}} := \eta^{(1)}_{[0,t_1]} \cup \eta^{(2)}_{[0,t_2]}$.

(CI) Conformal invariance: For all $a \in \mathbb{R}/2\pi\mathbb{Z}$ and $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$, the pullback measure $\rho_a^* \mathbb{P}_{(\theta_1,\theta_2)} = \mathbb{P}_{(\theta_1-a,\theta_2-a)}$. From this, we may extend the definition of $\mathbb{P}_{\theta_1,\theta_2}$ to any simply connected domain D with an interior marked point z_0 by pulling back using a uniformizing conformal map $D \to \mathbb{D}$ sending z_0 to 0.

(DMP) Domain Markov property: Conditioning on $\eta_{\mathbf{t}}$,

$$\left(g_{\mathbf{t}}(\eta^{(1)} \setminus \eta^{(1)}_{[0,t_1]}), g_{\mathbf{t}}(\eta^{(2)} \setminus \eta^{(2)}_{[0,t_2]})\right) \sim \mathbb{P}_{\left(\theta^{(1)}_{\mathbf{t}},\theta^{(2)}_{\mathbf{t}}\right)},$$

where $(\exp(\mathrm{i}\theta^{(1)}_{\mathbf{t}}), \exp(\mathrm{i}\theta^{(2)}_{\mathbf{t}})) = (g_{\mathbf{t}}(\eta^{(1)}_{t_1}), g_{\mathbf{t}}(\eta^{(2)}_{t_2}))$.

(MARG) Marginal laws: There exists $\kappa > 0$ such that the marginal local law of $\eta^{(j)}$ is "absolutely continuous" with respect to an SLE_κ for $j = 1, 2$.

(INT) Interchangeability condition: Let τ be the map which swaps $\eta^{(1)}$ and $\eta^{(2)}$. We have $\mathbb{P}_{(\theta_1,\theta_2)} = \tau^* \mathbb{P}_{(\theta_2,\theta_1)}$.

See Section 2.1 for a more precise statement of the axioms. We note that since $g_{\mathbf{t}}$ is parametrized by two times t_1, t_2, the condition **(DMP)** implies that $g_{\mathbf{t}}$ may be realized by first mapping out $\eta^{(1)}_{[0,t_1]}$ using $g^{(1)}_{t_1}$, then mapping out $g^{(1)}_{t_1}(\eta^{(2)}_{[0,t_2]})$, or vice versa. The image has the same law regardless of the order in which we map out the curves. This observation gives a **commutation relation** on the infinitesimal generators of the two curves (Proposition 2.2). Therefore, we call the family of the laws $(\mathbb{P}_{(\theta_1,\theta_2)})$ as an *interchangeable and locally commuting 2-radial SLE*$_\kappa$.

These conditions and our analysis are very close to the commutation relation of SLE studied by Dubédat in [9], which focuses on classifying all locally commuting chordal SLEs. Dubédat also derived the commutation relation in the radial setting in terms of the generators of radial SLE. Our main contribution is to find all solutions to the radial commutation relation and identify all interchangeable and locally commuting 2-radial SLE_κ. It is also natural to use SLE/GFF couplings, particularly, [28], to find examples of commuting 2-radial SLEs. See Remark 3.10. However, it is unclear to the authors a priori that *all* commuting 2-radial SLEs can be coupled to GFF.

1.3 Main result: A classification

We classify all locally commuting 2-radial SLE with the interchangeability condition **(INT)**. Similar to the analysis in [9], we also show that the conditions **(CI)**, **(DMP)**, and **(MARG)** imply that there exists $\mathcal{Z} : \{(\theta_1, \theta_2) \in \mathbb{R}^2 \,|\, \theta_1 < \theta_2 < \theta_1 + 2\pi\} \to \mathbb{R}_{>0}$, called *partition function* which encodes the family of distributions $\mathbb{P}_{(\theta_1,\theta_2)}$. See Section 2 for more details.

Theorem 1.1. *Fix $\kappa \in (0, 8)$. Suppose $\mathcal{Z}$ is the partition function for an interchangeable and locally commuting 2-radial* SLE_κ*. Then $\mathcal{Z}$ is one of the following two functions:*

1. *There exists $\mu \in \mathbb{R}$ such that, up to a multiplicative constant, $\mathcal{Z}$ is the same as*

$$\mathcal{G}_\mu(\theta_1, \theta_2) = (\sin((\theta_2 - \theta_1)/2))^{2/\kappa} \exp\left(\frac{\mu}{\kappa}(\theta_1 + \theta_2)\right). \tag{1.3}$$

In this case, the law of the corresponding locally commuting 2-radial SLE_κ *is the same as two-sided radial* SLE_κ *with spiraling rate* μ*, see Section 3.2 and Figure 2.*

2. *There exists* $\alpha < 1 - \kappa/8$ *such that, up to a multiplicative constant,* $\mathcal{Z}$ *is the same as*

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = \left(\sin\left((\theta_2 - \theta_1)/2\right)\right)^{(\kappa-6)/\kappa} \phi_\alpha\left(\left(\sin\left((\theta_2 - \theta_1)/4\right)\right)^2\right), \tag{1.4}$$

where ϕ_α *is the unique solution to the following Euler's hypergeometric differential equation*

$$\begin{cases} u(1-u)\phi''(u) - \frac{3\kappa-8}{2\kappa}(2u-1)\phi'(u) + \frac{8\alpha}{\kappa}\phi(u) = 0, \quad u \in (0,1); \\ \phi(1/2) = 1, \quad \phi'(1/2) = 0. \end{cases} \tag{1.5}$$

In this case, the law of the corresponding locally commuting 2-radial SLE_κ *is the same as* γ *chordal* SLE_κ *in* $\mathbb{D}$ *weighted by* $\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}$*, where* $\mathrm{CR}(\mathbb{D}\setminus\gamma)$ *denotes the conformal radius of the connected component of* $\mathbb{D}\setminus\gamma$ *containing the origin, see Section 3.4.*

Both $\mathcal{G}_\mu$ in (1.3) and $\mathcal{Z}_\alpha$ in (1.4) are solutions to "radial BPZ equations" (or "BPZ-Cardy equations" in [18], see also [19]):

$$\frac{\kappa}{2}\frac{\partial_{11}\mathcal{Z}}{\mathcal{Z}} + \cot\left((\theta_2-\theta_1)/2\right)\frac{\partial_2\mathcal{Z}}{\mathcal{Z}} - \frac{(6-\kappa)/(4\kappa)}{\left(\sin((\theta_2-\theta_1)/2)\right)^2} = F,$$
$$\frac{\kappa}{2}\frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} - \cot\left((\theta_2-\theta_1)/2\right)\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} - \frac{(6-\kappa)/(4\kappa)}{\left(\sin((\theta_2-\theta_1)/2)\right)^2} = F,$$

where ∂_i is the partial derivative with respect to θ_i and F is the constant:

$$F = \frac{\mu^2-3}{2\kappa} \geq -\frac{3}{2\kappa}, \qquad \text{when } \mathcal{Z} = \mathcal{G}_\mu,$$
$$F = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \alpha > -\frac{3}{2\kappa}, \qquad \text{when } \mathcal{Z} = \mathcal{Z}_\alpha.$$

In general, partition function $\mathcal{Z}_\alpha$ in (1.4) involves hypergeometric function and is not explicit; but in the following two special cases, such function has a simple explicit expression:

- When $\kappa = 4$, $\mathcal{Z}_\alpha$ can be written as trigonometric functions and hyperbolic functions in $\theta = \theta_2 - \theta_1$, see Remark 2.6.
- When $\alpha = \alpha_1(\kappa)$ which is the one-arm exponent for the conformal loop ensemble, $\mathcal{Z}_{\alpha_1(\kappa)}$ can be written as trigonometric functions in $\theta = \theta_2 - \theta_1$, see Remark 2.7. In this case, the partition function $\mathcal{Z}_{\alpha_1(\kappa)}$ with $\kappa \in (4, 8)$ is the limit of the partition function for interfaces in critical random-cluster model conditional on the one-arm event.

The two-sided radial SLE_κ with spiral is also closely related to the mixed multifractal spectrum introduced by Binder [5]. More precisely, the points on an SLE curve are classified in terms of the asymptotic behavior of the uniformizing conformal maps in the complement of the curve, according to its Hölder exponent and its winding exponent.

The Hausdorff dimension (i.e., mixed multifractal spectrum) of the set of points with a given behavior is computed in [11]. It was shown in [23] that conditioning a chordal SLE to pass through an interior point 0 gives the two-sided radial SLE (without spiral). Since the points with zero winding exponent have the maximal spectrum, this conditioning is equivalent to conditioning on the event where the 0 is a point on the SLE curve with zero winding exponent. We speculate that "conditioning" on the rare event where 0 is a point with winding rate μ, the curve obtained should satisfy the commutation relation and hence has to be the two-sided radial SLE that with the spiraling rate μ. In the terminology of the conformal field theory, this would correspond to inserting a curve-generating operator at 0 with complex charges and conformal weights [4].

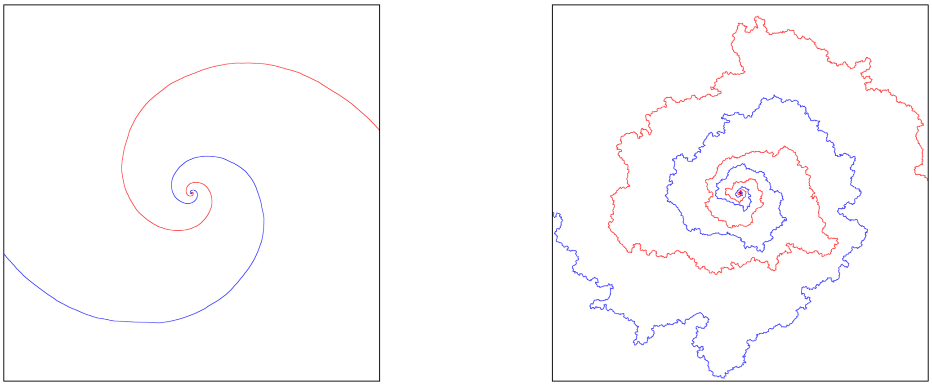

Figure 2: Simulation by Minjae Park. Two-sided radial SLE around the origin with $\kappa = 0.01$ and $\mu = 5$ in the left panel. Two-sided radial SLE around the origin with $\kappa = 2$ and $\mu = 5$ in the right panel.

The interchangeability condition **(INT)** is natural, given the reversibility of chordal SLE. It also simplifies our classification and in particular, the notation, see Section 2.3 and, in particular, Lemma 2.5. In Remark 3.16, we give an example of locally commuting radial 2-SLE without **(INT)**. In Proposition 3.17, we show the classification without the condition **(INT)**, where the second one-parameter family of solution (1.4) becomes a two-parameter family.

1.4 Resampling property

Let $D \subset \mathbb{C}$ be a simply connected domain and let x_1, x_2 be distinct prime ends of ∂D. We denote by $\mathfrak{X}(D; x_1, x_2)$ the set of continuous non-self-crossing curves in D from x_1 to x_2. We recall the definition of the chordal SLE in the upper half-plane $(\mathbb{H}; 0, \infty)$ in Appendix B. For other simply connected domain $(D; x_1, x_2)$, chordal SLE in $(D; x_1, x_2)$ is a random curve in $\mathfrak{X}(D; x_1, x_2)$ defined by mapping chordal SLE in $(\mathbb{H}; 0, \infty)$ conformally from $\mathbb{H}$ onto D sending 0 to x_1 and ∞ to x_2. Let $D \subset \mathbb{D}$ be a simply connected domain containing 0 and let x_1, x_2 be distinct prime ends of ∂D. We denote by $\mathfrak{X}(D; x_1, x_2; 0)$ the set of pairs of continuous simple curves $(\eta^{(1)}, \eta^{(2)})$ in D such that $\eta^{(1)}$ goes from x_1

to 0 and $\eta^{(2)}$ goes from x_2 to 0 and $\eta^{(1)} \cap \eta^{(2)} = \{0\}$. We say that a law on $(\eta^{(1)}, \eta^{(2)}) \in \mathfrak{X}(D; x_1, x_2; 0)$ satisfies *resampling property* if

- the conditional law of $\eta^{(2)}$ given $\eta^{(1)}$ is a chordal SLE_κ in $(D \setminus \eta^{(1)}; x_2, 0)$,
- and the conditional law of $\eta^{(1)}$ given $\eta^{(2)}$ is a chordal SLE_κ in $(D \setminus \eta^{(2)}; x_1, 0)$.

In Theorem 3.9 we show that for each $\mu \in \mathbb{R}$, two-sided radial SLE_κ with spiraling rate μ satisfies resampling property for $\kappa \in (0, 4]$. This result follows directly from the expression of the partition function $\mathcal{G}_\mu$. Another proof can be found in [28] using the coupling with GFF. Combining with Theorem 1.1, we find that commutation relation implies resampling property.

Corollary 1.2. *Fix $\kappa \in (0, 4]$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. An interchangeable and locally commuting 2-radial SLE_κ in $\mathfrak{X}(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}; 0)$ is a two-sided radial SLE_κ with spiral, hence, satisfies the resampling property. The converse is not true as we may take a linear combination of 2-radial SLE_κ with different spiraling rate.*

1.5 Semiclassical limits of partition functions

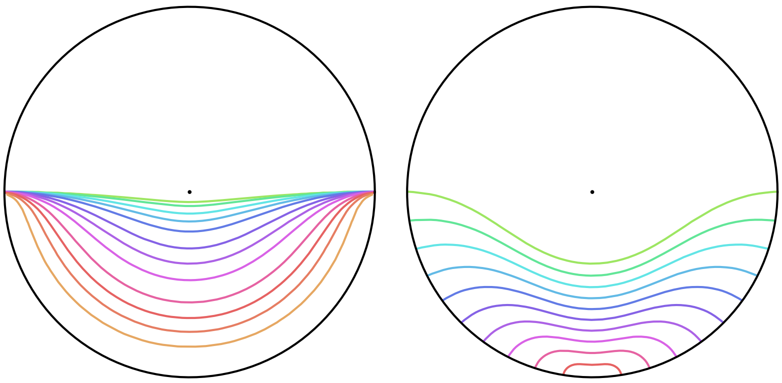

Figure 3: The curves generated by $\mathcal{U}^\lambda$ from Proposition 1.3. They correspond to the energy minimizers from Proposition 4.8. On the left, $\theta_2 - \theta_1 \approx \pi$, and $\lambda = 0.1,\ 0.2,\ 0.5,\ 1,\ 2,\ 5,\ 10,\ 20,\ 50,\ 100,\ 200,\ 500$ ($\lambda = 0.1$ corresponds to the innermost green curve and $\lambda = 500$ corresponds to the outermost orange curve). On the right $\lambda = 10$ while $\theta_2 - \theta_1$ varies.

We also discuss the commutation relation when $\kappa = 0$, the corresponding deterministic pair of curves, and the semiclassical limit $\kappa \to 0+$ in Section 4. In particular, semiclassical limits of partition functions in Theorem 1.1 have explicit formulas.

Proposition 1.3. *Fix $\theta_1 < \theta_2 < \theta_1 + 2\pi$ and denote $\theta = \theta_2 - \theta_1$.*

- *For the partition function $\mathcal{G}_\mu$ in (1.3), fix $\mu \in \mathbb{R}$, we have*

$$\lim_{\kappa \to 0} \kappa \log \mathcal{G}_\mu(\theta_1, \theta_2) = 2 \log \sin\left((\theta_2 - \theta_1)/2\right) + \mu(\theta_1 + \theta_2). \tag{1.6}$$

- *For the partition function $\mathcal{Z}_\alpha$ in (1.4), if $\alpha \sim -\lambda/\kappa$ for some $\lambda \geq 0$, then the following limit exists*

$$\mathcal{U}^\lambda(\theta) = \lim_{\kappa \to 0} \kappa \log \mathcal{Z}_\alpha(\theta_1, \theta_2);$$

and for $\theta \in (0, \pi]$,

$$\mathcal{U}^\lambda(\theta) = \mathcal{U}^\lambda(2\pi - \theta) = -2 \log \sin(\theta/2) + \int_\theta^\pi \sqrt{2\lambda + 4\cot^2(u/2)}\, \mathrm{d}u. \tag{1.7}$$

We mention that a similar semiclassical limit of the partition functions for multichordal SLE was considered in [3, 29] and that multiradial SLE_0 was considered in [2, 40]. We prove Proposition 1.3 in Section 4.

Proposition 1.4. *We use the same notations as in Proposition 1.3. The function $\mathcal{U}^\lambda$ can be expressed as*

$$\mathcal{U}^\lambda(\theta) = -6 \log \sin(\theta/2) - \inf_\gamma \Big(I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma) \Big) + C$$

where the infimum is attained and taken over all chords connecting $\mathrm{e}^{\mathrm{i}\theta_1}$ and $\mathrm{e}^{\mathrm{i}\theta_2}$, $I(\cdot)$ is the chordal Loewner energy, and C is a normalizing constant only depending on λ such that $\mathcal{U}^\lambda(\pi) = 0$. Moreover, the minimizer is unique when $\theta \neq \pi$ and there are exactly two minimizers when $\theta = \pi$.

See Lemma 4.7 for details and Figure 3 for a simulation of those energy minimizers. A related relation between the semiclassical limit of the partition function for multichordal SLE and the minimizers of the multichordal Loewner energy was proved in [29]. We point out in particular that $\mathcal{U}^\lambda$ in (1.7) when $\lambda > 0$ is not differentiable at $\theta = \pi$ since the minimizer of $I(\cdot) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \cdot)$ is non-unique when $\theta = \pi$. Such non-differentiable functions do not appear in earlier literature about the semiclassical limits of SLE partition functions [2, 3, 29, 40].

2 Partition functions of locally commuting 2-radial SLE

In this section, we give the precise definition of locally commuting 2-radial SLE. We also define and classify the partition functions.

2.1 Definition of locally commuting 2-radial SLE

Let $D \subset \mathbb{D}$ be a simply connected domain containing 0 and let x_1, x_2 be distinct prime ends of D. Let U_1, U_2 be, respectively, closed neighborhoods of x_1 and x_2 in D that do not contain 0 and such that $U_1 \cap U_2 = \emptyset$. We will consider probability measures $\mathbb{P}^{(U_1,U_2)}_{(D;x_1,x_2)}$ on pairs of unparametrized continuous curves in U_1 and U_2 starting from x_1 and x_2, and exiting U_1 and U_2 almost surely. We call that such a family of measures indexed by different choices of (U_1, U_2) *compatible* if for all $U_1 \subset U_1'$ and $U_2 \subset U_2'$, we have $\mathbb{P}^{(U_1,U_2)}_{(D;x_1,x_2)}$

is obtained from restricting the curves under $\mathbb{P}^{(U_1',U_2')}_{(D;\,x_1,x_2)}$ to the part before first exiting the subdomains U_1 and U_2.

The *locally commuting* 2-*radial* SLE_κ is a compatible family of measures $\mathbb{P}^{(U_1,U_2)}_{(D;\,x_1,x_2)}$ on pairs of continuous non-self-crossing curves $(\eta^{(1)},\eta^{(2)})$ for all D, (x_1,x_2), and (U_1,U_2) as above that satisfy additionally the **(CI),(DMP),(MARG)** conditions below. We say that a locally commuting 2-radial SLE_κ is *interchangeable* if it satisfies further the condition **(INT)**.

(CI) **Conformal invariance:** If $\varphi : D \to D'$ is a conformal map fixing 0, then the pullback measure

$$\varphi^*\mathbb{P}^{(\varphi(U_1),\varphi(U_2))}_{(D';\,\varphi(x_1),\varphi(x_2))} = \mathbb{P}^{(U_1,U_2)}_{(D;\,x_1,x_2)}.$$

Therefore, it suffices to describe the measure when $(D;\,x_1,x_2) = (\mathbb{D};\,\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})$.

(DMP) **Domain Markov property:** Let $(\eta^{(1)},\eta^{(2)}) \sim \mathbb{P}^{(U_1,U_2)}_{(\mathbb{D};\,\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})}$ and we parametrize $\eta^{(1)}$ and $\eta^{(2)}$ by their own capacity in $\mathbb{D}$. Let $\mathbf{t} = (t_1,t_2)$, such that t_j is a stopping time for $\eta^{(j)}$ and $\eta^{(j)}_{[0,t_j]}$ is contained in the interior of U_j. Let

$$\tilde{U}_j = U_j \setminus \eta^{(j)}_{[0,t_j]},\quad \tilde{\eta}^{(j)} = \eta^{(j)} \setminus \eta^{(j)}_{[0,t_j]},\quad j=1,2;\quad \tilde{D} = \mathbb{D} \setminus (\eta^{(1)}_{[0,t_1]} \cup \eta^{(2)}_{[0,t_2]}).$$

Then conditionally on $\eta^{(1)}_{[0,t_1]} \cup \eta^{(2)}_{[0,t_2]}$, we have

$$(\tilde{\eta}^{(1)},\tilde{\eta}^{(2)}) \sim \mathbb{P}^{(\tilde{U}_1,\tilde{U}_2)}_{(\tilde{D};\,\eta^{(1)}_{t_1},\eta^{(2)}_{t_2})}.$$

See Figure 4.

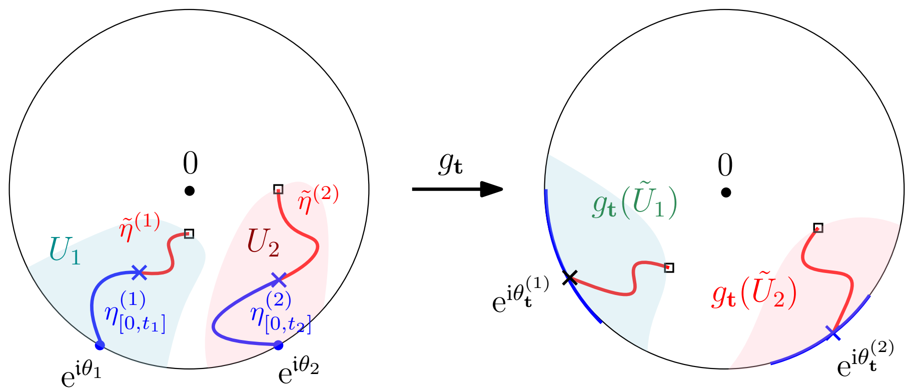

Figure 4: **(CI)** and **(DMP)** imply that a locally commuting 2-radial SLE satisfies $(\tilde{\eta}^{(1)},\tilde{\eta}^{(2)}) \sim \mathbb{P}^{(\tilde{U}_1,\tilde{U}_2)}_{(\tilde{D};\,\eta^{(1)}_{t_1},\eta^{(2)}_{t_2})} \sim g^*_{\mathbf{t}}\,\mathbb{P}^{(g_{\mathbf{t}}(\tilde{U}_1),g_{\mathbf{t}}(\tilde{U}_2))}_{(\mathbb{D};\mathrm{e}^{\mathrm{i}\theta^{(1)}_{\mathbf{t}}},\mathrm{e}^{\mathrm{i}\theta^{(2)}_{\mathbf{t}}})}$.

(MARG) **Marginal laws:** Let $(\eta^{(1)},\eta^{(2)}) \sim \mathbb{P}^{(U_1,U_2)}_{(\mathbb{D};\,\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})}$. We assume that there exist C^2 functions $b_j : S^1 \times S^1 \setminus \Delta \to \mathbb{R}$ where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and Δ is the diagonal $\{(\theta,\theta)\,|\,\theta \in S^1\}$

such that the capacity parametrized Loewner driving function $t \mapsto \xi_t^{(1)}$ of $\eta^{(1)}$ satisfies

$$\begin{cases} \xi_0^{(1)} = \theta_1, \quad V_0^{(2)} = \theta_2, \\ \mathrm{d}\xi_t^{(1)} = \sqrt{\kappa}\,\mathrm{d}B_t^{(1)} + b_1\left(\xi_t^{(1)}, V_t^{(2)}\right)\mathrm{d}t, \\ \mathrm{d}V_t^{(2)} = \cot\left((V_t^{(2)} - \xi_t^{(1)})/2\right)\mathrm{d}t, \end{cases} \tag{2.1}$$

where $B^{(1)}$ is one-dimensional standard Brownian motion. In other words, the radial Loewner chain $g_t^{(1)}$ associated with $\eta^{(1)}$ maps the tip $\eta_t^{(1)}$ to $\exp(\mathrm{i}\xi_t^{(1)})$ and $\mathrm{e}^{\mathrm{i}\theta_2}$ to $\exp(\mathrm{i}V_t^{(2)})$ by (1.2).

Similarly, the capacity parametrized Loewner driving function $t \mapsto \theta_t^{(2)}$ of $\eta^{(2)}$ satisfies

$$\begin{cases} V_0^{(1)} = \theta_1, \quad \xi_0^{(2)} = \theta_2, \\ \mathrm{d}\xi_t^{(2)} = \sqrt{\kappa}\,\mathrm{d}B_t^{(2)} + b_2(V_t^{(1)}, \xi_t^{(2)})\mathrm{d}t, \\ \mathrm{d}V_t^{(1)} = \cot\left((V_t^{(1)} - \xi_t^{(2)})/2\right)\mathrm{d}t. \end{cases} \tag{2.2}$$

In other words, the radial Loewner chain $g_t^{(2)}$ associated with $\eta^{(2)}$ maps the tip $\eta_t^{(2)}$ to $\exp(\mathrm{i}\xi_t^{(2)})$ and $\mathrm{e}^{\mathrm{i}\theta_1}$ to $\exp(\mathrm{i}V_t^{(1)})$.

(INT) **Interchangeability condition:** The two curves are unordered. In other words, let $\tau : (\eta^{(1)}, \eta^{(2)}) \mapsto (\eta^{(2)}, \eta^{(1)})$,

$$\mathbb{P}_{(D;\,x_1,x_2)}^{(U_1,U_2)} \sim \tau^* \mathbb{P}_{(D;\,x_2,x_1)}^{(U_2,U_1)}.$$

Remark 2.1. Despite the heavy notation, the only purpose of introducing the neighborhoods U_1, U_2 is to give a precise meaning of "local law" of the 2-radial SLE near their starting points. In particular, if we have a random pair of curves in $\mathfrak{X}(\mathbb{D}; x_1, x_2; 0)$, or a random simple curve in $\mathfrak{X}(\mathbb{D}; x_1, x_2)$, we obtain a compatible family of probability measures $\mathbb{P}_{(\mathbb{D};x_1,x_2)}^{(U_1,U_2)}$ by restricting the random curve (or the pair of random curve) to all possible pairs of neighborhoods (U_1, U_2). Note that a priori, a compatible family of local laws does not necessarily imply they can be coupled as the restriction of a random curve or pair of curves in all (U_1, U_2), but this will be a consequence of our classification Theorem 1.1.

To simplify notations, we also denote $\mathbb{P}_{(\mathbb{D};\,\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})}^{(U_1,U_2)}$ by $\mathbb{P}_{(\theta_1,\theta_2)}$ as in Section 1.2.

These axioms allow us to characterize the local law of the 2-radial SLE by considering the infinitesimal generators of the two-time driving function which is the subject of the next sections.

2.2 Commutation relations

In this section, we assume that $\kappa \in (0, \infty)$ and do not assume **(INT)**. We consider a *locally commuting* 2*-radial* SLE$_\kappa$, which is a compatible family of measures $\mathbb{P}_{(\theta_1,\theta_2)}$ satisfying the conditions **(CI), (DMP)**, and **(MARG)**. Let $b_1, b_2 : S^1 \times S^1 \setminus \Delta \to \mathbb{R}$ be C^2 functions as in the condition **(MARG)**.

We now derive the infinitesimal form of the radial commutation relation. Let

$$\begin{aligned}\mathcal{L}_1 &= \frac{\kappa}{2}\partial_{11} + b_1(\theta_1,\theta_2)\,\partial_1 + \cot\left(\frac{\theta_2-\theta_1}{2}\right)\partial_2 \\ \mathcal{L}_2 &= \frac{\kappa}{2}\partial_{22} + b_2(\theta_1,\theta_2)\,\partial_2 + \cot\left(\frac{\theta_1-\theta_2}{2}\right)\partial_1\end{aligned} \tag{2.3}$$

be the diffusion generators associated with (2.1) and (2.2).

Proposition 2.2. *The diffusion generators* (2.3) *of a locally commuting 2-radial* SLE_κ *satisfies the* infinitesimal commutation relation

$$[\mathcal{L}_1,\mathcal{L}_2] := \mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1 = \frac{\mathcal{L}_2-\mathcal{L}_1}{(\sin((\theta_2-\theta_1)/2))^2}. \tag{2.4}$$

The proof follows exactly the same steps as in [9] for chordal SLEs. The radial commutation relation was stated briefly in [9] but with an opposite sign. For the reader's convenience, we derive it here. We introduce the following notations. They will also be used in Section 3.2.

Fix $\theta_1 < \theta_2 < \theta_1 + 2\pi$. We will describe the growth of a pair of continuous non-self-crossing curves $(\eta^{(1)},\eta^{(2)})$ in $\mathbb{D}$ such that $\eta^{(1)}_0 = \mathrm{e}^{\mathrm{i}\theta_1}$ and $\eta^{(2)}_0 = \mathrm{e}^{\mathrm{i}\theta_2}$. For $\mathbf{t} = (t_1,t_2) \in \mathbb{R}^2_+$, suppose $\eta^{(1)}_{[0,t_1]}$ and $\eta^{(2)}_{[0,t_2]}$ are disjoint. We consider the following mapping-out functions:

- $g^{(j)}_{t_j} : \mathbb{D}\setminus\eta^{(j)}_{[0,t_j]} \to \mathbb{D}$ is conformal with $g^{(j)}_{t_j}(0) = 0$ and $\left(g^{(j)}_{t_j}\right)'(0) = \exp(t_j) > 0$, for $j = 1,2$.
- $g_{\mathbf{t}} : \mathbb{D}\setminus\left(\eta^{(1)}_{[0,t_1]}\cup\eta^{(2)}_{[0,t_2]}\right) \to \mathbb{D}$ is conformal with $g_{\mathbf{t}}(0) = 0$ and $g'_{\mathbf{t}}(0) > 0$.
- $g_{\mathbf{t},1} : \mathbb{D}\setminus g^{(1)}_{t_1}\left(\eta^{(2)}_{[0,t_2]}\right) \to \mathbb{D}$ is conformal with $g_{\mathbf{t},1}(0) = 0$ and $g'_{\mathbf{t},1}(0) > 0$.
- $g_{\mathbf{t},2} : \mathbb{D}\setminus g^{(2)}_{t_2}\left(\eta^{(1)}_{[0,t_1]}\right) \to \mathbb{D}$ is conformal with $g_{\mathbf{t},2}(0) = 0$ and $g'_{\mathbf{t},2}(0) > 0$.

Using such notations, we have $g_{\mathbf{t}} = g_{\mathbf{t},j}\circ g^{(j)}_{t_j}$ for $j = 1,2$. Let $\phi^{(j)}_{t_j}, \phi_{\mathbf{t}}, \phi_{\mathbf{t},j}$ be the covering maps of $g^{(j)}_{t_j}, g_{\mathbf{t}}, g_{\mathbf{t},j}$ respectively, i.e., the continuous function such that $g_\cdot(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\phi_\cdot(\theta)}$ and $\phi_0(\theta) = \theta$. Denote by $(\xi^{(j)}_{t_j}, t_j \geq 0)$ the driving function of $\eta^{(j)}$ as a radial Loewner chain for $j = 1,2$. Let

$$\theta^{(1)}_{\mathbf{t}} = \phi_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right), \qquad \theta^{(2)}_{\mathbf{t}} = \phi_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right).$$

The pair $\mathbf{t} \mapsto (\theta^{(1)}_{\mathbf{t}},\theta^{(2)}_{\mathbf{t}})$ may be viewed as the two-time driving function of the pair $(\eta^{(1)},\eta^{(2)})$. See Figure 5.

Proof of Proposition 2.2. The strategy of the proof consists of mapping out $\left(\eta^{(1)}_{[0,\varepsilon]},\eta^{(2)}_{[0,\varepsilon]}\right)$ in two ways, where both curves are parametrized by their intrinsic capacity seen in $\mathbb{D}$. We can either

- first map out $\eta^{(1)}_{[0,\varepsilon]}$ by the conformal map $g^{(1)}_\varepsilon$, then by $g_{(\varepsilon,\varepsilon),1}$,
- or first map out $\eta^{(2)}_{[0,\varepsilon]}$ by the conformal map $g^{(2)}_\varepsilon$, then by $g_{(\varepsilon,\varepsilon),2}$.

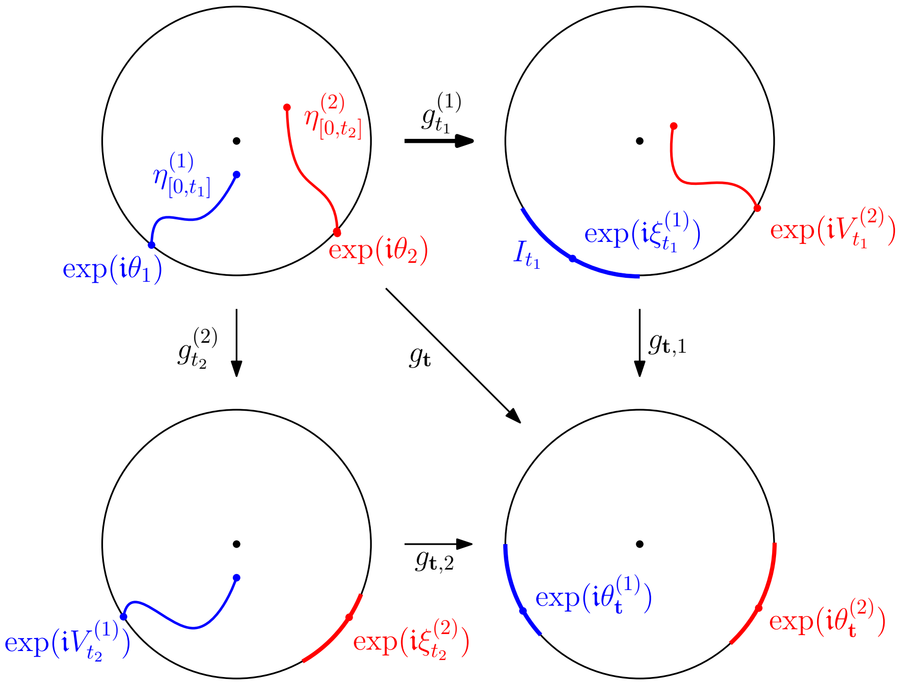

Figure 5: We have $g_{\mathbf{t}} = g_{\mathbf{t},1} \circ g^{(1)}_{t_1} = g_{\mathbf{t},2} \circ g^{(2)}_{t_2}$.

We then compare the expansions of $(\theta^{(1)}_{(\varepsilon,\varepsilon)}, \theta^{(2)}_{(\varepsilon,\varepsilon)})$ in ε which are expressed in terms of $\mathcal{L}_1$ and $\mathcal{L}_2$.

More precisely, we first follow the Loewner flow $t \mapsto g^{(1)}_t$ until $t = \varepsilon$. Under this flow, the radial driving function is $t \mapsto \xi^{(1)}_t$. And $t \mapsto V^{(2)}_t$ satisfies

$$\partial_t V^{(2)}_t = \cot\left((V^{(2)}_t - \xi^{(1)}_t)/2\right).$$

Hence, for any smooth test function $F\colon (\theta_1, \theta_2) \mapsto \mathbb{R}$, we have

$$\mathbb{E}_{(\theta_1,\theta_2)}\left[F(\xi^{(1)}_\varepsilon, V^{(2)}_\varepsilon)\right] = \left(1 + \varepsilon\mathcal{L}_1 + \frac{\varepsilon^2\mathcal{L}_1^2}{2}\right) F(\theta_1,\theta_2) + o(\varepsilon^2).$$

Now we use $g_{(\varepsilon,\varepsilon),1}$ to map out $\tilde{\eta}^{(2)}_{[0,\varepsilon]} := g^{(1)}_\varepsilon(\eta^{(2)}_{[0,\varepsilon]})$. We note that the capacity of $\tilde{\eta}^{(2)}_{[0,\varepsilon]}$ is not ε. We compute its capacity, we note that

$$\partial_t \left(g^{(1)}_t\right)'(z)|_{t=0} = -\frac{z + \mathrm{e}^{\mathrm{i}\theta_1}}{z - \mathrm{e}^{\mathrm{i}\theta_1}} + z\frac{2\mathrm{e}^{\mathrm{i}\theta_1}}{(z - \mathrm{e}^{\mathrm{i}\theta_1})^2} = -\frac{z^2 - \mathrm{e}^{2\mathrm{i}\theta_1} - 2z\mathrm{e}^{\mathrm{i}\theta_1}}{(z - \mathrm{e}^{\mathrm{i}\theta_1})^2}.$$

Therefore, for small ε,

$$\left(g^{(1)}_\varepsilon\right)'(\mathrm{e}^{\mathrm{i}\theta_2}) = 1 - \frac{\varepsilon}{2\left(\sin((\theta_2 - \theta_1)/2)\right)^2} + \mathrm{i}\varepsilon\cot((\theta_2 - \theta_1)/2) + O(\varepsilon^2).$$

So we have

$$\left|\left(g^{(1)}_\varepsilon\right)'(\mathrm{e}^{\mathrm{i}\theta_2})\right| = 1 - \frac{\varepsilon}{2\left(\sin((\theta_2 - \theta_1)/2)\right)^2} + o(\varepsilon).$$

It follows from (3.7) that the image under $g_\varepsilon^{(1)}$ of a set of capacity ε near $e^{i\theta_2}$ has capacity

$$\varepsilon' = \varepsilon \left(\left| \left(g_\varepsilon^{(1)} \right)' (e^{i\theta_2}) \right|^2 + o(\varepsilon) \right) = \varepsilon \left(1 - \frac{\varepsilon}{(\sin((\theta_2 - \theta_1)/2))^2} \right) + o(\varepsilon^2).$$

In particular, $\tilde{\eta}^{(2)}_{[0,\varepsilon]}$ has capacity ε'. Therefore, using the conditions **(CI)** and **(DMP)**, we have

$$\begin{aligned} &\mathbb{E}_{(\theta_1,\theta_2)} \left[F(\theta^{(1)}_{(\varepsilon,\varepsilon)}, \theta^{(2)}_{(\varepsilon,\varepsilon)}) \right] \qquad (2.5) \\ =& \left(1 + \varepsilon \mathcal{L}_1 + \frac{\varepsilon^2 \mathcal{L}_1^2}{2} \right) \left(1 + \varepsilon' \mathcal{L}_2 + \frac{(\varepsilon')^2 \mathcal{L}_2^2}{2} \right) F(\theta_1, \theta_2) + o(\varepsilon^2) \\ =& \left(1 + \varepsilon(\mathcal{L}_1 + \mathcal{L}_2) + \left(-\varepsilon\delta\mathcal{L}_2 + \frac{\varepsilon^2 \mathcal{L}_1^2}{2} + \frac{\varepsilon^2 \mathcal{L}_2^2}{2} + \varepsilon^2 \mathcal{L}_1 \mathcal{L}_2 \right) \right) F(\theta_1, \theta_2) + o(\varepsilon^2) \end{aligned}$$

where $\delta = \varepsilon\, (\sin((\theta_2 - \theta_1)/2))^{-2}$ so that $\varepsilon' = \varepsilon(1 - \delta)$.

If we first map out the second curve, then the first curve, and notice that the value of δ is unchanged, we obtain that the above expectation also equals

$$\left(1 + \varepsilon(\mathcal{L}_1 + \mathcal{L}_2) + \left(-\varepsilon\delta\mathcal{L}_1 + \frac{\varepsilon^2 \mathcal{L}_1^2}{2} + \frac{\varepsilon^2 \mathcal{L}_2^2}{2} + \varepsilon^2 \mathcal{L}_2 \mathcal{L}_1 \right) \right) F(\theta_1, \theta_2) + o(\varepsilon^2).$$

Comparing these two expansions, the coefficient of the ε^2-order terms have to coincide, we obtain the condition

$$[\mathcal{L}_1, \mathcal{L}_2] := \mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1 = \frac{\delta}{\varepsilon}(\mathcal{L}_2 - \mathcal{L}_1) = \frac{\mathcal{L}_2 - \mathcal{L}_1}{(\sin((\theta_2 - \theta_1)/2))^2}$$

as claimed. □

We note that the condition **(CI)** implies

$$b_j(\theta_1 + a, \theta_2 + a) = b_j(\theta_1, \theta_2), \qquad \forall j = 1,2 \text{ and } a \in \mathbb{R}. \tag{2.6}$$

Proposition 2.3 (Radial BPZ equations)**.** *Let $\kappa \in (0, \infty)$. Let $b_1, b_2 : S^1 \times S^1 \setminus \Delta \to \mathbb{R}$ be C^2 functions as in the condition **(MARG)**. Then* (2.4) *and* (2.6) *imply that there exists $\mathcal{Z} : \{(\theta_1, \theta_2) \in \mathbb{R}^2 \,|\, \theta_1 < \theta_2 < \theta_1 + 2\pi\} \to \mathbb{R}_{>0}$, called* partition function*, and a constant $F \in \mathbb{R}$ such that*

$$b_j = \kappa \partial_j \log \mathcal{Z}, \quad j = 1, 2,$$

and

$$\frac{\kappa}{2}\frac{\partial_{11}\mathcal{Z}}{\mathcal{Z}} + \cot(\theta_{21}/2)\frac{\partial_2\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\,(\sin(\theta_{21}/2))^2} = F, \tag{2.7}$$

$$\frac{\kappa}{2}\frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} - \cot(\theta_{21}/2)\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\,(\sin(\theta_{21}/2))^2} = F, \tag{2.8}$$

where $h = (6 - \kappa)/(2\kappa)$ and $\theta_{21} = \theta_2 - \theta_1 = -\theta_{12}$.

We note that $\mathcal{Z}$ does not always descend to a function on $(\mathbb{R}/2\pi\mathbb{Z})^2$.

Proof. We use the expression (2.3) and obtain

$$[\mathcal{L}_1,\mathcal{L}_2] = \left[-\frac{\kappa}{2\left(\sin(\theta_{12}/2)\right)^2}\right]\partial_{11} + \left[\kappa\partial_1 b_2 - \kappa\partial_2 b_1\right]\partial_{12} + \left[\frac{\kappa}{2\left(\sin(\theta_{12}/2)\right)^2}\right]\partial_{22}$$
$$+\left[\frac{\kappa-2}{4}\frac{\cot(\theta_{12}/2)}{\left(\sin(\theta_{12}/2)\right)^2} - \frac{b_1}{2\left(\sin(\theta_{12}/2)\right)^2} - \left(\frac{\kappa}{2}\partial_{22}b_1 + b_2\partial_2 b_1 + \cot(\theta_{12}/2)\partial_1 b_1\right)\right]\partial_1$$
$$+\left[\frac{\kappa-2}{4}\frac{\cot(\theta_{12}/2)}{\left(\sin(\theta_{12}/2)\right)^2} + \frac{b_2}{2\left(\sin(\theta_{12}/2)\right)^2} + \left(\frac{\kappa}{2}\partial_{11}b_2 + b_1\partial_1 b_2 - \cot(\theta_{12}/2)\partial_2 b_2\right)\right]\partial_2,$$

and

$$\frac{\mathcal{L}_2-\mathcal{L}_1}{\left(\sin(\theta_{12}/2)\right)^2} = \frac{1}{\left(\sin(\theta_{12}/2)\right)^2}\left[\frac{\kappa}{2}(\partial_{22}-\partial_{11}) + (\cot(\theta_{12}/2)-b_1)\partial_1 + (b_2+\cot(\theta_{12}/2))\partial_2\right].$$

Comparing the coefficients, then (2.4) shows

$$\partial_1 b_2 = \partial_2 b_1, \tag{2.9}$$
$$\frac{\kappa}{2}\partial_{22}b_1 + b_2\partial_2 b_1 + \cot(\theta_{12}/2)\partial_1 b_1 - \frac{b_1}{2\left(\sin(\theta_{12}/2)\right)^2} + \left(\frac{6-\kappa}{4}\right)\frac{\cot(\theta_{12}/2)}{\left(\sin(\theta_{12}/2)\right)^2} = 0, \tag{2.10}$$
$$\frac{\kappa}{2}\partial_{11}b_2 + b_1\partial_1 b_2 + \cot(\theta_{21}/2)\partial_2 b_2 - \frac{b_2}{2\left(\sin(\theta_{21}/2)\right)^2} + \left(\frac{6-\kappa}{4}\right)\frac{\cot(\theta_{21}/2)}{\left(\sin(\theta_{21}/2)\right)^2} = 0. \tag{2.11}$$

Eq. (2.9) shows that there exists a function $\mathcal{Z} : \{(\theta_1,\theta_2)\in\mathbb{R}^2 \mid \theta_1<\theta_2<\theta_1+2\pi\} \to \mathbb{R}_{>0}$ such that

$$b_1 = \kappa\,\partial_1\log\mathcal{Z} = \kappa\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}, \quad b_2 = \kappa\,\partial_2\log\mathcal{Z} = \kappa\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}.$$

Plugging it into (2.10) and (2.11), we have

$$\kappa\partial_1\left(\frac{\kappa}{2}\frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} - \cot(\theta_{21}/2)\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2}\right) = 0, \tag{2.12}$$
$$\kappa\partial_2\left(\frac{\kappa}{2}\frac{\partial_{11}\mathcal{Z}}{\mathcal{Z}} + \cot(\theta_{21}/2)\frac{\partial_2\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2}\right) = 0, \tag{2.13}$$

where $h = (6-\kappa)/(2\kappa)$.

Eq. (2.12) and (2.13) imply that, there exist functions F_1 and F_2:

$$\frac{\kappa}{2}\frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} - \cot(\theta_{21}/2)\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2} = F_2(\theta_2),$$
$$\frac{\kappa}{2}\frac{\partial_{11}\mathcal{Z}}{\mathcal{Z}} + \cot(\theta_{21}/2)\frac{\partial_2\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2} = F_1(\theta_1).$$

Using the identity

$$\frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} = \partial_2\left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right) + \left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right)^2,$$

Eq. (2.6) implies that F_1 and F_2 are constants and

$$\frac{\kappa}{2}\frac{\partial_{11}\mathcal{Z}}{\mathcal{Z}} + \cot(\theta_{21}/2)\frac{\partial_2\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2} = F_1, \tag{2.14}$$

$$\frac{\kappa}{2}\frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} - \cot(\theta_{21}/2)\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2} = F_2. \tag{2.15}$$

It remains to show $F_1 = F_2$. Taking the derivative of

$$a \mapsto b_j(\theta_1 + a, \theta_2 + a) = \kappa(\partial_j\mathcal{Z}/\mathcal{Z})(\theta_1 + a, \theta_2 + a)$$

and evaluate at $a = 0$, we get from **(CI)** and (2.6) that

$$0 = \partial_1\left(\frac{\partial_j\mathcal{Z}}{\mathcal{Z}}\right) + \partial_2\left(\frac{\partial_j\mathcal{Z}}{\mathcal{Z}}\right), \qquad \forall j = 1, 2. \tag{2.16}$$

From this, we have

$$\begin{aligned}
\frac{\partial_{11}\mathcal{Z}}{\mathcal{Z}} - \frac{\partial_{22}\mathcal{Z}}{\mathcal{Z}} &= \left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right)^2 - \left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right)^2 \\
\partial_1\left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right) &= \partial_2\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right) \\
\partial_1\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right) &= \partial_2\left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right).
\end{aligned}$$

If we take the difference (2.14) - (2.15), we get

$$\frac{\kappa}{2}\left(\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right)^2 - \left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right)^2\right) + \cot(\theta_{21}/2)\left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}} + \frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right) = F_1 - F_2. \tag{2.17}$$

Note that (2.16) also implies

$$0 = \partial_1\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right) + \partial_2\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right) = \partial_1\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}}\right) + \partial_1\left(\frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right) = \partial_1\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} + \frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right)$$

and

$$0 = \partial_2\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} + \frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right).$$

Hence there is $\mu \in \mathbb{R}$ such that

$$\frac{2\mu}{\kappa} \equiv \frac{\partial_1\mathcal{Z}}{\mathcal{Z}} + \frac{\partial_2\mathcal{Z}}{\mathcal{Z}}. \tag{2.18}$$

Plugging into (2.17) we get

$$\left(\frac{\partial_1\mathcal{Z}}{\mathcal{Z}} - \frac{\partial_2\mathcal{Z}}{\mathcal{Z}}\right)\mu + \cot(\theta_{21}/2)\frac{2\mu}{\kappa} = F_1 - F_2. \tag{2.19}$$

Combining (2.18) and (2.19), there are two cases.

Case 1: We have either $\mu = 0$, then $F_1 = F_2$ as desired.

Case 2: If $\mu \neq 0$, then

$$\begin{cases} \dfrac{\partial_1 \mathcal{Z}}{\mathcal{Z}} = -\frac{1}{\kappa}\cot(\theta_{21}/2) + \frac{\mu}{\kappa} + \frac{F_1 - F_2}{2\mu}, \\ \dfrac{\partial_2 \mathcal{Z}}{\mathcal{Z}} = \frac{1}{\kappa}\cot(\theta_{21}/2) + \frac{\mu}{\kappa} - \frac{F_1 - F_2}{2\mu}. \end{cases} \tag{2.20}$$

We solve (2.20) and obtain that, for some constant $C \in \mathbb{R}$,

$$\mathcal{Z}(\theta_1, \theta_2) = C\left(\sin(\theta_{21}/2)\right)^{2/\kappa} \exp\left(\frac{\mu_1}{\kappa}\theta_1 + \frac{\mu_2}{\kappa}\theta_2\right), \tag{2.21}$$

where

$$\mu_1 = \mu + \frac{\kappa(F_1 - F_2)}{2\mu}, \quad \mu_2 = \mu - \frac{\kappa(F_1 - F_2)}{2\mu}.$$

Plugging into (2.14)-(2.15), we obtain $\mu_1 = \mu_2$ which implies $F_1 = F_2$ as desired. Moreover, in this case we have

$$F_1 = F_2 = \frac{-3 + \mu^2}{2\kappa}, \qquad \mathcal{Z} \propto \mathcal{G}_\mu, \qquad \text{when } \mu \neq 0.$$

This completes the proof. □

2.3 Solutions to commutation relations with interchangeability

We now classify all possible partition functions $\mathcal{Z}$ with the additional condition **(INT)**, which is equivalent to

$$b_1(\theta_1, \theta_2) = b_2(\theta_2, \theta_1). \tag{2.22}$$

Theorem 2.4. *Let $\kappa \in (0, 8)$ and $\mathcal{Z}$ be a partition function from Proposition 2.3 and assume that* (2.22) *holds. Then, up to a multiplicative constant, $\mathcal{Z}$ is one of the following functions:*

1. *$\mathcal{Z} = \mathcal{G}_\mu$ for some $\mu \in \mathbb{R}$, where $\mathcal{G}_\mu$ is defined in* (1.3).
2. *$\mathcal{Z} = \mathcal{Z}_\alpha$ for some $\alpha < 1 - \kappa/8$, where $\mathcal{Z}_\alpha$ is defined in* (1.4).

Throughout this section, we assume that $\mathcal{Z}$ is a partition function from Proposition 2.3 and assume that (2.22) holds.

Lemma 2.5. *There exists $\lambda > 0$ such that for all $\theta_1 < \theta_2 < \theta_1 + 2\pi$,*

$$\lambda \mathcal{Z}(\theta_1, \theta_2) = \mathcal{Z}(\theta_2, \theta_1 + 2\pi). \tag{2.23}$$

Proof. Assumption (2.22) implies that

$$\kappa \partial_1 \log\left(\frac{\mathcal{Z}(\theta_2, \theta_1 + 2\pi)}{\mathcal{Z}(\theta_1, \theta_2)}\right) = b_2(\theta_2, \theta_1 + 2\pi) - b_1(\theta_1, \theta_2) = b_2(\theta_2, \theta_1) - b_1(\theta_1, \theta_2) = 0$$

and similarly,

$$\partial_2 \log\left(\frac{\mathcal{Z}(\theta_2, \theta_1 + 2\pi)}{\mathcal{Z}(\theta_1, \theta_2)}\right) = 0.$$

Therefore, $\mathcal{Z}(\theta_2, \theta_1 + 2\pi)/\mathcal{Z}(\theta_1, \theta_2)$ is a constant λ which also equals $\mathcal{Z}(\theta_1 + 2\pi, \theta_2 + 2\pi)/Z(\theta_2, \theta_1 + 2\pi)$. □

Proof of Theorem 2.4. From the proof of Proposition 2.3, combining (2.18) and (2.19), we have two cases: we have either $\mu \neq 0$, then

$$\mathcal{Z} \propto \mathcal{G}_\mu$$

which satisfies the interchangeability condition (2.22).

Or we have $\mu = 0$, then

$$\partial_1 \mathcal{Z} = -\partial_2 \mathcal{Z}. \tag{2.24}$$

Eq. (2.24) implies that $\partial_a \mathcal{Z}(\theta_1 + a, \theta_2 + a) = 0$, in other words, $\mathcal{Z}(\theta_1, \theta_2)$ only depends on the difference $\theta = \theta_2 - \theta_1$. Thus we write with a slight abuse of notation

$$\mathcal{Z}(\theta_1, \theta_2) = \mathcal{Z}(\theta), \quad \theta \in (0, 2\pi).$$

We want to solve the equations (2.7) and (2.8), they are simplified to the same equation:

$$\frac{\kappa}{2}\frac{\mathcal{Z}''}{\mathcal{Z}} + \cot(\theta/2)\frac{\mathcal{Z}'}{\mathcal{Z}} - \frac{h}{2\,(\sin(\theta/2))^2} = F. \tag{2.25}$$

Noticing that

$$\lambda^2 = \frac{\mathcal{Z}(\theta_1 + 2\pi, \theta_2 + 2\pi)}{\mathcal{Z}(\theta_1, \theta_2)} = 1,$$

we obtain $\lambda = 1$ and

$$\mathcal{Z}(\theta) = \mathcal{Z}(2\pi - \theta). \tag{2.26}$$

To solve (2.25) under the assumption (2.26), we may write

$$\theta \in (0, 2\pi), \quad u = (\sin(\theta/4))^2 \in (0,1), \quad \mathcal{Z}(\theta) = C(\sin(\theta/2))^{-2h}\phi(u), \tag{2.27}$$

satisfying for $u \in (0,1)$,

$$\begin{cases} u(1-u)\phi'' + \frac{3\kappa-8}{2\kappa}(1-2u)\phi' + \frac{8}{\kappa}\left(\frac{(6-\kappa)(\kappa-2)}{8\kappa} - F\right)\phi = 0, \\ \phi(1/2) = 1, \quad \phi'(1/2) = 0, \end{cases} \tag{2.28}$$

where C is a positive constant. Eq. (2.28) has a unique solution ϕ in $C^2(0,1)$ due to Lemma A.1. More precisely:

- When
$$\alpha := \frac{(6-\kappa)(\kappa-2)}{8\kappa} - F < 1 - \frac{\kappa}{8}, \qquad \text{i.e.,} \qquad F > -\frac{3}{2\kappa},$$
the unique solution $\phi(u)$ is positive for all $u \in (0,1)$.
- When $\alpha = 1 - \kappa/8$,
$$\phi(u) = (4u(1-u))^{4/\kappa - 1/2} = (\sin(\theta/2))^{8/\kappa - 1}$$
by (A.1). Hence,
$$\mathcal{Z}(\theta) = C\,(\sin(\theta/2))^{2/\kappa} \propto \mathcal{G}_0.$$

- When $\alpha > 1-\kappa/8$, Lemma A.1 shows that the unique solution is not always positive. Therefore it does not give any partition function in Proposition 2.3. See Figure 6 for the case when $\kappa = 4$.

This completes the proof. □

Remark 2.6. When $\kappa = 4$, the partition function $\mathcal{Z}_\alpha$ in (1.4) has an explicit expression. We denote $\theta = \theta_2 - \theta_1$.

- If $\alpha \in [0, 1/2)$, we have

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin(\theta/2))^{-1/2} \cos\left(\sqrt{\frac{\alpha}{2}}(\theta - \pi)\right).$$

- If $\alpha < 0$, we have

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin(\theta/2))^{-1/2} \cosh\left(\sqrt{\frac{|\alpha|}{2}}(\theta - \pi)\right).$$

See Figure 6.

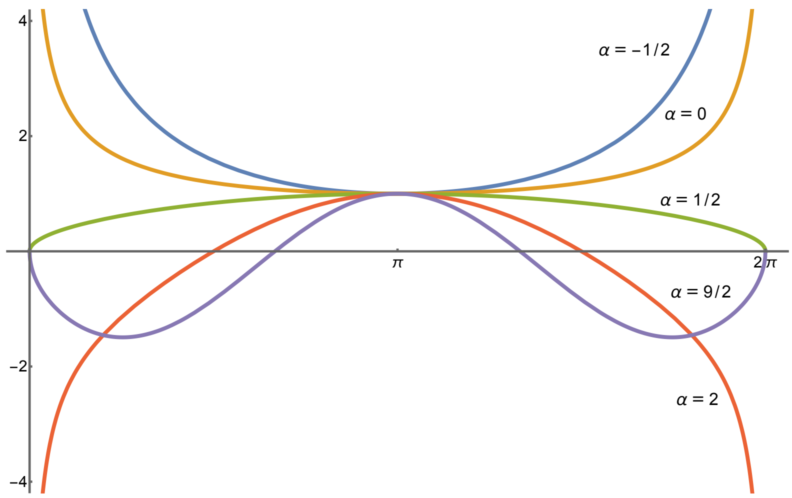

Figure 6: Plot of $\theta \mapsto \mathcal{Z}_\alpha(\theta)$ with $\kappa = 4$ and $\theta = \theta_2 - \theta_1$ for different α's. When $\alpha > 1/2$, it is not always positive.

Remark 2.7. When $\kappa \in (0, 8)$ and $\alpha = \alpha_1(\kappa)$ which is the one-arm exponent for conformal loop ensemble [35]:

$$\alpha_1(\kappa) = \frac{(3\kappa - 8)(8-\kappa)}{32\kappa},$$

the partition function $\mathcal{Z}_{\alpha_1(\kappa)}$ has an explicit expression: we denote $\theta = \theta_2 - \theta_1$,

$$\mathcal{Z}_{\alpha_1(\kappa)}(\theta_1, \theta_2) = 2^{4/\kappa - 3/2} (\sin(\theta/2))^{1-6/\kappa} \left((\sin(\theta/4))^{8/\kappa - 1} + (\cos(\theta/4))^{8/\kappa - 1} \right). \quad (2.29)$$

For $\kappa \in (4,8)$, this is the conjectured partition function for the scaling limit of interfaces in critical random-cluster models with Dobrushin boundary condition conditional on the one-arm event, see [12]. This conjecture holds for percolation (with $\kappa = 6$) and FK-Ising model (with $\kappa = 16/3$).

Remark 2.8. Let us give a comment on the C^2 requirement on b_1, b_2 in the assumption **(MARG)**. We assume such C^2 requirement as part of the axioms at the beginning. But in fact, this is not necessary. Suppose we relax the requirement and only assume b_1, b_2 in **(MARG)** are continuous, we are still able to derive (2.14) and (2.15) as weak solutions. The operators in the left-hand side of radial BPZ equations (2.14) and (2.15) are hypoelliptic, due to a general characterization by Hörmander [15], see also [10, Lemma 5] and [30, Sect. 2.3.3]. Therefore, the weak solutions are strong solutions which are in fact C^∞, and consequently b_1, b_2 are C^∞. Note that such analysis does not work for $\kappa = 0$ since the corresponding BPZ equation (4.1) is nonlinear.

3 Identification of locally commuting 2-radial SLEs

The goal of this section is to identify all locally commuting 2-radial SLEs, namely, those whose partition functions are classified in Theorem 2.4: We discuss the 2-sided radial SLE with spiral in Section 3.2 and chordal SLE weighted by conformal radius in Section 3.4.

3.1 Radial SLE

For $\theta \in [0, 2\pi)$, suppose $\eta : [0,T] \to \overline{\mathbb{D}}$ is a continuous non-self-crossing curve such that $\eta_0 = e^{i\theta}$ and $\eta_{(0,T)} \subset \mathbb{D}\setminus\{0\}$. We parameterize the curve by the capacity and denote by g_t the corresponding radial Loewner chain as in (1.1). Denote by ϕ_t the covering conformal map of g_t, i.e. $g_t(\exp(iw)) = \exp(i\phi_t(w))$ with $\phi_0(w) = w$ for $w \in \mathbb{H}$. Then the radial Loewner equation (1.1) is equivalent to

$$\partial_t \phi_t(w) = \cot\left((\phi_t(w) - \xi_t)/2\right), \quad \phi_0(w) = w.$$

Radial SLE_κ is the radial Loewner chain with $\xi_t = \sqrt{\kappa} B_t$ where B is one-dimensional Brownian motion. We also call it radial SLE_κ in $(\mathbb{D}; e^{i\theta}; 0)$. In the following lemma, we will describe the boundary perturbation property of radial SLE. We fix the parameters:

$$\kappa > 0, \quad h = \frac{6-\kappa}{2\kappa}, \quad \tilde{h} = \frac{(6-\kappa)(\kappa-2)}{8\kappa}, \quad c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}. \tag{3.1}$$

Lemma 3.1 (See [16, Prop. 5] or [14, Prop. 2.2])**.** *Fix $\kappa \in (0,8)$ and $\theta \in [0,2\pi)$. Suppose K is a compact subset of $\overline{\mathbb{D}}$ such that $\mathbb{D}\setminus K$ is simply connected and contains the origin and that K has a positive distance from $e^{i\theta}$. Suppose η is radial SLE_κ in $(\mathbb{D}; e^{i\theta}; 0)$ and define $\tau = \inf\{t : \eta_t \in K\}$. For $t < \tau$, denote by $g_{t,K}$ the unique conformal map $\mathbb{D} \setminus g_t(K) \to \mathbb{D}$ such that $g_{t,K}(0) = 0$ and $g'_{t,K}(0) > 0$. We denote by $\phi_{t,K}$ the covering map of $g_{t,K}$. Then the following process is a local martingale:*

$$M_t = 1\{t < \tau\} \phi'_{t,K}(\xi_t)^h g'_{t,K}(0)^{\tilde{h}} \exp\left(\frac{c}{2} m_t\right),$$

where m_t is defined through

$$\mathrm{d}m_t = -\frac{1}{3}\mathcal{S}\phi_{t,K}(\xi_t)\mathrm{d}t + \frac{1}{6}\left(1 - \phi'_{t,K}(\xi_t)^2\right)\mathrm{d}t,$$

and $\mathcal{S}\phi = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$ *denotes the Schwarzian derivative of* ϕ.

Moreover, when $\kappa \le 4$, the process M_t is a uniformly integrable martingale. The law of radial SLE_κ *in* $(\mathbb{D} \setminus K; \mathrm{e}^{\mathrm{i}\theta}; 0)$ *is the same as radial* SLE_κ *in* $(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta}; 0)$ *weighted by* M_t.

Remark 3.2. It is explained in the proof of [16, Prop. 5] that the term $m_t = m_{\mathbb{D}}(\eta_{[0,t]}, K)$ is the same as the Brownian loop measure of loops that intersect both $\eta_{[0,t]}$ and K when $\eta_{[0,t]} \cap K = \emptyset$.

Fix θ_1, θ_2 such that $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Let $\kappa \in [0, \infty)$, $\rho \in \mathbb{R}$ and $\mu \in \mathbb{R}$. A radial $\mathrm{SLE}^\mu_\kappa(\rho)$ in $\mathbb{D}$ starting from $\mathrm{e}^{\mathrm{i}\theta_1}$ with force point $\mathrm{e}^{\mathrm{i}\theta_2}$ and spiraling rate μ is the radial Loewner chain with driving function ξ_t that solves the following SDE:

$$\begin{cases} \xi_0 = \theta_1, V_0 = \theta_2, \\ \mathrm{d}\xi_t = \sqrt{\kappa}\mathrm{d}B_t + \frac{\rho}{2}\cot\left((\xi_t - V_t)/2\right)\mathrm{d}t + \mu\mathrm{d}t, \\ \mathrm{d}V_t = \cot\left((V_t - \xi_t)/2\right)\mathrm{d}t. \end{cases} \tag{3.2}$$

The solution to SDE (3.2) exists for all time when $\kappa \in (0, 8)$ and $\rho > -2$ and it is generated by a continuous curve from $\mathrm{e}^{\mathrm{i}\theta_1}$ to the origin.

Lemma 3.3. *For $\kappa \in (0, 8)$, $\rho > -2$, $\mu \in \mathbb{R}$, radial* $\mathrm{SLE}^\mu_\kappa(\rho)$ *in $\mathbb{D}$ is almost surely generated by a continuous curve η and* $\lim_{t\to\infty} \eta_t = 0$.

Proof. This is proved in [28, Prop. 3.30] and [28, Sect. 4]. See also [20]. □

Remark 3.4. The expression of $\mathcal{G}_\mu$ in (1.3) is such that the Loewner driving function of radial $\mathrm{SLE}^\mu_\kappa(2)$ can be rewritten as

$$\begin{cases} \xi_0 = \theta_1, V_0 = \theta_2, \\ \mathrm{d}\xi_t = \sqrt{\kappa}\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{G}_\mu(\xi_t, V_t)\mathrm{d}t, \\ \mathrm{d}V_t = \cot\left((V_t - \xi_t)/2\right)\mathrm{d}t. \end{cases}$$

3.2 Two-sided radial SLE with spiral

In this section, we introduce two-sided radial SLE_κ with spiral when $\kappa \in (0, 8)$ by reweighting two independent radial SLE_κ. The two-sided radial SLE analyzed in [13, 14, 20, 23] is a special case where the spiraling rate $\mu = 0$.

We use the same notations as in Figure 5.

Lemma 3.5. *Fix $\kappa \in (0, 8)$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. We fix the parameters $h, \tilde{h}, c$ as in (3.1). For $\mu \in \mathbb{R}$, we define $\mathcal{G}_\mu$ as in (1.3). Let $\mathbb{P}$ denote the probability measure* *under which* $(\eta^{(1)}, \eta^{(2)})$ *are two independent radial* SLE_κ *in* $\mathbb{D}$ *starting from* $\mathrm{e}^{\mathrm{i}\theta_1}$ *and* $\mathrm{e}^{\mathrm{i}\theta_2}$ *respectively. We define*

$$\begin{aligned} M_{\mathbf{t}}(\mathcal{G}_\mu) =& 1\left\{\eta^{(1)}_{[0,t_1]} \cap \eta^{(2)}_{[0,t_2]} = \emptyset\right\} g'_{\mathbf{t}}(0)^{\frac{3-\mu^2}{2\kappa}-\tilde{h}} \times \prod_{j=1}^{2} \phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)^h g'_{\mathbf{t},j}(0)^{\tilde{h}} \\ & \times \mathcal{G}_\mu\left(\theta^{(1)}_{\mathbf{t}}, \theta^{(2)}_{\mathbf{t}}\right) \exp\left(\frac{c}{2} m_{\mathbf{t}}\right), \end{aligned} \tag{3.3}$$

where $m_{\mathbf{t}}$ *is defined through*

$$\mathrm{d}m_{\mathbf{t}} = \sum_{j=1}^{2} \left(-\frac{1}{3}\mathcal{S}\phi_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right) + \frac{1}{6}\left(1 - \phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)^2\right)\right) \mathrm{d}t_j. \tag{3.4}$$

Then $M_{\mathbf{t}}(\mathcal{G}_\mu)$ *is a two-time-parameter local martingale with respect to* $\mathbb{P}$.

Lemma 3.1 and Remark 3.2 show that $m_{\mathbf{t}} = m_{\mathbb{D}}\left(\eta^{(1)}_{[0,t_1]}, \eta^{(2)}_{[0,t_2]}\right)$ is the Brownian loop measure of loops in $\mathbb{D}$ intersecting both $\eta^{(1)}_{[0,t_1]}$ and $\eta^{(2)}_{[0,t_2]}$ when $\eta^{(1)}_{[0,t_1]} \cap \eta^{(2)}_{[0,t_2]} = \emptyset$.

We note that $\mathcal{G}_\mu$ satisfies the "radial BPZ equations"

$$\frac{\kappa}{2}\frac{\partial_{11}\mathcal{G}_\mu}{\mathcal{G}_\mu} + \cot(\theta_{21}/2)\frac{\partial_2\mathcal{G}_\mu}{\mathcal{G}_\mu} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2} = \frac{\mu^2-3}{2\kappa}; \tag{3.5}$$

$$\frac{\kappa}{2}\frac{\partial_{22}\mathcal{G}_\mu}{\mathcal{G}_\mu} - \cot(\theta_{21}/2)\frac{\partial_1\mathcal{G}_\mu}{\mathcal{G}_\mu} - \frac{h}{2\left(\sin(\theta_{21}/2)\right)^2} = \frac{\mu^2-3}{2\kappa}. \tag{3.6}$$

Relations (3.4), (3.5) and (3.6) play an essential role in the proof of Lemma 3.5.

Proof of Lemma 3.5. Let us first compute the variations of terms appearing in (3.3). From standard calculations (see e.g. [14, Lem. 3.2]), we have the following variational formula of the capacity parameterizations:

$$\begin{aligned} \frac{\mathrm{d}g'_{\mathbf{t},1}(0)}{g'_{\mathbf{t},1}(0)} &= \left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2 - 1\right)\mathrm{d}t_1 + \phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2 \mathrm{d}t_2; \\ \frac{\mathrm{d}g'_{\mathbf{t},2}(0)}{g'_{\mathbf{t},2}(0)} &= \phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2 \mathrm{d}t_1 + \left(\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2 - 1\right)\mathrm{d}t_2; \\ \frac{\mathrm{d}g'_{\mathbf{t}}(0)}{g'_{\mathbf{t}}(0)} &= \sum_{j=1}^{2} \phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)^2 \mathrm{d}t_j. \end{aligned} \tag{3.7}$$

From the assumption that $\eta^{(1)}$ and $\eta^{(2)}$ are two independent radial SLE_κ under $\mathbb{P}$, we have that $\xi^{(1)} = \sqrt{\kappa}B^{(1)} + \theta_1$ and $\xi^{(2)} = \sqrt{\kappa}B^{(2)} + \theta_2$ where $B^{(1)}$ and $B^{(2)}$ are two independent Brownian motions. From this, Itô's calculus gives:

$$\begin{aligned} \mathrm{d}\theta^{(1)}_{\mathbf{t}} = \mathrm{d}\phi_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right) =& \phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)\mathrm{d}\xi^{(1)}_{t_1} - \kappa h \phi''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)\mathrm{d}t_1 \\ & + \cot\left((\theta^{(1)}_{\mathbf{t}} - \theta^{(2)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2 \mathrm{d}t_2, \end{aligned} \tag{3.8}$$

where we used the expansions

$$
\begin{aligned}
\partial_{t_1}\phi_{\mathbf{t},1}(w) &= \left(\phi^{(1)}_{\mathbf{t},1}\right)'\left(\xi^{(1)}_{t_1}\right)^2 \cot\left((\phi^{(1)}_{\mathbf{t},1}(w)-\theta^{(1)}_{\mathbf{t}})/2\right) - \left(\phi^{(1)}_{\mathbf{t},1}\right)'(w)\cot\left((w-\xi^{(1)}_{t_1})/2\right) \\
&= -3\left(\phi^{(1)}_{\mathbf{t},1}\right)''\left(\xi^{(1)}_{t_1}\right) + O(w-\xi^{(1)}_{t_1}), \quad \text{as } w\to\xi^{(1)}_{t_1}, \qquad (3.9)\\
\partial_{t_2}\phi_{\mathbf{t},1}(w) &= \cot\left((\phi^{(1)}_{\mathbf{t},1}(w)-\theta^{(2)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2. \qquad (3.10)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{d}\theta^{(2)}_{\mathbf{t}} = \mathrm{d}\phi_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right) =& \phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)\mathrm{d}\xi^{(2)}_{t_2} - \kappa h \phi''_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)\mathrm{d}t_2 \\
&+ \cot\left((\theta^{(2)}_{\mathbf{t}}-\theta^{(1)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2\mathrm{d}t_1. \qquad (3.11)
\end{aligned}
$$

Taking derivatives with respect to w in (3.9) and (3.10) and let $w\to\xi^{(1)}_{t_1}$ we obtain

$$
\begin{aligned}
\frac{\left(\partial_{t_1}\phi'_{\mathbf{t},1}\right)\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)} =& \frac{1}{2}\frac{\phi''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2} - \frac{4}{3}\frac{\phi'''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)} - \frac{1}{6}\left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2 - 1\right), \\
\frac{\left(\partial_{t_2}\phi'_{\mathbf{t},1}\right)\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)} =& -\frac{1}{2}\csc^2\left((\theta^{(1)}_{\mathbf{t}}-\theta^{(2)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2.
\end{aligned}
$$

Itô's formula gives

$$
\begin{aligned}
\frac{\mathrm{d}\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)} =& \frac{\phi''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}\mathrm{d}\xi^{(1)}_{t_1} - \frac{1}{2}\csc^2\left((\theta^{(1)}_{\mathbf{t}}-\theta^{(2)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2\mathrm{d}t_2 \\
&+ \left(\frac{\kappa}{2}\frac{\phi'''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)} + \frac{1}{2}\frac{\phi''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2} - \frac{4}{3}\frac{\phi'''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)} - \frac{1}{6}\left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2-1\right)\right)\mathrm{d}t_1, \\
\frac{\mathrm{d}\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)} =& \frac{\phi''_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}\mathrm{d}\xi^{(2)}_{t_2} - \frac{1}{2}\csc^2\left((\theta^{(2)}_{\mathbf{t}}-\theta^{(1)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2\mathrm{d}t_1 \\
&+ \left(\frac{\kappa}{2}\frac{\phi'''_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)} + \frac{1}{2}\frac{\phi''_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2}{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2} - \frac{4}{3}\frac{\phi'''_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)} - \frac{1}{6}\left(\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2-1\right)\right)\mathrm{d}t_2.
\end{aligned}
$$

Now we are ready to prove that $M_{\mathbf{t}}(\mathcal{G}_\mu)$ is a two-time-parameter local martingale. Combining with (3.4), (3.5) and (3.6), we have

$$
\begin{aligned}
\frac{\mathrm{d}M_{\mathbf{t}}(\mathcal{G}_\mu)}{M_{\mathbf{t}}(\mathcal{G}_\mu)} =& \left(\frac{3-\mu^2}{2\kappa} - \tilde{h}\right)\frac{\mathrm{d}g'_{\mathbf{t}}(0)}{g'_{\mathbf{t}}(0)} + \tilde{h}\sum_{j=1}^{2}\frac{\mathrm{d}g'_{\mathbf{t},j}(0)}{g'_{\mathbf{t},j}(0)} + \frac{c}{2}\mathrm{d}m_{\mathbf{t}} \\
&+ \sum_{j=1}^{2}\left(\frac{\partial_j\mathcal{G}_\mu}{\mathcal{G}_\mu}\mathrm{d}\theta^{(j)}_{\mathbf{t}} + \frac{\kappa}{2}\frac{\partial_{jj}\mathcal{G}_\mu}{\mathcal{G}_\mu}\phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)^2\mathrm{d}t_j\right) \\
&+ \sum_{j=1}^{2}\left(h\frac{\mathrm{d}\phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)}{\phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)} + \frac{\kappa h(h-1)}{2}\frac{\phi''_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)^2}{\phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)^2}\mathrm{d}t_j\right)
\end{aligned}
$$

$$+\kappa h\sum_{j=1}^{2}\frac{\partial_j\mathcal{G}_\mu}{\mathcal{G}_\mu}\phi''_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)\mathrm{d}t_j$$

$$=\sum_{j=1}^{2}\left(\frac{\partial_j\mathcal{G}_\mu}{\mathcal{G}_\mu}\phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)+h\frac{\phi''_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)}{\phi'_{\mathbf{t},j}\left(\xi^{(j)}_{t_j}\right)}\right)\mathrm{d}\xi^{(j)}_{t_j}. \tag{3.12}$$

In particular, $M_{\mathbf{t}}(\mathcal{G}_\mu)$ is a two-time-parameter local martingale under $\mathbb{P}$. □

Definition 3.6. *Fix* $\kappa\in(0,8)$ *and* $\theta_1<\theta_2<\theta_1+2\pi$. *For* $\mu\in\mathbb{R}$, *we define* $\mathcal{G}_\mu$ *as in* (1.3). *Let* $\mathbb{P}$ *denote the probability measure under which* $(\eta^{(1)},\eta^{(2)})$ *are two independent radial* SLE_κ *in* $\mathbb{D}$ *starting from* $\mathrm{e}^{\mathrm{i}\theta_1}$ *and* $\mathrm{e}^{\mathrm{i}\theta_2}$ *respectively. We define* $M_{\mathbf{t}}(\mathcal{G}_\mu)$ *as* (3.3) *in Lemma 3.5 and denote by* $\mathbb{P}(\mathcal{G}_\mu)$ *the probability measure obtained by tilting* $\mathbb{P}$ *by* $M_{\mathbf{t}}(\mathcal{G}_\mu)$ *and call it* two-sided radial SLE_κ with spiraling rate μ *in* $(\mathbb{D};\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2};0)$.

When $\kappa\le 4$, two-sided radial SLE_κ with spiral is well-defined for all time and the two curves do not touch each other before they reach the origin. When $\kappa\in(4,8)$, the above definition for two-sided radial SLE_κ with spiral is only defined up to the times the two curves touch each other. When the spiraling rate is 0, we obtain the standard two-sided radial SLE analyzed in [13, 14, 20, 23].

Corollary 3.7. *Under* $\mathbb{P}(\mathcal{G}_\mu)$, *for every* $\mathbf{t}=(t_1,t_2)$,

- $g_{\mathbf{t}}(\eta^{(1)})$ *is a radial* $\mathrm{SLE}^\mu_\kappa(2)$ *in* $\mathbb{D}$ *starting from* $\exp(\mathrm{i}\theta^{(1)}_{\mathbf{t}})$ *with force point at* $\exp(\mathrm{i}\theta^{(2)}_{\mathbf{t}})$,
- $g_{\mathbf{t}}(\eta^{(2)})$ *is a radial* $\mathrm{SLE}^\mu_\kappa(2)$ *in* $\mathbb{D}$ *starting from* $\exp(\mathrm{i}\theta^{(2)}_{\mathbf{t}})$ *with force point at* $\exp(\mathrm{i}\theta^{(1)}_{\mathbf{t}})$.

In particular, $\mathbb{P}(\mathcal{G}_\mu)$ *on pairs* $(\eta^{(1)},\eta^{(2)})$ *satisfies* ***(CI)***, ***(DMP)***, ***(MARG)*** *and* ***(INT)*** *with*

$$b_j=\kappa\partial_j\log\mathcal{G}_\mu,\quad j=1,2. \tag{3.13}$$

Proof. Combining (3.12) with (1.3), we have

$$\frac{\mathrm{d}M_{\mathbf{t}}(\mathcal{G}_\mu)}{M_{\mathbf{t}}(\mathcal{G}_\mu)}=\left(\frac{1}{\kappa}\cot\left((\theta^{(1)}_{\mathbf{t}}-\theta^{(2)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)+\frac{\mu}{\kappa}\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)+h\frac{\phi''_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}{\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)}\right)\mathrm{d}\xi^{(1)}_{t_1}$$
$$+\left(\frac{1}{\kappa}\cot\left((\theta^{(2)}_{\mathbf{t}}-\theta^{(1)}_{\mathbf{t}})/2\right)\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)+\frac{\mu}{\kappa}\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)+h\frac{\phi''_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)}\right)\mathrm{d}\xi^{(2)}_{t_2}.$$

From Girsanov's theorem and (3.8) and (3.11), under $\mathbb{P}(\mathcal{G}_\mu)$, we have

$$\mathrm{d}\theta^{(1)}_{\mathbf{t}}=\sqrt{\kappa}\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)\mathrm{d}\tilde{B}^{(1)}_{t_1}+\mu\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2\mathrm{d}t_1$$
$$+\cot\left((\theta^{(1)}_{\mathbf{t}}-\theta^{(2)}_{\mathbf{t}})/2\right)\left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2\mathrm{d}t_1+\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2\mathrm{d}t_2\right), \tag{3.14}$$

$$\mathrm{d}\theta^{(2)}_{\mathbf{t}}=\sqrt{\kappa}\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)\mathrm{d}\tilde{B}^{(2)}_{t_2}+\mu\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2\mathrm{d}t_2$$
$$+\cot\left((\theta^{(2)}_{\mathbf{t}}-\theta^{(1)}_{\mathbf{t}})/2\right)\left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2\mathrm{d}t_1+\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2\mathrm{d}t_2\right), \tag{3.15}$$

where $\tilde{B}^{(1)}$ and $\tilde{B}^{(2)}$ are two independent Brownian motions under $\mathbb{P}(\mathcal{G}_\mu)$.

Therefore, taking into account the variation of the capacity parametrization (3.7), (3.14) and (3.15) show that under $\mathbb{P}(\mathcal{G}_\mu)$, for every $\mathbf{t} = (t_1, t_2)$, $g_{\mathbf{t}}(\eta^{(1)})$ is a radial $\mathrm{SLE}_\kappa^\mu(2)$ in $\mathbb{D}$ starting from $\exp(\mathrm{i}\theta_{\mathbf{t}}^{(1)})$ with force point at $\exp(\mathrm{i}\theta_{\mathbf{t}}^{(2)})$. Similarly, $g_{\mathbf{t}}(\eta^{(2)})$ is a radial $\mathrm{SLE}_\kappa^\mu(2)$ in $\mathbb{D}$ starting from $\exp(\mathrm{i}\theta_{\mathbf{t}}^{(2)})$ with force point at $\exp(\mathrm{i}\theta_{\mathbf{t}}^{(1)})$. These imply **(CI)**, **(DMP)**, **(MARG)** and **(INT)**. Eq. (3.13) follows from Remark 3.4. □

Remark 3.8. Two-sided radial SLE_κ with spiral can be generalized to the multiple-sided case: N-sided radial SLE_κ with spiraling rate μ for $N \geq 2$ can be defined a similar way as in Definition 3.6 where the partition function $\mathcal{G}_\mu$ shall be replaced by

$$\mathcal{G}_\mu(\theta_1, \dots, \theta_N) = \prod_{1 \leq i < j \leq N} (\sin((\theta_j - \theta_i)/2))^{2/\kappa} \times \exp\left(\frac{\mu}{\kappa} \sum_{j=1}^{N} \theta_j\right),$$

for $\theta_1 < \cdots < \theta_N < \theta_1 + 2\pi$. When $\mu = 0$, it is the same as the partition function for N-sided radial SLE_κ in [14].

3.3 Resampling property of two-sided radial SLE with spiral

In this section, we will prove the resampling property of two-sided radial SLE with spiral as we described in Section 1.4 and in Corollary 1.2. We fix $\kappa \in (0, 4]$ in the following Theorem 3.9, because we will use the boundary perturbation property in Lemma 3.1 with $\kappa \in (0, 4]$ in the proof.

Theorem 3.9 (Resampling property)**.** *Fix $\kappa \in (0, 4]$, $\mu \in \mathbb{R}$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Suppose $(\eta^{(1)}, \eta^{(2)}) \sim \mathbb{P}(\mathcal{G}_\mu)$ is two-sided radial* SLE_κ *with spiraling rate μ as in Definition 3.6, we have the followings.*

- *The marginal law of $\eta^{(1)}$ is radial* $\mathrm{SLE}_\kappa^\mu(2)$ *in $\mathbb{D}$ starting from* $\mathrm{e}^{\mathrm{i}\theta_1}$ *with force point* $\mathrm{e}^{\mathrm{i}\theta_2}$ *and spiraling rate μ.*
- *Given $\eta^{(1)}$, the conditional law of $\eta^{(2)}$ is chordal* SLE_κ *in $\mathbb{D} \setminus \eta^{(1)}$ from* $\mathrm{e}^{\mathrm{i}\theta_2}$ *to* 0.

The same is true when we interchange $\eta^{(1)}$ and $\eta^{(2)}$.

Remark 3.10. Radial SLE with spiral appears as a flow line in the setup of imaginary geometry [28]. Furthermore, two-sided radial SLE_κ with spiraling rate μ can be viewed as a pair of flow lines of

$$\Gamma + \frac{(8-\kappa)}{2\sqrt{\kappa}} \arg(\cdot) + \frac{\mu}{\sqrt{\kappa}} \log|\cdot|,$$

where Γ is a GFF in $\mathbb{D}$ with properly chosen boundary data, and the angles of the two flow lines are also chosen properly. See Figure 2. Using such coupling, one is able to derive the resampling property in Theorem 3.9, see [28, Prop. 3.28]. However, our proof of the resampling property in Section 3.2 does not use the coupling with imaginary geometry. We derive it directly using a refined analysis of the Radon–Nikodym derivative $M_{\mathbf{t}}(\mathcal{G}_\mu)$ in Definition 3.6.

Proof of Theorem 3.9. Since $\mathcal{G}_\mu(\theta_1, \theta_2) = \lambda\, \mathcal{G}_\mu(\theta_2, \theta_1 + 2\pi)$ for some constant λ which does not depend on θ_1 and θ_2, $\eta^{(1)}$ and $\eta^{(2)}$ are interchangeable. Therefore, it suffices to show the bullet points in the statement. The marginal law of $\eta^{(1)}$ is a consequence of Corollary 3.7, and it remains to show the conditional law of $\eta^{(2)}$ given $\eta^{(1)}$.

The law of $\eta^{(1)}_{[0,t_1]}$ is the same as radial SLE_κ weighted by the following local martingale:

$$M_{(t_1,0)}(\mathcal{G}_\mu) = \left(g^{(1)}_{t_1}\right)'(0)^{\frac{3-\mu^2}{2\kappa}-\tilde{h}} \times \mathcal{G}_\mu\left(\xi^{(1)}_{t_1}, \phi^{(1)}_{t_1}(\theta_2)\right) \times \left(\phi^{(1)}_{t_1}\right)'(\theta_2)^h \left(g^{(1)}_{t_1}\right)'(0)^{\tilde{h}}.$$

The law of $(\eta^{(1)}, \eta^{(2)})$ is the same as two independent radial SLE_κ weighted by the local martingale $M_{\mathbf{t}}(\mathcal{G}_\mu)$. Therefore, the conditional law of $\eta^{(2)}_{[0,t_2]}$ given $\eta^{(1)}_{[0,t_1]}$ is the same as a radial SLE_κ weighted by

$$\frac{M_{(t_1,t_2)}(\mathcal{G}_\mu)}{M_{(t_1,0)}(\mathcal{G}_\mu)} = \underbrace{\frac{\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^h g'_{\mathbf{t},2}(0)^{\tilde{h}} \exp\left(\frac{c}{2} m_{\mathbf{t}}\right)}{\left(\phi^{(1)}_{t_1}\right)'(\theta_2)^h \left(g^{(1)}_{t_1}\right)'(0)^{\tilde{h}}}}_{=:P_{\mathbf{t}}} \times \underbrace{g'_{\mathbf{t},1}(0)^{\frac{3-\mu^2}{2\kappa}} \frac{\mathcal{G}_\mu\left(\theta^{(1)}_{\mathbf{t}}, \theta^{(2)}_{\mathbf{t}}\right)}{\mathcal{G}_\mu\left(\xi^{(1)}_{t_1}, \phi^{(1)}_{t_1}(\theta_2)\right)} \phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^h}_{=:R_{\mathbf{t}}}.$$

From the boundary perturbation property Lemma 3.1, a radial SLE_κ in $(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta_2}; 0)$ weighted by $P_{\mathbf{t}}$ has the same law as radial SLE_κ in $(\mathbb{D} \setminus \eta^{(1)}_{[0,t_1]}; \mathrm{e}^{\mathrm{i}\theta_2}; 0)$. Thus, the conditional law of $\eta^{(2)}_{[0,t_2]}$ given $\eta^{(1)}_{[0,t_1]}$ is the same as a radial SLE_κ in $(\mathbb{D} \setminus \eta^{(1)}_{[0,t_1]}; \mathrm{e}^{\mathrm{i}\theta_2}; 0)$ weighted by

$$R_{\mathbf{t}} = g'_{\mathbf{t},1}(0)^{\frac{3-\mu^2}{2\kappa}} \frac{\mathcal{G}_\mu\left(\theta^{(1)}_{\mathbf{t}}, \theta^{(2)}_{\mathbf{t}}\right)}{\mathcal{G}_\mu\left(\xi^{(1)}_{t_1}, \phi^{(1)}_{t_1}(\theta_2)\right)} \phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^h. \tag{3.16}$$

Combining Lemma 3.3 and Lemma 3.11, we see that radial SLE_κ in $(\mathbb{D} \setminus \eta^{(1)}_{[0,t_1]}; \mathrm{e}^{\mathrm{i}\theta_2}; 0)$ converges to chordal SLE_κ in $(\mathbb{D} \setminus \eta^{(1)}; \mathrm{e}^{\mathrm{i}\theta_2}, 0)$ as $t_1 \to \infty$; and we will show in Lemma 3.12 that $R_{\mathbf{t}} \to 1$ almost surely as $t_1 \to \infty$. Combining these two parts, the conditional law of $\eta^{(2)}_{[0,t_2]}$ given $\eta^{(1)}$ is the same as chordal SLE_κ as desired. □

Lemma 3.11. *Fix $\kappa \in (0,4]$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. We assume $\eta^{(1)} \in \mathfrak{X}(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta_1}; 0)$, namely a simple curve in $\mathbb{D}$ from $\mathrm{e}^{\mathrm{i}\theta_1}$ to 0.*

- *For $t_1 \in (0, \infty)$, we denote by Q_{t_1} the law of a radial* SLE_κ *in $(\mathbb{D} \setminus \eta^{(1)}_{[0,t_1]}; \mathrm{e}^{\mathrm{i}\theta_2}; 0)$.*
- *We denote by Q_∞ the law of a chordal* SLE_κ *in $(\mathbb{D} \setminus \eta^{(1)}; \mathrm{e}^{\mathrm{i}\theta_2}, 0)$.*

Then for any $t_1, t_2 \in (0, \infty)$, the law of the curve $\eta^{(2)}$ restricted to $[0, t_2]$ (under intrinsic capacity parametrization) under Q_{t_1} and Q_∞ are absolutely continuous. We have

$$\lim_{t_1 \to \infty} \frac{\mathrm{d}Q_{t_1}}{\mathrm{d}Q_\infty}\left(\eta^{(2)}_{[0,t_2]}\right) = 1, \quad Q_\infty - a.s.$$

Proof. It is proved in [17, Lem. 3.2] and [21]. To be self-contained, we include its short proof adapted to our setting in Appendix B. □

Lemma 3.12. *Assume the same notations as in the proof of Theorem 3.9 and recall that* $R_{\mathbf{t}}$ *is defined in* (3.16)*. We have*

$$\lim_{t_1\to\infty} R_{\mathbf{t}} = 1, \quad a.s.$$

Proof. We first argue that, almost surely, the difference

$$\Delta_{t_1} := \phi^{(1)}_{t_1}(\theta_2) - \xi^{(1)}_{t_1}$$

is bounded away from 0 and 2π. As the marginal law of $\eta^{(1)}$ is radial $\mathrm{SLE}^{\mu}_{\kappa}(2)$, we have

$$\mathrm{d}\Delta_{t_1} = -\sqrt{\kappa}\mathrm{d}B^{(1)}_{t_1} + 2\cot(\Delta_{t_1}/2)\mathrm{d}t_1 - \mu\mathrm{d}t_1,$$

where $B^{(1)}$ is one-dimensional Brownian motion. Roughly speaking, when Δ_{t_1} is close to zero, it is absolutely continuous with respect to the Bessel process of dimension $8/\kappa+1 \geq 3$; this explains that it is bounded away from zero. We will give more precise details below. Define

$$f(\theta) = \int_{\theta}^{2\pi-\theta} \exp(2\mu u/\kappa)\,(\sin(u/2))^{-8/\kappa}\,\mathrm{d}u, \quad \text{for } \theta \in (0, 2\pi),$$

then $f(\Delta_t)$ is a local martingale. Suppose $\Delta_0 \in (0, 2\pi)$. For $n \geq 1$, define $T_n = \inf\{t : \Delta_t = 2^{-n} \text{ or } \Delta_t = 2\pi - 2^{-n}\}$. For n large enough, we have $\Delta_0 \in (2^{-n}, 2\pi-2^{-n})$. Optional stopping theorem gives $\mathbb{E}[f(\Delta_{T_n})] = f(\Delta_0)$. Thus

$$\mathbb{P}[T_n < \infty] \sim 2^{-n(8/\kappa-1)}, \quad \text{for large } n.$$

In particular, we have $\sum_n \mathbb{P}[T_n < \infty] < \infty$, and Borel-Cantelli lemma tells that almost surely, there exists n_0 such that $T_{n_0} = \infty$. In other words, almost surely, there exists n_0 such that $\Delta_t \in (2^{-n_0}, 2\pi - 2^{-n_0})$ for all t.

Next, we show that $R_{\mathbf{t}} \to 1$ as $t_1 \to \infty$. We write I_{t_1} for the arc in $\partial\mathbb{D}$ that is the image of both sides of $\eta^{(1)}_{[0,t_1]}$ under the conformal map $g^{(1)}_{t_1}$ extended to the boundary, see Figure 5. It is easy to see the harmonic measure of $\partial\mathbb{D}$ seen from 0 of the domain $\mathbb{D} \setminus \eta^{(1)}_{[0,t_1]}$ is decreasing to 0 as $t_1 \to \infty$. Therefore,

$$|I^c_{t_1}| = |\partial\mathbb{D} \setminus I_{t_1}| \xrightarrow{t_1\to\infty} 0$$

where $I^c_{t_1} = \partial\mathbb{D} \setminus I_{t_1}$.

Lemma 3.3 shows that $\eta^{(2)}_{[0,t_2]}$ is at positive distance from $\eta^{(1)}_{[0,\infty)}$. Hence, there exists $\lambda \in (0, 1)$ such that the neighborhood

$$U = \left\{z \in \mathbb{D} \,|\, \mathbb{P}_z \left(\beta \text{ hits } \eta^{(1)}_{[0,\infty)} \text{ before exiting } \mathbb{D}\right) \geq \lambda\right\}$$

satisfies $\eta^{(2)}_{[0,t_2]} \cap U = \emptyset$, where $\mathbb{P}_z$ is the law of a two-dimensional Brownian motion β starting from z. Let

$$U_{t_1} = \left\{z \in \mathbb{D} \,|\, \mathbb{P}_z \left(\beta \text{ hits } \eta^{(1)}_{[0,t_1)} \text{ before exiting } \mathbb{D}\right) \geq \lambda\right\} \subset U.$$

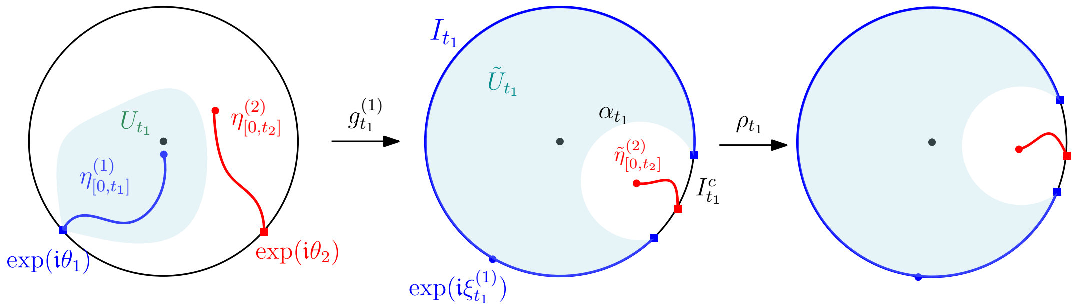

Figure 7: Illustration of the domains U_{t_1} and $\tilde{U}_{t_1} := g^{(1)}_{t_1}(U_{t_1})$ and the rotation map ρ_{t_1}.

The image $\tilde{U}_{t_1} := g^{(1)}_{t_1}(U_{t_1})$ is bounded by I_{t_1} and a circular arc α_{t_1} meeting the endpoints of I_{t_1} with angle λ/π. Since $|I^c_{t_1}| \to 0$, the diameter of the domain $\mathbb{D} \setminus \tilde{U}_{t_1}$, which contains $\tilde{\eta}^{(2)}_{[0,t_2]} = g^{(1)}_{t_1}\left(\eta^{(2)}_{[0,t_2]}\right)$, converges to 0.

Recall that the map $g_{\mathbf{t},1}$ maps out the curve $\tilde{\eta}^{(2)}_{[0,t_2]} = g^{(1)}_{t_1}\left(\eta^{(2)}_{[0,t_2]}\right)$. If we conjugate $g_{\mathbf{t},1}$ by the rotation $\rho_{t_1} : \mathbb{D} \to \mathbb{D}$ such that the image of the mid-point of α_{t_1} under ρ_{t_1} lies in $(0,1)$ (so that $R(\alpha_{t_1})$ is symmetric with respect to the real line and $\rho_{t_1}(\alpha_{t_1})$ shrinks to the point $1 \in \partial\mathbb{D}$), the map $\tilde{g}_{\mathbf{t},1} := \rho_{t_1} \circ g_{\mathbf{t},1} \circ \rho_{t_1}^{-1}$ converges in Carathéodory topology (namely, uniformly on compact subsets) to the identity map in $\mathbb{D}$ as $t_1 \to \infty$. If we Schwarz-reflect $g_{\mathbf{t},1}$ along $\partial\mathbb{D} \setminus g^{(1)}_{t_1}(\exp(\mathrm{i}\theta_2))$, we see that the convergence also extends to the boundary, more precisely, we obtain that $\tilde{g}_{\mathbf{t},1}$ converges uniformly on all compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$ (and the map is well-defined on every such compact subset for large enough t_1), so do the derivatives of $\tilde{g}_{\mathbf{t},1}$ with respect to z.

Finally, since Δ_{t_1} is bounded away from 0 and 2π almost surely, as we proved above, we obtain that $R_{\mathbf{t}} \to 1$ almost surely, which completes the proof. $\square$

3.4 Chordal SLE weighted by conformal radius

In this section, we show that the partition functions $\mathcal{Z}_\alpha$ correspond to the chordal SLE weighted by the conformal radius to the power $-\alpha$.

For this, we first calculate the Laplace transform of the conformal radius of the complement of chordal SLE. Usually, chordal SLE is defined in the upper-half plane as in Appendix B. It is more convenient here to describe it in the unit disc $\mathbb{D}$ via a change of coordinate. Fix $\kappa \in (0,8)$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Suppose γ is chordal SLE$_\kappa$ in $(\mathbb{D}; e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2})$. We parameterize it by the capacity and define g_t, ξ_t accordingly as in Section 3.1. Denote by T the first time γ disconnects $e^{\mathrm{i}\theta_2}$ from the origin. A chordal SLE$_\kappa$ in $(\mathbb{D}; e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2})$, up to T, has the same law as radial SLE$_\kappa(\kappa-6)$ starting from $e^{\mathrm{i}\theta_1}$ with force point $e^{\mathrm{i}\theta_2}$, up to the same time, see [36]. In other words, its driving function ξ_t

solves the following SDE:

$$\begin{cases} \xi_0 = \theta_1, V_0 = \theta_2, \\ \mathrm{d}\xi_t = \sqrt{\kappa}\mathrm{d}B_t + \dfrac{\kappa-6}{2}\cot\left((\xi_t - V_t)/2\right)\mathrm{d}t, \\ \mathrm{d}V_t = \cot\left((V_t-\xi_t)/2\right)\mathrm{d}t. \end{cases} \tag{3.17}$$

Note that the conformal radius $\mathrm{CR}(\mathbb{D}\setminus\gamma)$ is the same as e^{-T}; thus its Laplace transform can be derived from the SDE (3.17).

Lemma 3.13. *Fix $\kappa \in (0,8)$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. We denote $\theta = \theta_2 - \theta_1 \in (0, 2\pi)$. Suppose γ is chordal SLE_κ in $(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2})$ and denote by $\mathbb{E}_\theta$ the expectation with respect to γ. Denote by $\mathrm{CR}(\mathbb{D}\setminus\gamma)$ the conformal radius of $\mathbb{D}\setminus\gamma$ seen from the origin. For $\alpha\in\mathbb{R}$, we define*

$$\Phi(\kappa,\alpha;u) := \mathbb{E}_\theta\left[\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}\right], \quad \text{where } u = (\sin(\theta/4))^2 \in (0,1). \tag{3.18}$$

Then $\Phi(\kappa,\alpha;u)$ is finite for $u\in(0,1)$ if only and if $\alpha < 1-\kappa/8$. Moreover, when $\alpha < 1-\kappa/8$, $\Phi(u) = \Phi(\kappa,\alpha;u)$ satisfies the following ODE

$$u(1-u)\Phi'' + \frac{3\kappa-8}{2\kappa}(1-2u)\Phi' + \frac{8\alpha}{\kappa}\Phi = 0, \tag{3.19}$$

and the symmetry

$$\Phi(u) = \Phi(1-u), \quad u\in(0,1). \tag{3.20}$$

Proof. We first show that $\Phi(\kappa,\alpha;u)$ is finite as long as $\alpha < 1-\kappa/8$. This is done in [31, Proof of Prop. 3.5]. For the reader's convenience, we summarize its proof here.

When $\alpha \le 0$, since $\mathrm{CR}(\mathbb{D}\setminus\gamma)\le 1$ by Schwarz lemma, we obtain immediately that $\Phi(\kappa,\alpha;u)<\infty$.

When $\alpha\in(0,1-\kappa/8)$, we will derive Φ in terms of hypergeometric functions. We set

$$A = 1-\frac{4}{\kappa}+\sqrt{\left(1-\frac{4}{\kappa}\right)^2+\frac{8\alpha}{\kappa}}, \quad B = 1-\frac{4}{\kappa}-\sqrt{\left(1-\frac{4}{\kappa}\right)^2+\frac{8\alpha}{\kappa}}, \quad C = \frac{3}{2}-\frac{4}{\kappa}.$$

Assume $C\notin\mathbb{Z}$ and define

$$f_1(u) := {}_2\mathrm{F}_1(A,B,C;u), \quad f_2(u) := u^{1-C}\,{}_2\mathrm{F}_1(1+A-C, 1+B-C, 2-C, u),$$

where ${}_2\mathrm{F}_1$ is the hypergeometric function (see e.g. [1, Eq.(15.1.1)]). Note that f_1, f_2 are two linearly independent solutions to ODE (3.19). Let us check the values of f_1, f_2 at the endpoints $u=0$ or $u=1$. Since $\kappa\in(0,8)$ and $\alpha\in(0,1-\kappa/8)$ and $C\notin\mathbb{Z}$, we have

$$A<1, \quad B\in(1-8/\kappa, 1-4/\kappa], \quad C\in(-\infty,1)\setminus\mathbb{Z}, \quad C > A+B.$$

From [1, Eq.(15.1.20)], we have

$$f_1(0) = 1, \quad f_1(1) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} = \frac{\cos\left(\pi\sqrt{\left(1-\frac{4}{\kappa}\right)^2+\frac{8\alpha}{\kappa}}\right)}{\cos\left(\pi\left(1-\frac{4}{\kappa}\right)\right)};$$

$$f_2(0) = 0, \quad f_2(1) = \frac{\Gamma(2-C)\Gamma(1-C)}{\Gamma(1-A)\Gamma(1-B)} \in (0,\infty).$$

We parameterize γ by the capacity, then its driving function ξ_t solves SDE (3.17). We denote $\theta_t = V_t - \xi_t$. The process θ_t satisfies the SDE:

$$\mathrm{d}\theta_t = \sqrt{\kappa}\mathrm{d}B_t + \frac{\kappa-4}{2}\cot(\theta_t/2)\mathrm{d}t. \tag{3.21}$$

The disconnection time T is the first time that θ_t hits 0 or 2π. Suppose f is an analytic function defined on $(0,1)$. Then $\mathrm{e}^{\alpha t} f\left((\sin(\theta_t/4))^2\right)$ is a local martingale if and only if f satisfies (3.19). Since f_1, f_2 are solutions to this ODE, the processes

$$\mathrm{e}^{\alpha t} f_1\left((\sin(\theta_t/4))^2\right) \quad \text{and} \quad \mathrm{e}^{\alpha t} f_2\left((\sin(\theta_t/4))^2\right)$$

are local martingales. These martingales are also considered in [35]. Since f_1, f_2 are finite at $u=0$ and $u=1$, and the lifetime T has finite expectation, we may conclude that these two local martingales are martingales up to T. Then the optional stopping theorem gives

$$\begin{cases} \mathbb{E}_\theta\left[\mathrm{e}^{\alpha T} 1_{\{\theta_T = 0\}}\right] + f_1(1)\mathbb{E}_\theta\left[\mathrm{e}^{\alpha T} 1_{\{\theta_T = 2\pi\}}\right] = f_1\left((\sin(\theta/4))^2\right); \\ f_2(1)\mathbb{E}_\theta\left[\mathrm{e}^{\alpha T} 1_{\{\theta_T = 2\pi\}}\right] = f_2\left((\sin(\theta/4))^2\right). \end{cases}$$

As $\mathrm{CR}(\mathbb{D}\setminus\gamma) = \mathrm{e}^{-T}$, the above relation gives

$$\mathbb{E}_\theta\left[\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}\right] = f_1\left((\sin(\theta/4))^2\right) + \frac{1-f_1(1)}{f_2(1)} f_2\left((\sin(\theta/4))^2\right). \tag{3.22}$$

In particular, this implies that $\Phi(\kappa,\alpha;u)$ is finite for $u \in (0,1)$ when

$$\kappa \in (0,8), \quad C = \frac{3}{2} - \frac{4}{\kappa} \notin \mathbb{Z}, \quad \alpha \in (0, 1-\kappa/8).$$

As $\Phi(\kappa,\alpha;u)$ is continuous in $\kappa \in (0,8)$ and is increasing in α, we conclude that $\Phi(\kappa,\alpha;u)$ is finite for $u \in (0,1)$ when

$$\kappa \in (0,8), \quad \alpha < 1-\kappa/8.$$

Moreover, when $\alpha \in (0, 1-\kappa/8)$ and $C = 3/2 - 4/\kappa \notin \mathbb{Z}$,

$$\Phi(u) = \Phi(\kappa,\alpha;u) = \mathbb{E}_\theta[\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}] = f_1(u) + \frac{1-f_1(1)}{f_2(1)} f_2(u)$$

satisfies (3.19). In fact, Φ satisfies (3.19) for all $\kappa \in (0,8)$ and $\alpha < 1-\kappa/8$. Note that

$$\mathrm{e}^{\alpha t}\Phi\left((\sin(\theta_t/4))^2\right) = \mathbb{E}_\theta\left[\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha} \mid \gamma_{[0,t]}\right] \tag{3.23}$$

is a martingale and θ_t satisfies (3.21). Thus, Φ is a weak solution for (3.19) and

$$(\sin((\theta_2-\theta_1)/2))^{-2h}\,\Phi\left((\sin((\theta_2-\theta_1)/4))^2\right)$$

is a weak solution to the radial BPZ equations. As the operators in the radial BPZ equations are hypoelliptic, see Remark 2.8, weak solutions are strong solutions. Thus Φ

is a C^2 solution to (3.19) for all $\kappa \in (0,8)$ and $\alpha < 1-\kappa/8$. The symmetry in (3.20) is clear from the definition.

Finally, let us consider the case when $\alpha \geq 1-\kappa/8$. Fix $\kappa \in (0,8)$ with $C \notin \mathbb{Z}$. As $\alpha \uparrow (1-\kappa/8)$, we have

$$\begin{aligned}
&A \to 1, \quad B \to (1-8/\kappa), \\
&f_1(u) \to {}_2\mathrm{F}_1(1, 1-8/\kappa, 3/2-4/\kappa; u) \in (-\infty, \infty), && f_1(1) \to -1, \\
&f_2(u) \to u^{1-C}\, {}_2\mathrm{F}_1(1+4/\kappa, 1/2-4/\kappa, 1/2+4/\kappa; u) \neq 0, && f_2(1) \to 0.
\end{aligned}$$

Plugging into (3.22), we see that

$$\Phi(\kappa, \alpha; u) \uparrow \infty, \quad \text{as } \alpha \uparrow (1-\kappa/8).$$

Note that $\Phi(\kappa, \alpha; u)$ is increasing in α. This completes the proof. □

Corollary 3.14. *Fix $\kappa \in (0,8)$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. We denote $\theta = \theta_2 - \theta_1 \in (0, 2\pi)$. Suppose γ is chordal SLE_κ in $(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2})$ and denote by $\mathbb{E}_\theta$ the expectation with respect to γ. Recall that $\mathcal{Z}_\alpha$ is defined in (1.4) for $\alpha < 1-\kappa/8$ and $\mathcal{G}_\mu$ is defined in (1.3) for $\mu \in \mathbb{R}$. Recall that $h = \frac{6-\kappa}{2\kappa}$ from (3.1). Then we have*

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin(\theta/2))^{-2h} \frac{\mathbb{E}_\theta[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]}{\mathbb{E}_\pi[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]}, \quad \textit{for } \alpha < 1-\kappa/8. \tag{3.24}$$

Moreover, we have

$$\frac{\mathbb{E}_\theta[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]}{\mathbb{E}_\pi[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]} = (\sin(\theta/2))^{2h}\, \mathcal{Z}_\alpha(\theta_1, \theta_2) \to (\sin(\theta/2))^{2h}\, \mathcal{G}_0(\theta_1, \theta_2) \tag{3.25}$$

as $\alpha \uparrow (1-\kappa/8)$.

Proof. We denote $u = (\sin(\theta/4))^2$. Recall from (1.4) that ϕ_α is the unique solution to (1.5). Comparing with (3.19) and (3.20), it is clear that

$$\phi_\alpha(\cdot) = \frac{\Phi(\kappa, \alpha; \cdot)}{\Phi(\kappa, \alpha; 1/2)}.$$

Thus

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) := (\sin(\theta/2))^{-2h}\, \phi_\alpha(u) = (\sin(\theta/2))^{-2h} \frac{\mathbb{E}_\theta[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]}{\mathbb{E}_\pi[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]},$$

as desired in (3.24). Moreover, we have

$$\frac{\mathbb{E}_\theta[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]}{\mathbb{E}_\pi[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]} = (\sin(\theta/2))^{2h}\, \mathcal{Z}_\alpha(\theta_1, \theta_2) = \phi_\alpha(u)$$

which converges to $\phi_{\alpha_0}(u) = (\sin(\theta/2))^{2h}\, \mathcal{G}_0(\theta_1, \theta_2)$ as $\alpha \to \alpha_0 = 1-\kappa/8$ by Lemma A.1. This gives (3.25). □

Corollary 3.15. *Fix $\kappa \in (0,8)$ and $\alpha < 1-\kappa/8$. Denote by $\mathbb{P}$ the law of γ chordal SLE_κ in $(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ with $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Let $\eta^{(1)}$ be γ and let $\eta^{(2)}$ be the time-reversal of γ and still denote by $\mathbb{P}$ the induced law on $(\eta^{(1)}, \eta^{(2)})$. We define $\mathcal{Z}_\alpha$ as in (1.4). Denote by $\mathbb{P}(\mathcal{Z}_\alpha)$ the probability measure obtained by weighting $\mathbb{P}$ by $\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}$. Then, under $\mathbb{P}(\mathcal{Z}_\alpha)$, the family of local laws obtained by restricting the pair $(\eta^{(1)}, \eta^{(2)})$ in disjoint neighborhoods satisfies **(CI)**, **(DMP)**, **(MARG)** and **(INT)** with*

$$b_j = \kappa \partial_j \log \mathcal{Z}_\alpha, \quad j = 1,2.$$

More precisely, the driving function of $\eta^{(1)}$ solves the following SDE, up to the first time $e^{i\theta_2}$ is disconnected from the origin:

$$\begin{cases} \xi_0^{(1)} = \theta_1, V_0^{(2)} = \theta_2, \\ \mathrm{d}\xi_t^{(1)} = \sqrt{\kappa}\mathrm{d}\tilde{B}_t^{(1)} + \kappa\partial_1(\log \mathcal{Z}_\alpha)(\xi_t^{(1)}, V_t^{(2)})\mathrm{d}t, \\ \mathrm{d}V_t^{(2)} = \cot\left((V_t^{(2)} - \xi_t^{(1)})/2\right)\mathrm{d}t, \end{cases} \tag{3.26}$$

where $\tilde{B}^{(1)}$ is Brownian motion under $\mathbb{P}(\mathcal{Z}_\alpha)$. Similarly, the driving function of $\eta^{(2)}$ solves the following SDE, up to the first time $e^{i\theta_1}$ is disconnected from the origin:

$$\begin{cases} V_0^{(1)} = \theta_1, \xi_0^{(2)} = \theta_2, \\ \mathrm{d}\xi_t^{(2)} = \sqrt{\kappa}\mathrm{d}\tilde{B}_t^{(2)} + \kappa\partial_2(\log \mathcal{Z}_\alpha)(V_t^{(1)}, \xi_t^{(2)})\mathrm{d}t, \\ \mathrm{d}V_t^{(1)} = \cot\left((V_t^{(1)} - \xi_t^{(2)})/2\right)\mathrm{d}t, \end{cases} \tag{3.27}$$

where $\tilde{B}^{(2)}$ is Brownian motion under $\mathbb{P}(\mathcal{Z}_\alpha)$.

Proof. The fact that the local laws obtained by restricting the pair $(\eta^{(1)}, \eta^{(2)}) \sim \mathbb{P}(\mathcal{Z}_\alpha)$ in disjoint neighborhoods satisfies **(CI)**, **(DMP)**, **(MARG)** and **(INT)** follows from the reversibility of SLE (proved in [26, 27, 39]): suppose γ is chordal SLE_κ in $\mathbb{D}$ from $e^{i\theta_1}$ to $e^{i\theta_2}$ with $\kappa \in (0,8)$, the time-reversal of γ has the same law as chordal SLE_κ in $\mathbb{D}$ from $e^{i\theta_2}$ to $e^{i\theta_1}$. It remains to check (3.26) and (3.27). As the pair $(\eta^{(1)}, \eta^{(2)})$ is interchangeable, it suffices to check (3.26).

Denote $\Phi(\cdot) = \Phi(\kappa, \alpha; \cdot)$ as in (3.18). Using the same notations as in the proof of Lemma 3.13, we denote the martingale in (3.23) by

$$M_t(\alpha) := e^{\alpha t}\Phi\left((\sin(\theta_t/4))^2\right).$$

Then $\mathbb{P}(\mathcal{Z}_\alpha)$ is the same as $\mathbb{P}$ tilting by $M_t(\alpha)$. Recall from (3.21), under $\mathbb{P}$, we have

$$\mathrm{d}\theta_t = \sqrt{\kappa}\mathrm{d}B_t + \frac{\kappa-4}{2}\cot(\theta_t/2)\mathrm{d}t.$$

Thus, under $\mathbb{P}$, we have

$$\frac{\mathrm{d}M_t(\alpha)}{M_t(\alpha)} = \frac{\sqrt{\kappa}}{4}\frac{\Phi'}{\Phi}\sin(\theta_t/2)\mathrm{d}B_t.$$

Girsanov's theorem tells that

$$\tilde{B}_t = B_t - \frac{\sqrt{\kappa}}{4}\frac{\Phi'}{\Phi}\sin(\theta_t/2)\mathrm{d}t$$

is Brownian motion under $\mathbb{P}(\mathcal{Z}_\alpha)$. Combining with (3.24), we obtain (3.26). □

Now, we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The conclusion follows from Theorem 2.4, Corollary 3.7 and Corollary 3.15. □

3.5 Commuting SLEs without interchangeability

We may also classify the commuting SLEs without interchangeability. Let us first give an example.

Remark 3.16. Using the same notations as in Lemma 3.13, for $\kappa \in (0,8)$ and $\alpha < 1-\kappa/8$, we consider

$$\begin{aligned} \Phi^L(\kappa,\alpha;u) :=& \mathbb{E}_\theta\left[\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}1_{\{0 \text{ is to the left of } \gamma\}}\right]; \\ \Phi^R(\kappa,\alpha;u) :=& \mathbb{E}_\theta\left[\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}1_{\{0 \text{ is to the right of } \gamma\}}\right]. \end{aligned}$$

Then $\Phi^L(\cdot) = \Phi^L(\kappa,\alpha;\cdot)$ and $\Phi^R(\cdot) = \Phi^R(\kappa,\alpha;\cdot)$ also satisfy the ODE (3.19), but they do not enjoy the symmetry (3.20) anymore. This gives an example of locally commuting 2-radial SLE without interchangeability.

Moreover, we denote $\theta = \theta_2 - \theta_1$ and set

$$\begin{aligned} \mathcal{Z}^L(\theta_1,\theta_2) &= (\sin(\theta/2))^{-2h}\,\Phi^L(\kappa,\alpha;(\sin(\theta/4))^2); \\ \mathcal{Z}^R(\theta_1,\theta_2) &= (\sin(\theta/2))^{-2h}\,\Phi^R(\kappa,\alpha;(\sin(\theta/4))^2). \end{aligned}$$

Then both $\mathcal{Z}^L$ and $\mathcal{Z}^R$ satisfy (2.14) and (2.15) with

$$F = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \alpha.$$

Proposition 3.17. *If one removes the interchangeability condition in Theorem 2.4, one obtains partition functions, $\mathcal{Z}$, of the form:*

1. *$\mathcal{Z} = \mathcal{G}_\mu$ for some $\mu \in \mathbb{R}$, where $\mathcal{G}_\mu$ is defined as in* (1.3)*.*
2. *$\mathcal{Z} = \mathcal{Z}_{\alpha,\beta}$, for $\alpha < 1-\kappa/8$ and $\beta \in [0,1]$, where*

$$\mathcal{Z}_{\alpha,\beta}(\theta_1,\theta_2) = (\sin(\theta_{21}/2))^{-2h}(\beta\Phi^L((\sin(\theta_{21}/4))^2) + (1-\beta)\Phi^R((\sin(\theta_{21}/4))^2)).$$

The locally commuting 2-radial SLE_κ corresponding to the second case above is obtained analogously to Corollary 3.15. That is, one weights the law of a chordal SLE_κ in $(\mathbb{D};e^{i\theta_1},e^{i\theta_2})$, denoted by γ, by

$$\mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\alpha}(\beta 1_{\{0 \text{ is to the left of } \gamma\}} + (1-\beta)1_{\{0 \text{ is to the right of } \gamma\}}),$$

then lets $\eta^{(1)}$ be γ and $\eta^{(2)}$ be the time-reversal of γ, and finally restricts the law of $(\eta^{(1)},\eta^{(2)})$ to disjoint neighborhoods. One sees that the obtained family of local laws satisfies **(CI)**, **(DMP)**, and **(MARG)** by using the reversibility of SLE: this makes $\eta^{(2)}$ a chordal SLE in $(\mathbb{D};e^{i\theta_2},e^{i\theta_1})$ weighted by

$$\mathrm{CR}(\mathbb{D}\setminus\eta^{(2)})^{-\alpha}(\beta 1_{\{0 \text{ is to the right of } \eta^{(2)}\}} + (1-\beta)1_{\{0 \text{ is to the left of } \eta^{(2)}\}}).$$

Proof of Proposition 3.17. If we in the proof of Theorem 2.4 assume that $\mu = 0$ without assuming interchangeability, then $\mathcal{Z}$ is a positive solution of (2.25). By changing variables by (2.27), we find that $\mathcal{Z}$ corresponds to a positive solution ϕ of (3.19). Lemma A.2, shows that (up to a multiplicative constant)

$$\phi = \beta\Phi^L + (1-\beta)\Phi^R, \quad \beta \in [0,1], \tag{3.28}$$

if $\alpha < 1 - \kappa/8$, that there are no positive solutions if $\alpha > 1 - \kappa/8$, and that there is only one positive solution, corresponding to $\mathcal{Z} = \mathcal{G}_0$, if $\alpha = 1 - \kappa/8$. □

4 Semiclassical limits of commutation relation

4.1 Commutation relation when $\kappa = 0$

We now consider the commutation relation when $\kappa = 0$. In the multichordal SLE case, a similar semiclassical limit of partition functions was considered in [29] and [3].

It is not hard to see that the infinitesimal commutation relation (Proposition 2.2) also holds when $\kappa = 0$. However, we need to derive the BPZ equation and classify the partition functions differently, which we summarize in the following proposition.

Proposition 4.1. *We consider an interchangeable and locally commuting* 2*-radial* SLE_0*. Let* $b_1, b_2 : S^1 \times S^1 \setminus \Delta \to \mathbb{R}$ *be* C^2 *functions as in the condition* ***(MARG)****. Then* (2.4) *and* (2.6) *imply that there exists* $\mathcal{U} : \{(\theta_1,\theta_2) \in \mathbb{R}^2 \,|\, \theta_1 < \theta_2 < \theta_1 + 2\pi\} \to \mathbb{R}$ *and a constant* C *such that*

$$b_j = \partial_j \mathcal{U}, \quad j = 1,2$$

and

$$\begin{aligned} (\partial_2\mathcal{U})^2 + 2\cot(\theta_{12}/2)\partial_1\mathcal{U} - \frac{3}{(\sin(\theta_{12}/2))^2} &= C, \\ (\partial_1\mathcal{U})^2 + 2\cot(\theta_{21}/2)\partial_2\mathcal{U} - \frac{3}{(\sin(\theta_{21}/2))^2} &= C, \end{aligned} \tag{4.1}$$

where $\theta_{21} = \theta_2 - \theta_1 = -\theta_{12}$*. The only solutions are*

$$\mathcal{U}(\theta_1,\theta_2) = \mathcal{U}_\mu(\theta_1,\theta_2) := 2\log\sin(\theta_{21}/2) + \mu(\theta_1+\theta_2) \tag{4.2}$$

for some $\mu \in \mathbb{R}$ *or*

$$\mathcal{U}(\theta_1,\theta_2) = -6\log\sin(\theta_{21}/2) \tag{4.3}$$

up to an additive constant.

Proof. The proof of Proposition 2.3 up to (2.11) holds verbatim when $\kappa = 0$.

From (2.9) we know that there is a function $\mathcal{U} : \{(\theta_1,\theta_2) \,|\, \theta_1 < \theta_2 < \theta_1 + 2\pi\} \to \mathbb{R}$ such that we can write $b_1 = \partial_1\mathcal{U}$ and $b_2 = \partial_2\mathcal{U}$. Equations (2.10) and (2.11) give:

$$\partial_1\left((\partial_2\mathcal{U})^2 + 2\cot(\theta_{12}/2)\partial_1\mathcal{U} - \frac{3}{(\sin(\theta_{12}/2))^2}\right) = 0 \tag{4.4}$$

$$\partial_2 \left((\partial_1 \mathcal{U})^2 + 2\cot(\theta_{21}/2)\partial_2 \mathcal{U} - \frac{3}{(\sin(\theta_{21}/2))^2} \right) = 0. \tag{4.5}$$

Hence, there exist functions $C_1(\theta_1)$ and $C_2(\theta_2)$ such that

$$\begin{aligned} (\partial_2 \mathcal{U})^2 + 2\cot(\theta_{12}/2)\partial_1 \mathcal{U} - \frac{3}{(\sin(\theta_{12}/2))^2} &= C_2(\theta_2) \\ (\partial_1 \mathcal{U})^2 + 2\cot(\theta_{21}/2)\partial_2 \mathcal{U} - \frac{3}{(\sin(\theta_{21}/2))^2} &= C_1(\theta_1). \end{aligned} \tag{4.6}$$

Since b_j is invariant under the rotation $(\theta_1, \theta_2) \mapsto (\theta_1 + a, \theta_2 + a)$, so are the left-hand sides of (4.6). Thus, $C_1(\theta_1) = C_1$ and $C_2(\theta_2) = C_2$ are constant. The rotational invariance of b_j also gives

$$0 = \partial_1(\partial_1 \mathcal{U} + \partial_2 \mathcal{U}) = \partial_2(\partial_1 \mathcal{U} + \partial_2 \mathcal{U}).$$

We let $\mu \in \mathbb{R}$ such that

$$2\mu \equiv \partial_1 \mathcal{U} + \partial_2 \mathcal{U}.$$

Taking the difference of the equations in (4.6), we obtain

$$(\partial_1 \mathcal{U} - \partial_2 \mathcal{U} + 2\cot(\theta_{21}/2))\, 2\mu = C_1 - C_2 \tag{4.7}$$

Case 1: If $\mu \neq 0$, then

$$\begin{cases} \partial_1 \mathcal{U} = \frac{C_1 - C_2}{4\mu} + \mu - \cot(\theta_{21}/2), \\ \partial_2 \mathcal{U} = \frac{C_2 - C_1}{4\mu} + \mu + \cot(\theta_{21}/2). \end{cases}$$

Plugging this back into (4.6) shows that $C_1 = C_2 =: C$ and $C = \mu^2 - 3$. Therefore, up to an additive constant, we have

$$\mathcal{U}(\theta_1, \theta_2) = 2\log\sin(\theta_{21}/2) + \mu(\theta_1 + \theta_2).$$

Case 2: If $\mu = \partial_1 \mathcal{U} + \partial_2 \mathcal{U} = 0$, then it follows directly that $C_1 = C_2 =: C$. Moreover, $\mathcal{U}$ only depends on $\theta := \theta_2 - \theta_1$ and writing $\mathcal{U}(\theta) = \mathcal{U}(\theta_1, \theta_2)$ with a slight abuse of notation, we have from the interchangeability condition that

$$\mathcal{U}(\theta) = \mathcal{U}(2\pi - \theta). \tag{4.8}$$

Thus, (4.1) becomes

$$(\mathcal{U}'(\theta) + \cot(\theta/2))^2 = C + \frac{3 + (\cos(\theta/2))^2}{(\sin(\theta/2))^2}. \tag{4.9}$$

Eq. (4.9) gives the solutions

$$\mathcal{U}'(\theta) = \pm\sqrt{C + \frac{3 + (\cos(\theta/2))^2}{(\sin(\theta/2))^2}} - \cot(\theta/2). \tag{4.10}$$

Since $\mathcal{U}'(\pi) = 0$ from (4.8), this implies that $C = -3$ and

$$\mathcal{U}'(\theta) = \cot(\theta/2) \text{ or } -3\cot(\theta/2).$$

Namely,

$$\mathcal{U}(\theta) = 2\log\sin(\theta/2) \text{ or } -6\log\sin(\theta/2)$$

up to an additive constant. □

We have defined the two-sided radial SLE_κ with spiraling rate μ for $\kappa \in (0, 8)$ in Definition 3.6, we now extend the definition to the case $\kappa = 0$.

Definition 4.2. We use the same notations as in Figure 5, we say that a deterministic pair of continuous simple curves $(\eta^{(1)}, \eta^{(2)})$ is the two-sided radial SLE_0 with spiraling rate μ in $(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2}; 0)$ if we have for all $\mathbf{t} = (t_1, t_2)$,

$$\begin{cases} \theta^{(1)}_{(0,0)} = \theta_1, \quad \theta^{(2)}_{(0,0)} = \theta_2, \\ d\theta^{(1)}_{\mathbf{t}} = \mu\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2 dt_1 + \cot\left((\theta^{(1)}_{\mathbf{t}} - \theta^{(2)}_{\mathbf{t}})/2\right)\left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2 dt_1 + \phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2 dt_2\right), \\ d\theta^{(2)}_{\mathbf{t}} = \mu\phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2 dt_2 + \cot\left((\theta^{(2)}_{\mathbf{t}} - \theta^{(1)}_{\mathbf{t}})/2\right)\left(\phi'_{\mathbf{t},1}\left(\xi^{(1)}_{t_1}\right)^2 dt_1 + \phi'_{\mathbf{t},2}\left(\xi^{(2)}_{t_2}\right)^2 dt_2\right). \end{cases}$$

To justify such a pair exists, we note that when $t_2 \equiv 0$, we write $\theta^{(1)}_{\mathbf{t}} = \xi^{(1)}_{t_1}$ and $\theta^{(2)}_{\mathbf{t}} = V^{(2)}_{t_1}$, then

$$\begin{cases} \xi^{(1)}_0 = \theta_1, \quad V^{(2)}_0 = \theta_2, \\ d\xi^{(1)}_{t_1} = \mu dt_1 + \cot\left((\xi^{(1)}_{t_1} - V^{(2)}_{t_1})/2\right) dt_1 = \partial_1\mathcal{U}_\mu(\xi^{(1)}_{t_1}, V^{(2)}_{t_1})dt_1, \\ dV^{(2)}_{t_1} = \cot\left((V^{(2)}_{t_1} - \xi^{(1)}_{t_1})/2\right) dt_1, \end{cases} \tag{4.11}$$

which shows $\eta^{(1)}$ is the radial $\mathrm{SLE}_0^\mu(2)$ with force point $e^{i\theta_2}$. Similarly, $\eta^{(2)}$ is the radial $\mathrm{SLE}_0^\mu(2)$ with force point $e^{i\theta_1}$. As in Corollary 3.7, the system of equations in Definition 4.2 is simply the two-time version of (4.11) starting from $\mathbf{t}$ after capacity-reparametrization.

Since $\mathcal{U}_\mu$ satisfies the commutation relation as we showed in Proposition 4.1, we know that $(\eta^{(1)}, \eta^{(2)})$ gives a pair in Definition 4.2 if we show that $\theta^{(2)}_{\mathbf{t}} \notin \{\theta^{(1)}_{\mathbf{t}}, \theta^{(1)}_{\mathbf{t}} + 2\pi\}$ for all $\mathbf{t}$. Indeed, the map $t_2 \mapsto \theta^{(2)}_{(0,t_2)} - \theta^{(1)}_{(0,t_2)}$ is repulsive away from $\{0, 2\pi\}$, so is well-defined and for all $t_2 \geq 0$. Similarly, for fixed t_2, the function $t_1 \mapsto \theta^{(2)}_{(t_1,t_2)} - \theta^{(1)}_{(t_1,t_2)}$ is repulsive away from $\{0, 2\pi\}$, so $\theta^{(2)}_{\mathbf{t}} \notin \{\theta^{(1)}_{\mathbf{t}}, \theta^{(1)}_{\mathbf{t}} + 2\pi\}$ for all $t_1, t_2 \geq 0$.

Corollary 4.3. *The only interchangeable and locally commuting* 2*-radial* SLE_0 *are the two-sided radial* SLE_0 *with spiraling rate* $\mu \in \mathbb{R}$ *and the chordal* SLE_0.

Proof. When $\mathcal{U} = \mathcal{U}_\mu$, the corresponding commuting SLE_0 is the two-sided radial SLE_0 (with spiraling rate μ).

When $\mathcal{U}(\theta_1, \theta_2) = -6\log\sin(\theta_{21}/2)$, the corresponding commuting SLE_0 is the chordal SLE_0 in $(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ and its time-reversal. □

Remark 4.4. We note that when $\mathcal{U}(\theta_1, \theta_2) = \mathcal{U}_0(\theta_1, \theta_2) = 2 \log \sin(\theta_{21}/2)$, the corresponding commuting SLE_0 is the two-sided radial SLE_0 (with zero spiraling rate). This is also the geodesic pair in $\mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2}; 0)$ studied in [24, 25, 38], or in other words, the concatenation of $\eta^{(1)}$ and $\eta^{(2)}$ is the chord minimizing the chordal Loewner energy among all chords in $(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ passing through 0, hence can be viewed as the chordal SLE_0 "conditioned" to pass through 0.

We remark that, unlike the $\kappa > 0$ case, we do not have the second one-parameter family given by weighting by conformal radius as described in Section 3.4.

4.2 Semiclassical limits of partition functions

To understand the reason why the second one-parameter family given by $\mathcal{Z}_\alpha$ disappears when $\kappa = 0$, we now take a closer look at the semiclassical limit ($\kappa \to 0$) of $\mathcal{G}_\mu$ and $\mathcal{Z}_\alpha$.

For $\mu \in \mathbb{R}$, recall that $\mathcal{G}_\mu$ is defined in (1.3):

$$\mathcal{G}_\mu(\theta_1, \theta_2) = (\sin((\theta_2 - \theta_1)/2))^{2/\kappa} \exp\left(\frac{\mu}{\kappa}(\theta_1 + \theta_2)\right).$$

Lemma 4.5. *Fix $\mu \in \mathbb{R}$ and $0 < \theta_1 < \theta_2 < \theta_1 + 2\pi$, we have (1.6):*

$$\lim_{\kappa \to 0} \kappa \log \mathcal{G}_\mu(\theta_1, \theta_2) = 2 \log \sin((\theta_2 - \theta_1)/2) + \mu(\theta_1 + \theta_2)$$

which is the solution (4.2) in Proposition 4.1.

Proof. This is immediate from the expression of $\mathcal{G}_\mu$. □

Lemma 4.5 gives the first part of Proposition 1.3. We will prove its second part below. For $\alpha < 1 - \kappa/8$, recall that $\mathcal{Z}_\alpha$ is defined in (1.4):

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin((\theta_2 - \theta_1)/2))^{(\kappa-6)/\kappa} \phi_\alpha\left((\sin((\theta_2 - \theta_1)/4))^2\right),$$

where ϕ_α is the unique solution to (1.5). Before we derive semiclassical limit of $\mathcal{Z}_\alpha$, let us first address chordal Loewner energy [29]. Recall that $\mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ is the space of all continuous curves in $\mathbb{D}$ connecting $e^{i\theta_1}$ and $e^{i\theta_2}$.

Lemma 4.6. *Fix $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Suppose $\gamma \in \mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$. We parameterize it by the capacity and define g_t, ξ_t accordingly as in Section 3.1 and set $T = -\log \mathrm{CR}(\mathbb{D} \setminus \gamma)$. Suppose $t \mapsto \xi_t$ is absolutely continuous and denote its derivative by $\dot{\xi}_t$. We denote by $t \mapsto V_t$ the solution to*

$$\dot{V}_t = \cot((V_t - \xi_t)/2), \quad V_0 = \theta_2. \tag{4.12}$$

Then the chordal Loewner energy $I(\gamma)$ can be written as

$$I(\gamma) = \frac{1}{2} \int_0^T \left(\dot{\xi}_s - 3 \cot((V_s - \xi_s)/2)\right)^2 \mathrm{d}s. \tag{4.13}$$

Furthermore, the infimum of $I(\gamma)$ in $\mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ is zero and is attained by the hyperbolic geodesic.

Proof. Eq. (4.13) is a standard calculation by changing coordinates. □

Lemma 4.7. *If we choose* $\alpha = \alpha(\kappa)$ *such that* $\alpha = o(1/\kappa)$*, then*

$$\lim_{\kappa \to 0} \kappa \log \mathcal{Z}_\alpha(\theta_1, \theta_2) = -6 \log \sin(\theta_{21}/2) \tag{4.14}$$

which is the solution (4.3) *in Proposition 4.1.*

If $\alpha \sim -\lambda/\kappa$ *for some* $\lambda > 0$*, then the limit* $\lim_{\kappa\to 0} \kappa \log \mathcal{Z}_\alpha(\theta_1, \theta_2)$ *exists and equals*

$$\begin{aligned} \mathcal{U}^\lambda(\theta_1, \theta_2) := -6 \log \sin(\theta_{21}/2) - &\inf_{\gamma \in \mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})} (I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma)) \\ + &\inf_{\gamma \in \mathfrak{X}(\mathbb{D}; -1, 1)} (I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma)), \end{aligned} \tag{4.15}$$

where $I(\gamma)$ *is the chordal Loewner energy of* γ *in* $\mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ *and the infimums are attained.*

We note that the constraint $\alpha < 1 - \kappa/8$ implies that we can only choose $\lambda \geq 0$ and the last term in (4.15) is a constant such that $\mathcal{U}^\lambda(\theta_1, \theta_1 + \pi) = 0$ as we have normalized $\mathcal{Z}_\alpha$ such that $\mathcal{Z}_\alpha(\theta_1, \theta_1 + \pi) = 1$.

Proof. For $\alpha < 1 - \kappa/8$, recall from Lemma 3.13 and (3.24) that $\mathcal{Z}_\alpha$ is defined as

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin(\theta/2))^{(\kappa-6)/\kappa} \frac{\mathbb{E}_\theta[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]}{\mathbb{E}_\pi[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}]},$$

where $\mathbb{E}_\theta$ is the expectation with respect to the law of γ which is a chordal SLE_κ in $(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ and $\theta = \theta_2 - \theta_1$.

Then the result follows from the large deviation principle for chordal SLE as $\kappa \to 0+$ [29]. In fact, the Loewner energy is the large deviation rate function of chordal SLE_{0+} for the Hausdorff metric and $\gamma \mapsto -\log \mathrm{CR}(\mathbb{D} \setminus \gamma)$ is a continuous function $\mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2}) \to [0, \infty]$. Varadhan's lemma [8, Lem. 4.3.4 and 4.3.6] shows if $\alpha \sim -\lambda/\kappa$, then

$$\lim_{\kappa \to 0} \kappa \log \mathbb{E}_\theta[\mathrm{CR}(\mathbb{D} \setminus \gamma)^{-\alpha}] = -\inf_{\gamma \in \mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})} (I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma))$$

which proves the limit (4.15). Since the large deviation rate function I of chordal SLE_{0+} is good, the infimum in (4.15) is attained.

Similarly, an easy bound and Varadhan's lemma also show the limit (4.14). □

Proposition 4.8. *For* $\theta_1 < \theta_2 \leq \theta_1 + \pi$*, we denote* $\theta = \theta_2 - \theta_1$ *and we have*

$$\inf_{\gamma \in \mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})} (I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma)) = \int_0^\theta \left(\sqrt{2\lambda + 4\cot^2(u/2)} - 2\cot(u/2) \right) \mathrm{d}u. \tag{4.16}$$

- *If* $\theta_1 < \theta_2 < \theta_1 + \pi$*, the infimum in* (4.16) *is attained for a unique curve* $\gamma^* \in \mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2})$ *whose radial driving function* $\xi^*_\cdot$*, defined on* $[0, T]$*, satisfies*

$$\begin{cases} \xi^*_0 = \theta_1, V^*_0 = \theta_2, \\ \dot{\xi}^*_t = \cot((V^*_t - \xi^*_t)/2) + \sqrt{2\lambda + 4\cot^2((V^*_t - \xi^*_t)/2)}, \\ \dot{V}^*_t = \cot((V^*_t - \xi^*_t)/2), \end{cases} \tag{4.17}$$

  and $\lim_{t \to T-}(V^*_t - \xi^*_t) = 0$.

- *If instead $\theta_2 = \theta_1 + \pi$, then the infimum in* (4.16) *is attained for two curves, γ^* and γ^{**}. One of the corresponding driving functions, say $\xi^*_\cdot$, satisfies* (4.17) *so that* $\lim_{t\to T-}(V_t^* - \xi_t^*) = 0$, *while the other $\xi^{**}_\cdot$, satisfies $\dot\xi_t^{**} = -\dot\xi_t^*$ for all $t \in [0,T)$, so that* $\lim_{t\to T-}(V_t^{**} - \xi_t^{**}) = 2\pi$.

Proof. We denote the right-hand side of (4.16) by

$$H_\lambda(\theta) = \int_0^\theta \Big(\sqrt{2\lambda + 4\cot^2(u/2)} - 2\cot(u/2)\Big)\mathrm{d}u, \tag{4.18}$$

for $\theta \in [0,\pi]$, and $H_\lambda(\theta) = H_\lambda(2\pi - \theta)$ for $\theta \in (\pi, 2\pi]$. Let $\gamma_{[0,t]} : [0,t] \to \mathbb{D} \setminus \{0\}$ be a simple curve, parametrized by capacity, with an absolutely continuous driving function $t \mapsto \xi_t$ and set V_t as in (4.12). Define

$$J^\lambda_{(\theta_1,\theta_2)}\left(\gamma_{[0,t]}\right) := \frac{1}{2}\int_0^t \Big(\dot\xi_s - 3\cot((V_s - \xi_s)/2) - H'_\lambda(V_s - \xi_s)\Big)^2 \mathrm{d}s. \tag{4.19}$$

Since H_λ is not differentiable at $\theta = \pi$ (the left and right derivatives differ by a sign), the integrand on the right-hand side of Eq. (4.19) is not necessarily well-defined for s such that $V_s - \xi_s = \pi$. However, this is not an issue: Let $E = \{s \in [0,t] : V_s - \xi_s = \pi\}$. In the interior of E, we have $\dot\xi_s = 0$ and $\dot V_s = 0$. Hence, the integrand is well-defined and takes the value 2λ for $s \in E^\circ$. Since ∂E has measure zero the integrand need not be well-defined there.

Let us connect $J^\lambda_{(\theta_1,\theta_2)}$ to the chordal Loewner energy I in Lemma 4.6. We have the following two observations.

- Denote by $\hat\gamma_t$ the union of $\gamma_{[0,t]}$ and the hyperbolic geodesic from γ_t to $\mathrm{e}^{\mathrm{i}\theta_2}$ in $\mathbb{D} \setminus \gamma_{[0,t]}$. Then Lemma 4.6 gives the energy of $\hat\gamma_t$:

$$I(\hat\gamma_t) = \frac{1}{2}\int_0^t \Big(\dot\xi_s - 3\cot((V_s - \xi_s)/2)\Big)^2 \mathrm{d}s.$$

- For H_λ, we have

$$H_\lambda(V_t - \xi_t) - H_\lambda(\theta_2 - \theta_1) = \int_0^t H'_\lambda(V_s - \xi_s)\Big(\cot((V_s - \xi_s)/2) - \dot\xi_s\Big)\mathrm{d}s.$$

Plugging these two observations into (4.19), we have

$$J^\lambda_{(\theta_1,\theta_2)}\left(\gamma_{[0,t]}\right) = I(\hat\gamma_t) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma_t) + H_\lambda(V_t - \xi_t) - H_\lambda(\theta_2 - \theta_1). \tag{4.20}$$

Suppose $\gamma : (0,T) \to \mathbb{D} \setminus \{0\}$, with $\gamma(0+) = \mathrm{e}^{\mathrm{i}\theta_1}$ and $\gamma(T-) = \mathrm{e}^{\mathrm{i}\theta_2}$, has finite chordal Loewner energy. Then the associated radial driving function, in the capacity parametrization, $\xi_\cdot$, is absolutely continuous on $[0,t]$ for all $t \in [0,T)$. Furthermore, a harmonic measure argument shows that

$$V_t - \xi_t \to 0 \quad \text{or} \quad V_t - \xi_t \to 2\pi, \quad \text{as } t \to T-$$

(depending on, on which side of the origin γ passes). Hence, Eq. (4.20) implies

$$I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma) = H_\lambda(\theta_2 - \theta_1) + \lim_{t \to T-} J^\lambda_{(\theta_1, \theta_2)}(\gamma_t) \geq H_\lambda(\theta_2 - \theta_1). \tag{4.21}$$

Now let $\theta_1 < \theta_2 \leq \theta_1 + \pi$, and let (ξ_t^*, V_t^*) be the solution to (4.17). Then,

$$\partial_t (V_t^* - \xi_t^*) = -\sqrt{2\lambda + 4\cot^2((V_t^* - \xi_t^*)/2)},$$

so that there does, indeed, exist $T \in (0, \infty)$ so that $\lim_{t \to T-}(V_t^* - \xi_t^*) = 0$. It follows from (4.21) that $\xi_\cdot^*$ is the driving function of a simple curve γ^*, with $\gamma^*(0+) = \mathrm{e}^{\mathrm{i}\theta_1}$ and $\gamma^*(T-) = \mathrm{e}^{\mathrm{i}\theta_2}$, and that

$$I(\gamma^*) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma^*) = H_\lambda(\theta_2 - \theta_1).$$

Thus, the infimum of $I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma)$ equals $H_\lambda(\theta_2 - \theta_1)$ and is attained by γ^*. Furthermore, any $\tilde{\gamma} \in \mathfrak{X}(\mathbb{D}; \mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2})$ which minimizes $I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D} \setminus \gamma)$ must satisfy $J^\lambda_{(\theta_1,\theta_2)}\left(\tilde{\gamma}_{[0,t]}\right) = 0$ for all t, or equivalently, its driving function must satisfy

$$\dot{\xi}_t = 3\cot((V_t - \xi_t)/2)) + H_\lambda'(V_t - \xi_t), \quad \text{a.e.} \tag{4.22}$$

If $\theta_2 < \theta_1 + \pi$ the unique continuous solution of (4.22) is $\xi_\cdot^*$. If $\theta_2 = \theta_1 + \pi$, then (4.22) has two continuous solutions, $\xi_\cdot^*$ and $\xi_\cdot^{**}$, where $\dot{\xi}_t^{**} = -\dot{\xi}_t^*$. ☐

Since $\mathcal{Z}_\alpha$ is a function of $\theta := \theta_2 - \theta_1$, so we may write $\mathcal{U}^\lambda(\theta) = \mathcal{U}^\lambda(\theta_1, \theta_2)$. The next corollary explains why we do not find this solution in Proposition 4.1.

Corollary 4.9. *We let $\theta = \theta_2 - \theta_1$ and write $\mathcal{U}^\lambda(\theta) = \mathcal{U}^\lambda(\theta_1, \theta_2)$. Then we have (1.7): for $\theta \in (0, \pi)$,*

$$\mathcal{U}^\lambda(\theta) = \mathcal{U}^\lambda(2\pi - \theta) = -2\log\sin(\theta/2) + \int_\theta^\pi \sqrt{2\lambda + 4\cot^2(u/2)}\,\mathrm{d}u.$$

In particular, it satisfies

$$(\mathcal{U}'(\theta))^2 + 2\cot(\theta/2)\mathcal{U}'(\theta) - \frac{3}{(\sin(\theta/2))^2} = -3 + 2\lambda \tag{4.23}$$

and has the left derivative $-\sqrt{2\lambda}$ and right derivative $\sqrt{2\lambda}$ at $\theta = \pi$.

In other words, $\mathcal{U}^\lambda$ is not differentiable at π. That is why we do not see it in Proposition 4.1. Moreover, the driving function ξ^* in (4.17) satisfies $\dot{\xi}_t^* = \partial_1 \mathcal{U}^\lambda(\xi_t^*, V_t^*)$ which is analogous to (4.11).

Proof. The expression (1.7) follows directly from (4.15) and Proposition 4.8. It is straightforward to check that it satisfies (4.23). ☐

Proof of Proposition 1.3 and Proposition 1.4. This is a collection of Lemma 4.5, Lemma 4.7, Proposition 4.8 and Corollary 4.9. ☐

Remark 4.10. When $\lambda = 2$, Corollary 4.9 is consistent with Remark 2.7: when $\alpha = \alpha_1(\kappa)$ as in (2.29), we have $\alpha_1(\kappa) \sim -2/\kappa$ and thus

$$\begin{aligned}\mathcal{U}^{\lambda=2}(\theta) &= \lim_{\kappa\to 0+} \kappa \log \mathcal{Z}_{\alpha_1(\kappa)}(\theta_1,\theta_2) \\ &= \begin{cases} 4\log 2 - 6\log\sin(\theta/2) + 8\log\cos(\theta/4), & \text{if } \theta \in (0,\pi]; \\ 4\log 2 - 6\log\sin(\theta/2) + 8\log\sin(\theta/4), & \text{if } \theta \in [\pi, 2\pi). \end{cases}\end{aligned}$$

Then

$$\partial_\theta \mathcal{U}^{\lambda=2}(\theta) = \begin{cases} -3\cot(\theta/2) - 2\tan(\theta/4), & \text{if } \theta \in (0,\pi); \\ -3\cot(\theta/2) + 2\cot(\theta/4), & \text{if } \theta \in (\pi, 2\pi). \end{cases}$$

Note that, when $\theta \in (0,\pi)$, the derivative of the right-hand side of (1.7) is

$$-\cot(\theta/2) - \frac{2}{\sin(\theta/2)} = -3\cot(\theta/2) - 2\tan(\theta/4).$$

Remark 4.11. If one removes the assumption of interchangeability from Proposition 4.1, then one obtains, in addition to the solutions (4.2) and (4.3), the solutions

$$\mathcal{U}^{\lambda,\pm}(\theta_1,\theta_2) := -2\log\sin(\theta_{21}/2) \pm \int_{\theta_{21}}^{\pi} \sqrt{2\lambda + 4\cot^2(u/2)}du, \quad \theta_1 < \theta_2 < \theta_1 + 2\pi, \quad \lambda > 0.$$

(These solutions correspond to Eq. (4.10) with $C > -3$ and $\lambda = (3+C)/2$.) Note that $\mathcal{U}^{\lambda,+}(\theta_1,\theta_2) = \mathcal{U}^{\lambda}(\theta_1,\theta_2)$ for $\theta_1 < \theta_2 \le \theta_1 + \pi$ and $\mathcal{U}^{\lambda,-}(\theta_1,\theta_2) = \mathcal{U}^{\lambda}(\theta_1,\theta_2)$ for $\theta_1 + \pi \le \theta_2 \le \theta_1 + 2\pi$. As in the case with interchangeability, we do not have a one-to-one correspondence with the solutions from the $\kappa > 0$ case. Following Lemma 4.7, one can take the semiclassical limit of $\mathcal{Z}_{\alpha,\beta}$ from Proposition 3.17. If $\lambda > 0$ and $\alpha \sim -\lambda/\kappa$ as $\kappa \to 0$, then one has

$$\lim_{\kappa\to 0} \kappa \log \frac{\mathcal{Z}_{\alpha,\beta}(\theta_1,\theta_2)}{\mathcal{Z}_{\alpha,\beta}(0,\pi)} = \begin{cases} \mathcal{U}^{\lambda,+}(\theta_1,\theta_2), & \text{if } \beta = 1; \\ \mathcal{U}^{\lambda}(\theta_1,\theta_2), & \text{if } \beta \in (0,1); \\ \mathcal{U}^{\lambda,-}(\theta_1,\theta_2), & \text{if } \beta = 0. \end{cases}$$

The case $\beta = 1$ follows from

$$\inf_{\substack{\gamma \in \mathfrak{X}(\mathbb{D}; e^{i\theta_1}, e^{i\theta_2}) \\ 0 \text{ is to the left of } \gamma}} (I(\gamma) - \lambda \log \mathrm{CR}(\mathbb{D}\setminus\gamma)) = \int_0^{\theta_{21}} \left(\sqrt{2\lambda + 4\cot^2(u/2)} - 2\cot(u/2)\right) \mathrm{d}u,$$

which can be seen by examining the proof of Proposition 4.8. The case $\beta = 0$ is analogous.

A Euler's hypergeometric differential equations

Lemma A.1. *Fix $\kappa \in (0,8)$ and $\alpha \in \mathbb{R}$, we consider Euler's hypergeometric differential equation* (1.5)*:*

$$\begin{cases} u(1-u)\phi''(u) - \frac{3\kappa-8}{2\kappa}(2u-1)\phi'(u) + \frac{8\alpha}{\kappa}\phi(u) = 0, & u \in (0,1); \\ \phi(1/2) = 1, \quad \phi'(1/2) = 0. \end{cases}$$

There is a unique solution ϕ_α to (1.5) in $C^2(0,1)$ and the solution ϕ_α is continuous in α. Furthermore,

- *when $\alpha = \alpha_0 := 1 - \kappa/8$, we have*

$$\phi_{\alpha_0}(u) = (4u(1-u))^{4/\kappa - 1/2}; \tag{A.1}$$

- *when $\alpha < 1 - \kappa/8$, we have $\phi_\alpha(u) > 0$ for all $u \in (0,1)$;*
- *when $\alpha > 1 - \kappa/8$, there exists $u \in (0,1)$ such that $\phi_\alpha(u) \le 0$.*

Proof. By direct calculation, we see that (A.1) satisfies (1.5) when $\alpha = \alpha_0$. We write

$$\alpha_0 = 1 - \kappa/8, \quad \phi_{\alpha_0}(u) = (4u(1-u))^{4/\kappa - 1/2}, \quad \phi(u) = \phi_{\alpha_0}(u) f(u).$$

Then Eq. (1.5) becomes

$$\begin{cases} u(1-u)f''(u) - \dfrac{\kappa+8}{2\kappa}(2u-1)f'(u) + \dfrac{8}{\kappa}(\alpha - \alpha_0) f(u) = 0, \quad u \in (0,1); \\ f(1/2) = 1, \quad f'(1/2) = 0. \end{cases} \tag{A.2}$$

We write $y_1 = f$ and $y_2 = f'$, then Eq. (A.2) becomes

$$\begin{cases} y_1' = \mathcal{F}_1(u, y_1, y_2) := y_2; \\ y_2' = \mathcal{F}_2(u, y_1, y_2) := \dfrac{8}{\kappa}\dfrac{\alpha_0 - \alpha}{u(1-u)} y_1 + \dfrac{\kappa+8}{2\kappa}\dfrac{(2u-1)}{u(1-u)} y_2; \\ y_1(1/2) = 1, \quad y_2(1/2) = 0. \end{cases} \tag{A.3}$$

The functional $(\mathcal{F}_1, \mathcal{F}_2)$ is continuous in $\Lambda = (0,1) \times \mathbb{R} \times \mathbb{R}$ and satisfies a local Lipschitz condition with respect to (y_1, y_2) in Λ. Thus, the initial value problem (A.3) has exactly one solution, and the solution can be extended up to the boundary of Λ. This gives the unique solution ϕ_α to (1.5) and the solution ϕ_α is continuous in α.

Next, let us check the positivity of the solution. There are two cases.

Case 1: $\alpha \le \alpha_0$. In this case, $\mathcal{F}_1$ is increasing in y_2 and $\mathcal{F}_2$ is increasing in y_1. Thus we have a comparison principle for the unique solution to (A.3). In particular, the solution f to (A.2) is decreasing in α as long as $\alpha \le \alpha_0$. Consequently, the unique solution ϕ_α to (1.5) is decreasing in α as long as $\alpha \le \alpha_0$ and $\phi_\alpha(u) \ge \phi_{\alpha_0}(u) > 0$ for all $u \in (0,1)$ when $\alpha \le \alpha_0$.

Case 2: $\alpha > \alpha_0$. We prove by contradiction and assume $f(u) \ge 0$ for all $u \in (1/2, 1)$. As $f'(1/2) = 0$ and $f''(1/2) = 32(\alpha_0 - \alpha)/\kappa < 0$, there exists $\varepsilon > 0$ small such that

$$-\delta := f'(1/2 + \varepsilon) < 0.$$

The ODE in (A.2) implies that

$$f''(u) \le \left(\frac{\kappa+8}{2\kappa}\right) \frac{2u-1}{u(1-u)} f'(u), \quad u \in (1/2, 1).$$

Using Grönwall's inequality, we obtain, for $u \in (1/2+\varepsilon, 1)$,

$$f'(u) \le f'(1/2+\varepsilon)\exp\left(\int_{1/2+\varepsilon}^{u}\left(\frac{\kappa+8}{2\kappa}\right)\frac{2s-1}{s(1-s)}\mathrm{d}s\right) = -\delta\left(\frac{u(1-u)}{1/4-\varepsilon^2}\right)^{-\frac{\kappa+8}{2\kappa}}.$$

Since $\kappa \in (0,8)$, we have $\frac{\kappa+8}{2\kappa} > 1$. The derivative f' is not integrable as $u \to 1$. This implies that $f(u) \to -\infty$ as $u \to 1$, which contradicts our assumption and completes the proof. □

Lemma A.2. *Fix $\kappa \in (0,8)$ and $\alpha \in \mathbb{R}$. Consider Euler's hypergeometric differential equation:*

$$u(1-u)\phi''(u) - \frac{3\kappa-8}{2\kappa}(2u-1)\phi'(u) + \frac{8\alpha}{\kappa}\phi'(u) = 0, \quad u \in (0,1). \tag{A.4}$$

We have the following:

- *When $\alpha > 1-\kappa/8$, there exists, for each solution ϕ to* (A.4)*, $u \in (0,1)$ such that $\phi(u) \le 0$.*
- *When $\alpha = 1-\kappa/8$, the only positive solution (up to a multiplicative constant) of* (A.4) *is*

$$\phi_{\alpha_0}(u) = (4u(1-u))^{4/\kappa-1/2}.$$

- *When $\alpha < 1-\kappa/8$, the positive solutions of* (A.4)*, are (up to a multiplicative constant) all of the form*

$$\phi(u) = \beta\Phi^L(u) + (1-\beta)\Phi^R(u), \quad \beta \in [0,1],$$

  with Φ^L and Φ^R as in Remark 3.16.

Proof. We first consider $\alpha \ge 1-\kappa/8$. Let ϕ_α be as in Lemma A.1, and let ψ_α be the solution of (A.4) satisfying $\psi_\alpha(1/2) = 0$ and $\psi'_\alpha(1/2) = 1$. It is easily verified that

$$\psi_\alpha(u) = -\psi_\alpha(1-u), \quad u \in (0,1). \tag{A.5}$$

Since ϕ_α and ψ_α are linearly independent, any solution of (A.4) can be expressed as a linear combination of them. When $\alpha > 1-\kappa/8$, it follows from Lemma A.1 and Eq. (A.5) that there is no positive solution of (A.4). Similarly, when $\alpha = \alpha_0 = 1-\kappa/8$, it follows from Lemma A.1 and Eq. (A.5) that the only positive solutions of (A.4) are $C\phi_{\alpha_0}$, with $C > 0$.

Now consider $\alpha < 1-\kappa/8$. As stated in Remark 3.16, Φ^L and Φ^R are C^2-solutions to (A.4) ($\Phi^L, \Phi^R \in C^2((0,1))$ follows from the same argument as for Φ in Lemma 3.13 using Remark 2.8). Moreover, $0 < \Phi^L(u), \Phi^R(u) \le \Phi(u)$, for $u \in (0,1)$, and

$$\Phi^L(0) = 1, \quad \Phi^L(1) = 0, \quad \text{and} \quad \Phi^R(0) = 0, \quad \Phi^R(1) = 1.$$

Since Φ^L and Φ^R are linearly independent, any solution ϕ of (A.4) can be decomposed as $\phi = C_L\Phi^R + C_R\Phi^R$, for some real constants C_L and C_R. If $C_L < 0$, then $\phi(u) < 0$ for $u > 0$ small. So, for a positive solution ϕ we must have $C_L \ge 0$. Similarly, we deduce that $C_R \ge 0$, and clearly we may not have $C_L = C_R = 0$. This finishes the proof. □

B Convergence of radial SLE to chordal SLE

Suppose $\gamma : [0, \infty) \to \mathbb{H}$ is a continuous non-self-crossing curve in $\mathbb{H}$ such that $\gamma_0 = 0$. Let H_t be the unbounded connected component of $\mathbb{H} \setminus \gamma_{[0,t]}$. Let $g_t : H_t \to \mathbb{H}$ be the unique conformal map with $\lim_{z\to\infty} |g_t(z) - z| = 0$. We say that the curve is parameterized by half-plane capacity if

$$g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{|z|}\right), \quad \text{as } z \to \infty.$$

Then g_t satisfies the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

We call W_t the driving function. Chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ is the chordal Loewner chain with driving function $W_t = \sqrt{\kappa} B_t$ where B is one-dimensional Brownian motion. For a general simply connected domain D with two distinct prime ends $x_1, x_2 \in \partial D$, we define chordal SLE$_\kappa$ in D from x_1 to x_2 to be the image of chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ under the conformal map $\varphi : \mathbb{H} \to D$ sending 0 to x_1 and ∞ to x_2.

Proof of Lemma 3.11. Recall that Q_{t_1} denotes the law of radial SLE$_\kappa$ in $(\mathbb{D}\setminus\eta^{(1)}_{[0,t_1]}; e^{i\theta_2}; 0)$, and denote by Q_∞ the law of chordal SLE$_\kappa$ in $(\mathbb{D} \setminus \eta^{(1)}; e^{i\theta_2}, 0)$. We fix a sequence of conformal maps $f_{t_1} : \mathbb{D} \setminus \eta^{(1)}_{[0,t_1]} \to \mathbb{H}$ and a conformal map $f_\infty : \mathbb{D} \setminus \eta^{(1)} \to \mathbb{H}$ such that

$$f_{t_1}\left(\eta^{(1)}_{t_1}\right) = -1, \quad f_{t_1}(e^{i\theta_2}) = 0; \qquad f_\infty(0) = -1, \quad f_\infty(e^{i\theta_2}) = 0;$$

and that f_{t_1} converges to f_∞ locally uniformly. Note that $(f_{t_1})_*(Q_{t_1})$ is the same as radial SLE$_\kappa$ in $(\mathbb{H}; 0; w = f_{t_1}(0))$, and $(f_\infty)_*(Q_\infty)$ is the same as chordal SLE$_\kappa$ in $(\mathbb{H}; 0, -1)$.

Suppose γ is chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ and denote by W_t its driving function and by g_t the corresponding conformal maps. From [36], the law of chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to -1 is the same as γ weighted by the local martingale

$$|g_t'(-1)|^h (W_t - g_t(-1))^{-2h}, \tag{B.1}$$

where $h = (6 - \kappa)/(2\kappa)$ as in (3.1). The law of radial SLE$_\kappa$ in $\mathbb{H}$ from 0 to $w = f_{t_1}(0)$ is the same as γ weighted by the local martingale

$$|g_t'(w)|^{\tilde{h}} \times \left(\frac{\mathrm{Im}(g_t(w))}{\mathrm{Im}(w)}\right)^{(\kappa-6)^2/(8\kappa)} \times \left|\frac{W_t - \mathrm{Re}(g_t(w))}{\mathrm{Re}(w)}\right|^{-2h}, \tag{B.2}$$

where $\tilde{h} = (\kappa - 2)(6 - \kappa)/(8\kappa)$ as in (3.1).

Combining (B.1) and (B.2), $(f_{t_1})_*(Q_{t_1})$ (radial SLE$_\kappa$ in $\mathbb{H}$ from 0 to $w = f_{t_1}(0)$) is the same as $(f_\infty)_*(Q_\infty)$ (chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to -1) weighted by

$$M_t = \frac{|g_t'(w)|^{\tilde{h}}}{|g_t'(-1)|^h} \times \left(\frac{\mathrm{Im}(g_t(w))}{\mathrm{Im}(w)}\right)^{(\kappa-6)^2/(8\kappa)} \times \left|\frac{W_t - \mathrm{Re}(g_t(w))}{(W_t - g_t(-1))\,\mathrm{Re}(w)}\right|^{-2h}.$$

As $t_1 \to \infty$, we have $w = f_{t_1}(0) \to -1$ and $M_t \to 1$ almost surely, and f_{t_1} converges to f_∞ locally uniformly. These give the desired conclusion. □

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