Reduced Variance by Robust Design of Boundary Conditions for an Incompletely Parabolic System of Equations

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The study of partial differential equations with uncertainty in the boundary and initial data is an important task in

- Climatology
- Turbulent combustion
- Flow in porous media
- Electromagnetics
- Seismic activity
An Incompletely parabolic system:

\[
\begin{align*}
    u_t + Au_x - \epsilon Bu_{xx} &= F(x, t, \xi) & 0 < x < 1, & t > 0 \\
    H_0 u &= g_0(t, \xi) & x = 0, & t \geq 0 \\
    H_1 u &= g_1(t, \xi) & x = 1, & t \geq 0 \\
    u(x, 0, \xi) &= f(x, \xi) & 0 \leq x \leq 1, & t = 0.
\end{align*}
\]  

- \( A \) and \( B \) are symmetric \( M \times M \) matrices
- \( B \) is positive semi-definite
- \( \epsilon \) is a positive constant
- \( H_0 \) and \( H_1 \) are boundary operators
- \( g_0, g_1, f\) and \( F \) are data
Taking the expected value of (1) and letting $v = \mathbb{E}[u]$

$$
\begin{align*}
\nu_t + A\nu_x - \epsilon B\nu_{xx} &= \mathbb{E}[F](x, t) & 0 < x < 1, \quad t > 0 \\
H_0 \nu &= \mathbb{E}[g_0](t) & x = 0, \quad t \geq 0 \\
H_1 \nu &= \mathbb{E}[g_1](t) & x = 1, \quad t \geq 0 \\
\nu(x, 0, \xi) &= \mathbb{E}[f](x) & 0 \leq x \leq 1, \quad t = 0.
\end{align*}
$$

(2)

Now taking the difference between (1) and (2)

$$
\begin{align*}
e_t + A e_x - \epsilon B e_{xx} &= \delta F(x, t, \xi) & 0 < x < 1, \quad t > 0 \\
H_0 e &= \delta g_0(t, \xi) & x = 0, \quad t \geq 0 \\
H_1 e &= \delta g_1(t, \xi) & x = 1, \quad t \geq 0 \\
e(x, 0, \xi) &= \delta f(x, \xi) & 0 \leq x \leq 1, \quad t = 0,
\end{align*}
$$

(3)

where $e = u - v$. 

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Reduced Variance by Robust Design of Boundary Conditions for
Variance formulation

Multiplying (3) with $e^T$ and integrating in space gives

$$\|e\|^2_t + 2\epsilon \int e^T B e_x \, dx = [e^T A e - 2\epsilon e^T B e_x]^1_0. \quad (4)$$

Taking the expected value of (4) and notice that $E[\|e\|^2] = \|Var[u]\|_1$ we obtain

$$\frac{d}{dt} \|Var[u]\|_1 + 2\epsilon \int E[e^T B e_x] \, dx = [E[e^T A e] - 2\epsilon E[e^T B e_x]]^1_0. \quad (5)$$

Finally, by imposing boundary conditions in (5) we are able to analyze their effects on the variance of the solution.