Solution framework for Adaptive Finite Element methods

Maya Neytcheva, Christian Karlsson
Department of Information technology, Uppsala, Sweden

Beräkningsmatematikcirkus, May 28, 2014
AFEM framework
AFEM framework

AFEM
MARK
REFINE
ASSEMBLY
ESTIMATE
SOLVE
Mesh refinement
Mesh refinement

In blue - old points, in red - new points
The division in new’-’old’ imposes a two-by-two block structure on the finite element stiffness matrix.
By construction, $A^{(\ell)}$ has a two-by-two block structure

$$A^{(\ell)} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \ \text{red} \ \text{blue}$$

It can be preconditioned by a block-factorized form by approximating some of the matrix blocks.

$$B^{(\ell)} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S^{(\ell)} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}$$

where $S^{(\ell)}$ is some approximation of the exact Schur complement $S^{(\ell)}_A$ of $A^{(\ell)}$, $S^{(\ell)}_A = A_{22} - A_{21}A_{11}^{-1}A_{12}$. 
What to do with $S_A^{(\ell)}$?

For regular refinements and Hierarchical basis functions (HBF) it is well known that $A^{(k-1)}$ is a good approximation of $S^{(k)}$. Thus, the preconditioner is

$$B^{(\ell)} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A^{(\ell-1)} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix}$$

In practice:

$$B^{(\ell)} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & B^{(\ell-1)} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix}$$
Take-with-you message:

• In the AFEM framework for a reasonable broad class of problems (anisotropic, convection-diffusion-advection, parabolic problems, discontinuous coefficients) the matrix at the previous refinement level is a good approximation of the Schur complement.
• No construction costs!
• Recursively applicable.
• Mesh-independent estimates can be shown.
<table>
<thead>
<tr>
<th>Problem 1 (nonsymmetric)</th>
<th>Problem 2 (anisotropy)</th>
<th>Problem 3 ('Moon'-like domain)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem size</td>
<td>Iter.</td>
<td>Problem size</td>
</tr>
<tr>
<td>545</td>
<td>15</td>
<td>537</td>
</tr>
<tr>
<td>2112</td>
<td>18</td>
<td>2037</td>
</tr>
<tr>
<td>8278</td>
<td>20</td>
<td>7504</td>
</tr>
<tr>
<td>32030</td>
<td>23</td>
<td>26770</td>
</tr>
<tr>
<td>118720</td>
<td>25</td>
<td>96779</td>
</tr>
<tr>
<td>403811</td>
<td>28</td>
<td>275989</td>
</tr>
<tr>
<td></td>
<td></td>
<td>486307</td>
</tr>
</tbody>
</table>
Work to do:

- Estimate the spectral equivalence constant for AFEM
- Test more problems