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Radial Basis Functions generated Finite Differences to solve High-Dimensional PDEs in Finance

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June 1, 2014
Stockholm

Option Pricing

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- The standard Black-Scholes-Merton model

$$dB(t) = rB(t)dt, \tag{1}$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{2}$$

  where B is the *bond value*, S is the *stock value*, r is the *interest rate*, μ is the *drift constant*, σ is the *volatility* and W is a *Wiener process*.

- How to price a *contingent claim* issued on a stock S, *maturing* at time T, with a *payoff* function $g(S(T))$?
  - Itō calculus and Feynman-Kac theory

$$u(S(t), t) = e^{-r(T-t)}\mathbb{E}^{Q}_{S(t),t}[g(S(T))]. \tag{3}$$

  - Under Q measure the underlying dynamics is the following

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t). \tag{4}$$

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- The Black-Scholes-Merton equation

$$\begin{cases} u_t + rsu_s + \frac{1}{2}s^2\sigma^2 u_{ss} - ru = 0, \\ u(s,T) = g(s). \end{cases} \tag{5}$$

  - Analytical solution exists for certain contracts.
- Options

  A *European call/put option* is a financial contract which gives the **right** to its owner, but **not the obligation**, to *buy/sell* a particular financial instrument (i.e. a stock S) at a certain expiration time T for a certain strike price K.
- Payoff function for European call

$$g(s) = \max(s - K,\ 0) = (s - K)^+. \tag{6}$$

- Boundary conditions?

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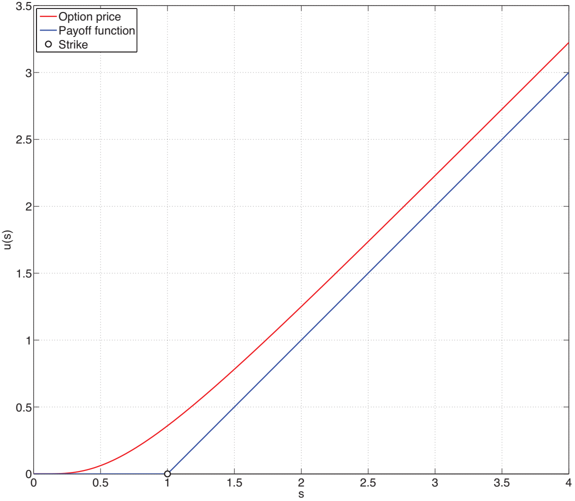

Figure 1 : The solution of the Black-Scholes-Merton equation for a call option with $r = 0.05$, $\sigma = 0.3$ and $T = 5$.

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- In higher dimensions

$$\begin{cases} dB(t) &= rB(t)dt, \\ dS_1(t) &= \mu_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t), \\ dS_2(t) &= \mu_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t), \\ \vdots \\ dS_D(t) &= \mu_D S_D(t)dt + \sigma_D S_D(t)dW_D(t). \end{cases} \tag{7}$$

- The Black-Scholes-Merton equation

$$\begin{cases} u_t + r\sum_{i}^{D} s_i u_{s_i} + \frac{1}{2}\sum_{i,j}^{D} [\Sigma \cdot \Sigma^T]_{i,j} s_i s_j u_{s_i s_j} - ru = 0, \\ u(s_1, s_2, ..., s_D, T) = g(s_1, s_2, ..., s_D). \end{cases} \tag{8}$$

Test Problem

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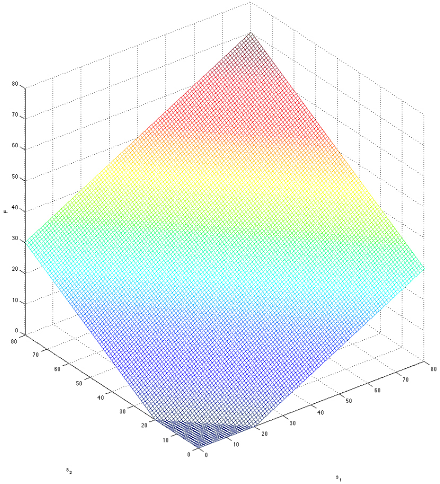

Figure 2 : The terminal condition.

Test Problem

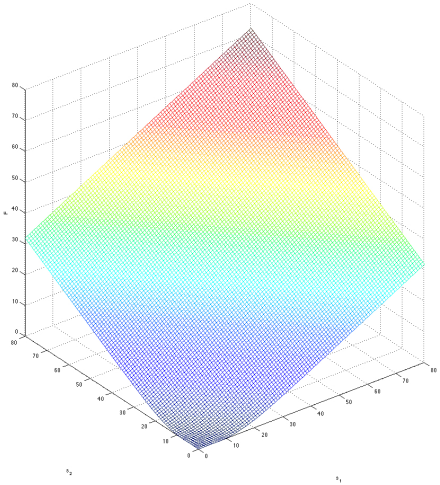

Figure 3 : The computed solution with $T = 1$, $K = 1$, $r = 0.05$, $\Sigma = [0.3, 0.05; 0.05, 0.3]$.

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- Solutions
  - Lower-dimensional problems are solved either analytically or using *finite difference* methods (FD).
  - Higher-dimensional problems are solved using *Monte-Carlo* methods (MC).
- Problems
  - MC converges slowly.
  - FD becomes harder to implement in higher dimensions and suffers from the curse of dimensionality.
- Goals
  - Price options using mesh-free methods whose complexity does not increase severely with the dimensionality of the problem.

Radial basis functions methods (RBF)?

Radial Basis Functions Method

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- Discretize space using N nodes.
- Approximate solution

$$u(s,t) \approx \sum_{k=1}^{N} \lambda_k(t)\phi(\varepsilon\|s - s_k\|),\ k = 1, 2, \ldots, N, \quad (9)$$

where ϕ is a radial basis function and ε is a shape parameter.

- The linear combination constants λ_k are found by enforcing the interpolation condition.
- This global approximation leads to a dense linear system of equations which tends to be ill-conditioned when ε is small.

A localized RBF method might be better!

Radial Basis Functions generated Finite Differences

- Try to exploit the best properties from both FD and RBF with the minimal loss.
- For each point s_i in space, define its neighborhood of $M-1$ points.
- Approximate the differential operator at every point

$$[Lu(s)]_i \approx \sum_{k=1}^{M} w_k^{(i)} u_k^{(i)}. \tag{10}$$

- Compute the weights and put them in the matrix W

$$\begin{bmatrix} \phi(\|s_1^{(i)} - s_1^{(i)}\|) & \dots & \phi(\|s_1^{(i)} - s_M^{(i)}\|) \\ \vdots & \ddots & \vdots \\ \phi(\|s_M^{(i)} - s_1^{(i)}\|) & \dots & \phi(\|s_M^{(i)} - s_M^{(i)}\|) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_M \end{bmatrix} = \begin{bmatrix} [L\phi(\|s - s_1^{(i)}\|)]_{s=s_i} \\ \vdots \\ [L\phi(\|s - s_M^{(i)}\|)]_{s=s_i} \end{bmatrix}.$$

Implementation

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- Discretize the Black-Scholes-Merton equation operator in space using RBF-FD

$$u_t = -\left[r \sum_{i}^{D} s_i u_{s_i} + \frac{1}{2} \sum_{i,j}^{D} \left[\Sigma \cdot \Sigma^T \right]_{i,j} s_i s_j u_{s_i s_j} - ru \right] \approx Wu.$$

- Integrate in time using the standard implicit schemes
  - BDF-1,
  - BDF-2.
- How to choose a stencil, boundary conditions, an RBF kernel and a shape parameter ε?

Results

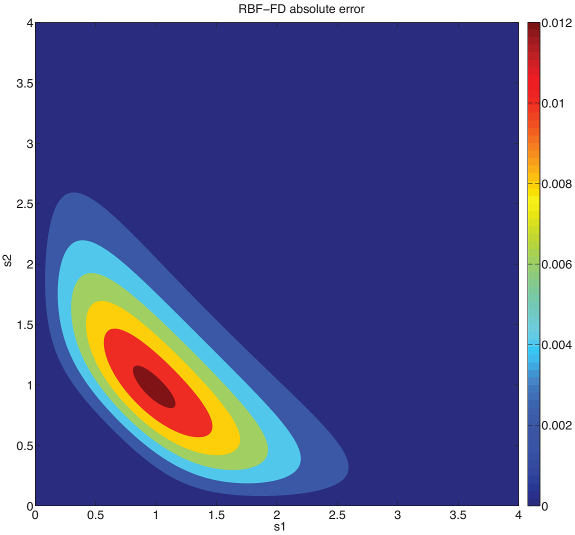

Figure 4 : The absolute error computed using 41 point in each dimension and a 5-point stencil.

Summary

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- The method shows to be reliable with an expected error distribution.
- Performance of the method is high due to the very sparse linear system.
- The method promises competitiveness with the standard methods in the field.

Thank you for your attention!

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