Circus presentation

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Dispersion Relation Preservation

Source of wave speed and dissipative errors

Plane wave, $u(x,t) = e^{i(kx-\omega t)}$, $u_t + cu_x = 0 \Rightarrow \omega = ck$

Discretise in space

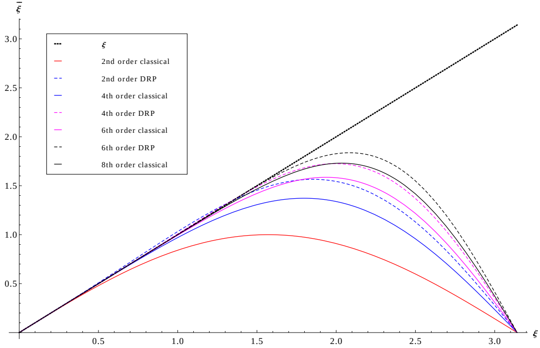
$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + \frac{c}{\Delta x}\sum_{j=-L}^{R} a_j u_{i+j} = 0$$

$$\Rightarrow \omega = -i\frac{c}{\Delta x}\sum_{j=-L}^{R} a_j e^{ijk\Delta x} = c\bar{k}$$

Want $\bar{\xi} = \bar{k}\Delta x \approx k\Delta x = \xi$

$0 \le \xi \le \xi_{max} \le \pi$

Idea[1]: Sacrifice one order of accuracy and use free parameter, α, to minimise $\int_0^{\xi_{max}} |\xi - \bar{\xi}|^2 \mathrm{d}\xi$



[1] C. Tam, J. Webb , JCP 23 (1993) 262-281

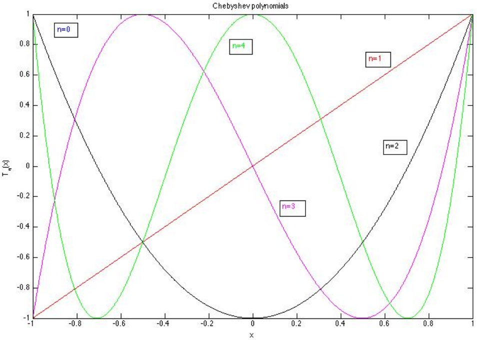
1. Given α and ξ_{max}, how large is $\|\xi - \bar{\xi}\|_\infty$?
2. Given ξ_{max}, how to choose α so $\|\xi - \bar{\xi}\|_\infty$ is small?
3. Given $\|\xi - \bar{\xi}\|_\infty < \epsilon$, how small must ξ_{max} be?

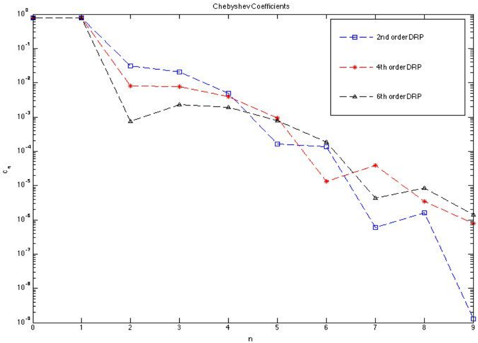
Chebyshev expansion - *nearly* best polynomial approximation in $\|\cdot\|_\infty$.

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(x),$$

$$T_n(\cos(\theta)) = \cos(n\theta)$$

$$c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}} dx$$





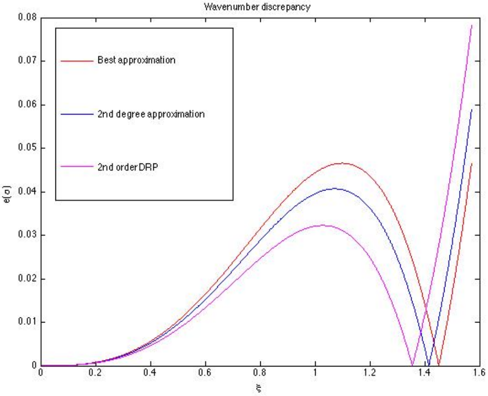
Example

2^{nd} order central difference

For general $\bar{\xi}$, Chebyshev coefficients,

$$c_n = 2\sum_{j=1}^{N} a_j J_n(j\xi_{max}/2) \begin{cases} \sin(j\xi_{max}/2)(-1)^{n/2} & \text{for } n \text{ even} \\ \cos(j\xi_{max}/2)(-1)^{(n-1)/2} & \text{for } n \text{ odd} \end{cases}$$

- $0 \leq \xi \leq \xi_{max} = \pi/2$
- $\bar{\xi}(\xi,\alpha) = 2\alpha\sin(\xi) + \left(\frac{1}{2} - \alpha\right)\sin(2\xi)$
- $e(\xi,\alpha) := \xi - \bar{\xi}(\xi,\alpha)$
- Truncate Chebyshev expansion at n=2.
- Assume extrema at $\xi = \frac{\pi}{2}$ and $\xi = \xi_M < \frac{\pi}{2}$.
- Concavity $\Rightarrow$ $e(\xi_M,\alpha) = -e(\pi/2,\alpha)$

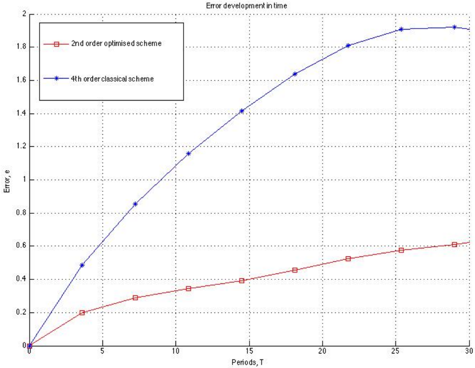


Application

Summation-by-Parts operator

- $u_x \approx P^{-1}Qu$
- $P = P^T > 0$
- $Q + Q^T$
  $= \text{diag}(-1, 0, \ldots 0, 1)$
- Provable stability.
- Consider periodic
  interface problem

$u_t + u_x = 0, \ -1 \leq x \leq 0$
$v_t + v_x = 0, \ 0 \leq x \leq 1$
$u(0, t) = v(0, t)$



Send in wavefunction $\sin(9\pi(x - t))$
$\Delta x = 1/20 \Rightarrow \xi \lessapprox \frac{\pi}{2}$