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Computable error estimates for finite element methods for PDEs with log-normal data

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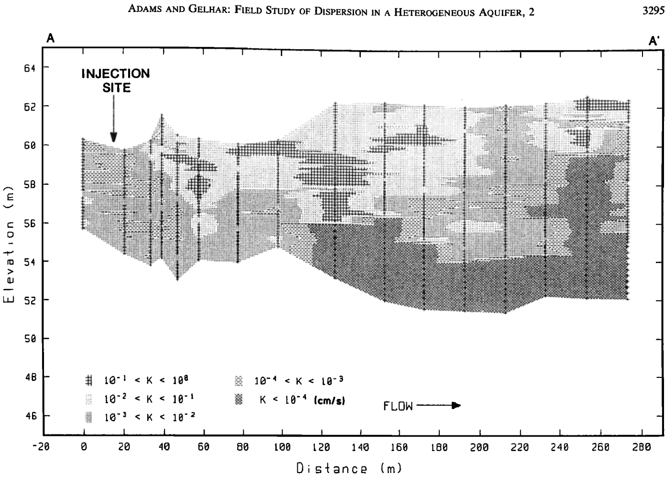
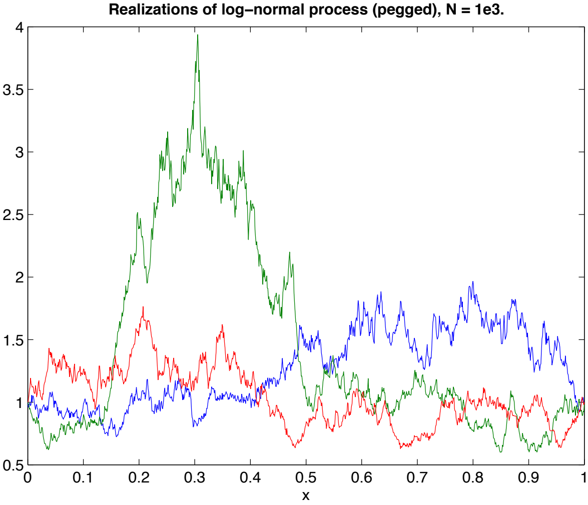
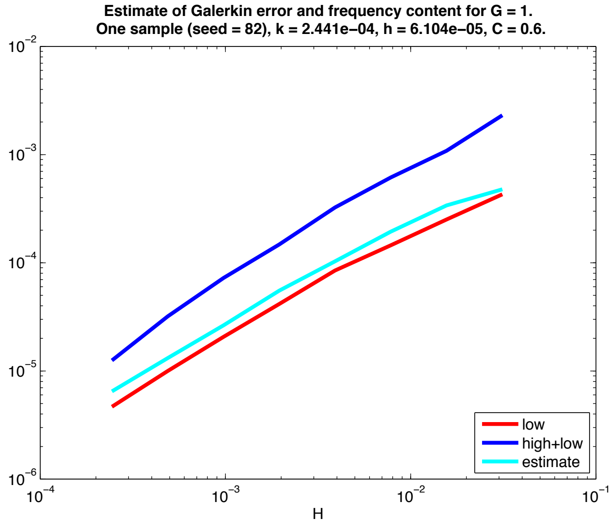
Geophysics - subsurface flow and transport



Log-normal processes, 1-D



Computable error estimates



COMPUTABLE ERROR ESTIMATES FOR PDE WITH RANDOM LOG-NORMAL DATA

0° INTRODUCTION

1° We consider the isotropic diffusion equation

$$\boxed{-\operatorname{div}(a \nabla u) = f}$$ with appropriate boundary conditions.

Why? Such equations arise in Geophysics in the study of subsurface flow and transport, for example, when one would like to study the TIME-INDEPENDENT flow of groundwater (tracking dispersal of a contaminant in a water table).

* SLIDE — this picture shows the conductivity on a scale of 100's of meters. You can see that the conductivity is highly irregular. ~~and that~~ In practice, measurements for a can only be made at a few/finite places — simply interpolating between a few measurements ~~is~~ provides a very poor model for a.

Instead, the uncertainty in the problem data is incorporated into the model by assuming that a and f are random fields.

In GEOPHYSICS APPLICATIONS, the law of a is assumed to be log-normal.

* SLIDE

Just to fix some ideas, we consider the 1-D two point boundary value problem

$$(*) \quad \begin{cases} -(a(x)u'(x))' = 0, & x \in [0,1] \\ u(0) = 0, & u'(1) = 1 \end{cases} \qquad \underline{a.s.}$$

where $\boxed{a(x) = e^{B(x)}}$ for $B(x)$ a Brownian Bridge

In this slide I have plotted 3 realizations of such an a.

AN IMPORTANT FEATURE THEN of the problem we are trying to solve is the low regularity of a. a is only 'almost' 1/2-Hölder continuous.

2° Our aim is to give COMPUTABLE ERROR ESTIMATES for standard (piecewise linear) FE methods for (*)

We consider the problem of estimating OBSERVABLES (u,G), a linear functional of the solution u for a given function G.

Then our aim is to estimate error functionals

$$\boxed{\mathcal{E}(G) = (u-u_h, G) = \int_0^1 (u-u_h)(x)\, G(x)\, dx}$$ or expectations.

A simple calculation, using integration by parts and Galerkin orthogonality, yields

$$(u-u_h, G) = \int_0^1 \underbrace{(a(x)\, u_h'(x))}_{R(u)} (\lambda - v_h)'(x)\, dx. \quad , \quad u_h, v_h \in V_h \subset V$$

$R(u)$ a distribution on $C(0,1)$
and where λ solves the dual problem.

~~SLIDE~~ SO – how would one estimate this quantity?

☆ SLIDE

What makes this problem different from an aposteriori estimate for a standard FE approximation of an equation with a smooth a is that the ~~ER~~ FREQUENCY CONTENT OF THE ERROR FUNCTIONAL EXHIBITS ~~VER~~ VERY DIFFERENT PROPERTIES.

Namely, the high frequency component of the error functional for ~~this~~ our problem is non negligible.

So what ~~one~~ we are displaying in this slide is as follows:

~~$\frac{1}{2} < \alpha, \beta < \frac{3}{2}$~~

Provided one assumes

$$\int_0^1 a\,(u'_{h_1} - u'_{h_1/2})\,(\lambda'_{h_2} - \lambda'_{h_2/2})\, dx \leq C\, h_1^{\alpha}\, h_2^{\beta}$$

which is equivalent to assumming that the high-frequency content of the error functional has a particular decay [which we have verified computationally]

then it is possible to estimate the high frequency content in terms of low frequency content and we estimate the low frequency content by

$$(u-u_H, G) \simeq \sum a\, D^2 u_H\, D^2 \lambda_H \frac{1}{\tilde{C} H} \quad = \text{"Estimate" in plot.}$$

<u>SUMMARY</u> Some interesting problems in Geophysics suggest solving elliptic PDE with log-normal data. The low regularity of the data ~~makes giv~~ poses certain difficulties in providing ~~a p~~ computable estimates for the error in FE methods which can be overcome by understanding ~~and carefully~~ the frequency content of the error functional. In particular the high frequency content of the error is non-neglibile.