Computable error estimates for finite element methods for PDEs with log-normal data

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Geophysics - subsurface flow and transport
Log-normal processes, 1-D

Realizations of log-normal process (pegged), $N = 1e3$. 
Estimate of Galerkin error and frequency content for $G = 1$.
One sample (seed = 82), $k = 2.441 \times 10^{-04}$, $h = 6.104 \times 10^{-05}$, $C = 0.6$. 

Diagram showing the estimate of Galerkin error and frequency content.
INTRODUCTION

We consider the isotropic diffusion equation

\[-\text{div}(a \nabla u) = f,\]

with appropriate boundary conditions.

Why? Such equations arise in Geophysics, in the study of subsurface flow and transport, for example, when one would like to study the time-independent flow of groundwater (tracing dispersion of a contaminant in a water table).

SLIDE - this picture shows the conductivity on a scale of 100's of meters. You can see that the conductivity is highly irregular, and that in practice, measurements can only be made at a few finite places - simply interpolating between a few measurements it provides a very poor model for a.

Instead, the uncertainty in the problem data is incorporated into the model by assuming that \(a\) and \(f\) are random fields.

In GEOPHYSICS, APPLICATIONS, the law of \(a\) is assumed to be log-normal.

SLIDE

Just to fix some ideas, we consider the 1-D two-point boundary value problem

\[
\begin{align*}
-\mu' & = 0, & x \in \{0, 1\} \\
\mu(0) & = 0, & \mu'(1) = 1
\end{align*}
\]

where \(a(x) = e^{B(x)}\) for \(B(x)\) a Brownian Bridge

In this slide I have plotted 3 realizations of such an \(a\).

AN IMPORTANT FEATURE THEN of the problem we are trying to solve is the low regularity of \(a\). \(a\) is only almost \(1/2\)-Hölder continuous.

Our aim is to give COMPUTABLE ERROR ESTIMATES for standard (piecewise linear) FE methods for \(\mu\).

We consider the problem of estimating OBSERVABLES \((\mu, G)\), a linear functional of the solution \(u\) for a given function \(G\).

Then our aim is to estimate error functionals

\(E(G) = (u - u_h, G) = \int_0^1 (u - u_h)(x)G(x)\ dx\)

or expectations.
A simple calculation, using integration by parts and Galerkin orthogonality, yields:

\[(u - u_h, G) = \int_0^1 (a(x) u_h'(x)) (\lambda - v_h)'(x) \, dx, \quad u_h, v_h \in V_h, G \in V\]

\[R(u) \text{ a distribution on } C(0, 1)\]

and where \( \lambda \) solves the dual problem.

**Slide**: So - how would one estimate this quantity?

**Slide**

What makes this problem different from an a posteriori estimate for a standard FE approximation of an equation with a smooth a is that the error frequency content of the error functional exhibits very different properties.

Namely, the high frequency component of the error functional for our problem is non-negligible.

So what we are displaying in this slide is as follows:

Provided one assumes:

\[
\int_0^1 a(u_h - u_{h/2}) (\lambda_h^{1/2} - \lambda_{h/2}^{1/2}) \, dx \leq C h^\alpha \sqrt{n}
\]

which is equivalent to assuming that the high frequency content of the error functional has a particular decay [which we have verified computationally].

Then it is possible to estimate the high frequency content in terms of low frequency content and we estimate the low frequency content by:

\[
(u - u_{h+1}, G) \lesssim \sum a D^2 u_h D^2 \lambda_h \frac{1}{\hat{C} H} = \text{"Estimate" in plot.}
\]

**Summary**: Some interesting problems in Geophysics suggest solving elliptic PDE with log-normal data.

The low regularity of the data makes classical poses certain difficulties in providing computable estimates for the error in FE method, which can be overcome by understanding and carefully the frequency content of the error functional. In particular, the high frequency content of the error is non-negligible.