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Time-domain numerical modeling of poroelastic waves: the Biot-JKD model with fractional derivatives

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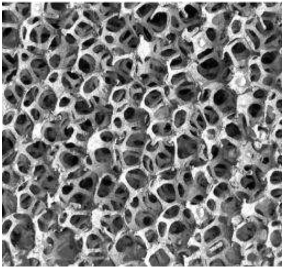
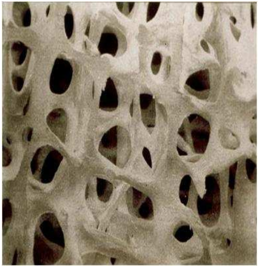
Introduction

Ceramic



Cancellous bone



- Homogeneized medium: Biot model (1956)
- 2 frequency regimes LF and HF : $f_c = \dfrac{\eta\phi}{2\pi a \kappa_0 \rho_f}$
- Viscous dissipation in the HF regime: Johnson, Koplik and Dashen (1987)

2D Biot-JKD model

- 2D transversely isotropic media: 17 positive physical parameters
- 8 variables: $\xi = -\nabla.\phi(\boldsymbol{u_f} - \boldsymbol{u_s})$, $\underline{\varepsilon} = \frac{1}{2}\left(\nabla\, \boldsymbol{u_s} + \nabla\, \boldsymbol{u_s}^T\right)$, $\underline{\boldsymbol{\sigma}}$, p
- Constitutive laws and conservation of momentum:

$$\begin{cases} \boldsymbol{\sigma} = \boldsymbol{C}^u\, \varepsilon - m\,\boldsymbol{\beta}\,\xi \\ p = -m\left(\boldsymbol{\beta}^T\,\varepsilon - \xi\right) \\ \rho\,\dfrac{\partial\, \boldsymbol{v}_s}{\partial\, t} + \rho_f\,\dfrac{\partial\, \boldsymbol{w}}{\partial\, t} = \nabla.\underline{\boldsymbol{\sigma}} \\ \rho_f\,\dfrac{\partial\, \boldsymbol{v}_s}{\partial\, t} + \begin{pmatrix} \rho_{w1} & 0 \\ 0 & \rho_{w3} \end{pmatrix} \dfrac{\partial\, \boldsymbol{w}}{\partial\, t} + \begin{pmatrix} \dfrac{\eta}{\kappa_1}\,F_1(t) & 0 \\ 0 & \dfrac{\eta}{\kappa_3}\,F_3(t) \end{pmatrix} * \boldsymbol{w} = -\nabla\, p \end{cases}$$

Low-frequency (Biot-LF)

$$\widehat{F}_i^{LF}(\omega) = 1$$

$$F_i^{LF}(t) * w_i(t) = w_i$$

High-frequency (Biot-JKD)

$$\widehat{F}_i^{JKD}(\omega) = \frac{1}{\sqrt{\Omega_i}}\,(j\,\omega + \Omega_i)^{1/2}, \quad \Omega_i = \frac{\Lambda_i^2\,\phi^2\,\eta}{4\,\mathcal{T}_i^2\,\kappa_i^2\,\rho_f}$$

$$F_i^{JKD}(t) * w_i(t) = \frac{1}{\sqrt{\Omega_i}}\,(D + \Omega_i)^{1/2}\,w_i$$

$$= \frac{e^{-\Omega_i t}}{\sqrt{\pi\,\Omega_i\,t}} * \left(\frac{\partial w_i}{\partial t} + \Omega_i\,w_i\right)$$

Fractional derivative and diffusive representation

Diffusive representation: $$\frac{1}{\sqrt{t}} = \int_0^{\infty} \frac{1}{\sqrt{\pi\,\theta}} e^{-\theta\, t}\, d\theta$$

$$\begin{aligned}
(D+\Omega)^{1/2} w &= \frac{e^{-\Omega t}}{\sqrt{\pi\, t}} * \left(\frac{\partial w}{\partial t} + \Omega w\right) \quad \text{non local-in-time}\\
&= \int_0^{\infty} \frac{1}{\pi\sqrt{\theta}} \int_0^t e^{-(\theta+\Omega)\,(t-\tau)} \left(\frac{\partial w}{\partial t} + \Omega w\right) d\tau\, d\theta\\
&= \int_0^{\infty} \frac{1}{\pi\sqrt{\theta}}\, \psi(\theta, t) d\theta \quad \text{diffusive representation}
\end{aligned}$$

Ordinary differential equation local-in-time satisfied by the diffusive variable ψ :

$$\begin{cases}
\dfrac{\partial\,\psi}{\partial\, t} = -(\theta+\Omega)\,\psi + \dfrac{\partial w}{\partial t} + \Omega w\\[2ex]
\psi(0) = 0
\end{cases}$$

Diffusive approximation (DA): $$(D+\Omega)^{1/2} w \simeq \sum_{\ell=1}^{N} a_\ell\, \psi(\theta_\ell, t)$$

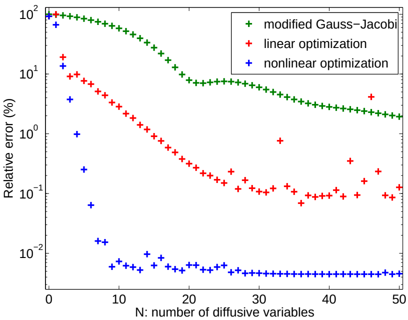
Determination of the quadrature coefficients (1/2)

- Goal: determination of θ_ℓ and a_ℓ
- The dispersion relation depends on the physical parameters and on the viscous operator

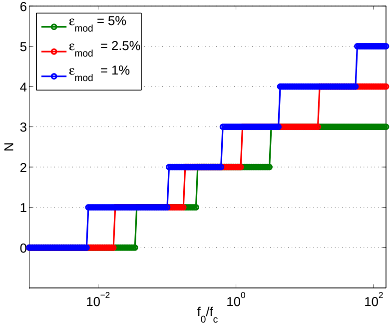
$$\begin{cases} \widehat{F}^{JKD}(\omega) & = \dfrac{1}{\sqrt{\Omega}}(j\,\omega + \Omega)^{1/2} & \text{Biot-JKD} \\ \widehat{F}^{DA}(\omega) & = \dfrac{j\,\omega + \Omega}{\sqrt{\Omega}} \displaystyle\sum_{\ell=1}^{N} \dfrac{a_\ell}{\theta_\ell + j\,\omega + \Omega} & \text{Biot-DA} \end{cases}$$

- Frequency range of interest $[\omega_0/10\,,\,10\,\omega_0]$
- Three methods:
  - Gaussian quadrature
  - classical linear least-squares minimization
  - nonlinear constrained minimization

Relative error ε_{mod}



Parametric determination of N



Error of model ε_{mod} at $N = 5$: 84.42%, 6.78%, 0.25%

Best method: nonlinear optimization

Numerical modeling

- $8 + 2\,N$ variables. Velocity-stress formulation

$$\boldsymbol{U} = (v_{s,x}\,,\;v_{s,z}\,,\;w_x\,,\;w_z\,,\;\sigma_{xx}\,,\;\sigma_{xz}\,,\;\sigma_{zz}\,,\;p\,,\;\psi_1^x\,,\;\psi_1^z, \cdots\,,\;\psi_N^x\,,\;\psi_N^z)^T$$

- First-order hyperbolic system with source term

$$\frac{\partial\,\boldsymbol{U}}{\partial\,t} + \boldsymbol{A}\,\frac{\partial\,\boldsymbol{U}}{\partial\,x} + \boldsymbol{B}\,\frac{\partial\,\boldsymbol{U}}{\partial\,z} = -\boldsymbol{S}\,\boldsymbol{U}$$

- Strang splitting. Successive resolutions of

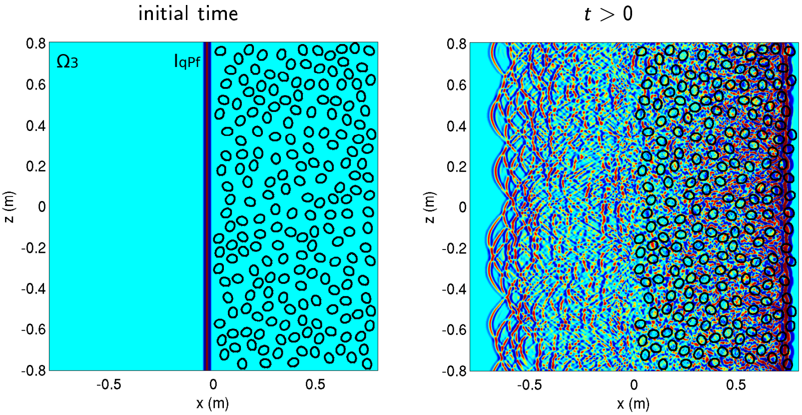
$$\frac{\partial\,\boldsymbol{U}}{\partial\,t} + \boldsymbol{A}\,\frac{\partial\,\boldsymbol{U}}{\partial\,x} + \boldsymbol{B}\,\frac{\partial\,\boldsymbol{U}}{\partial\,z} = \boldsymbol{0} \quad \text{fourth-order ADER scheme}$$

$$\frac{\partial\,\boldsymbol{U}}{\partial\,t} = -\boldsymbol{S}\,\boldsymbol{U} \quad \text{exact integration}$$

- Piecewise homogeneous media: Immersed Interface Method
- Optimal condition of stability $CFL = \max\limits_{\varphi\in[0,\pi/2]} c_{pf}^{\infty}(\varphi)\,\frac{\Delta\,t}{\Delta\,x} \leqslant 1$

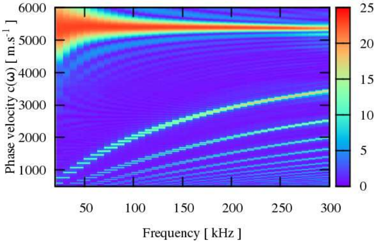
2D numerical experiments: multiple scattering (1/2)

- Wave propagation in complex media
- Plane wave in transversely isotropic medium
- Ellipsoidal scatterers randomly distributed: concentration 25%
- Properties of effective medium

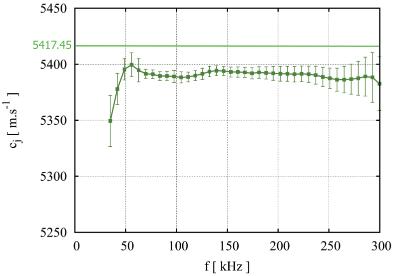


2D numerical experiments: multiple scattering (2/2)

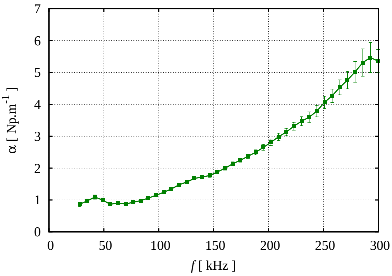
- M. Chekroun (Université du Maine)
- $p - \omega$ transform of the coherent field
- Phase velocity and attenuation of the effective medium in terms of the frequency



effective phase velocity



effective attenuation



Thank you for your attention!