Homework assignment

Random Matrix Theory, Spring 11.

1) Let $f \in L^1(\mathbb{T})$ be a function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} ; |z| = 1\}$. The Fourier coefficient f_k, $k \in \mathbb{Z}$ are defined by

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} \, d\theta .$$

Show that

$$D_n(f) := \det(f_{j-k})_{1 \le j,k \le n} = \frac{1}{(2\pi)^n n!} \int_{[-\pi,\pi]^n} \prod_{1 \le j < k} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) \, d^n\theta . \quad (1)$$

The left side is the Toeplitz determinant with symbol f.

2) Let U be a random unitary matrix with respect to normalized Haar measure on $U(n)$, and let $e^{i\theta_j}$, $1 \le j \le n$, be the eigenvalues of U. Define the random probability measure

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{e^{i\theta_j}}$$

on $\mathbb{T}$, the unit circle. Take $f(e^{i\theta}) = 1_{[-a,a]}(\theta)$, where $0 < a < \pi$ is fixed, the indicator function for $[-a,a]$. Show that

$$\frac{1}{n}\#\{\text{the number of eigenvalues in } [-a,a]\}$$
$$= \frac{1}{n}\int_{\mathbb{T}} f(e^{i\theta})\,d\mu_n(\theta) \to \frac{a}{\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,d\theta$$

almost surely as $n\to\infty$.

Comment: The probability space is $\Omega = U(1)\times U(2)\times U(3)\times\ldots$ and the probability measure on Ω is the product measure of the normalized Haar measures.

Hint: Show that for $f\in L^2(\mathbb{T})$,

$$\mathbb{E}\left[\left(\int_{\mathbb{T}} f(e^{i\theta})\,d\mu_n(\theta) - f_0\right)^2\right] = \frac{2}{n}\sum_{k=n}^{\infty}|f_k|^2 + \frac{2}{n^2}\sum_{k=1}^{n} k|f_k|^2.$$

The eigenvalue measure induced by the normalized Haar measure is the one in the right side of (1). Show that this is a determinantal point process and compute the correlation kernel. Use the Borel-Cantelli lemma.

3) a) Consider a determinantal point process on $[N] = \{0, 1, \dots, N\}$ with correlation kernel $K(x,y)$, $x,y \in [N]$. Each particle configuration $x_1, \dots, x_n$ in the point process has an associated dual particle configuration $y_1, \dots, y_{N-n}$ of holes defined by

$$\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_{N-n}\} = [N].$$

Show that the holes $y_1, \dots, y_{N-n}$ also form a determinantal point process with correlation kernel

$$\tilde{K}(x,y) = \delta_{xy} - K(x,y) \quad , x,y \in [N],$$

where δ_{xy} is Kronecker's delta. We call this the <u>dual point process</u>.

b) The construction in a) also works for a determinantal point process on $\mathbb{Z}$. Consider the discrete sine kernel point process on $\mathbb{Z}$ with correlation kernel

$$K^a_{\text{dsine}}(x,y) = \frac{\sin \pi a(x-y)}{\pi(x-y)},$$

$0 < a < 1$, $x,y \in \mathbb{Z}$. Let $a = 1 - 1/L$ and consider the corresponding dual point process of holes. Show that an appropriate scaling limit of this dual process converges in distribution to the (continuous) sine kernel point process.