The Aztec diamond

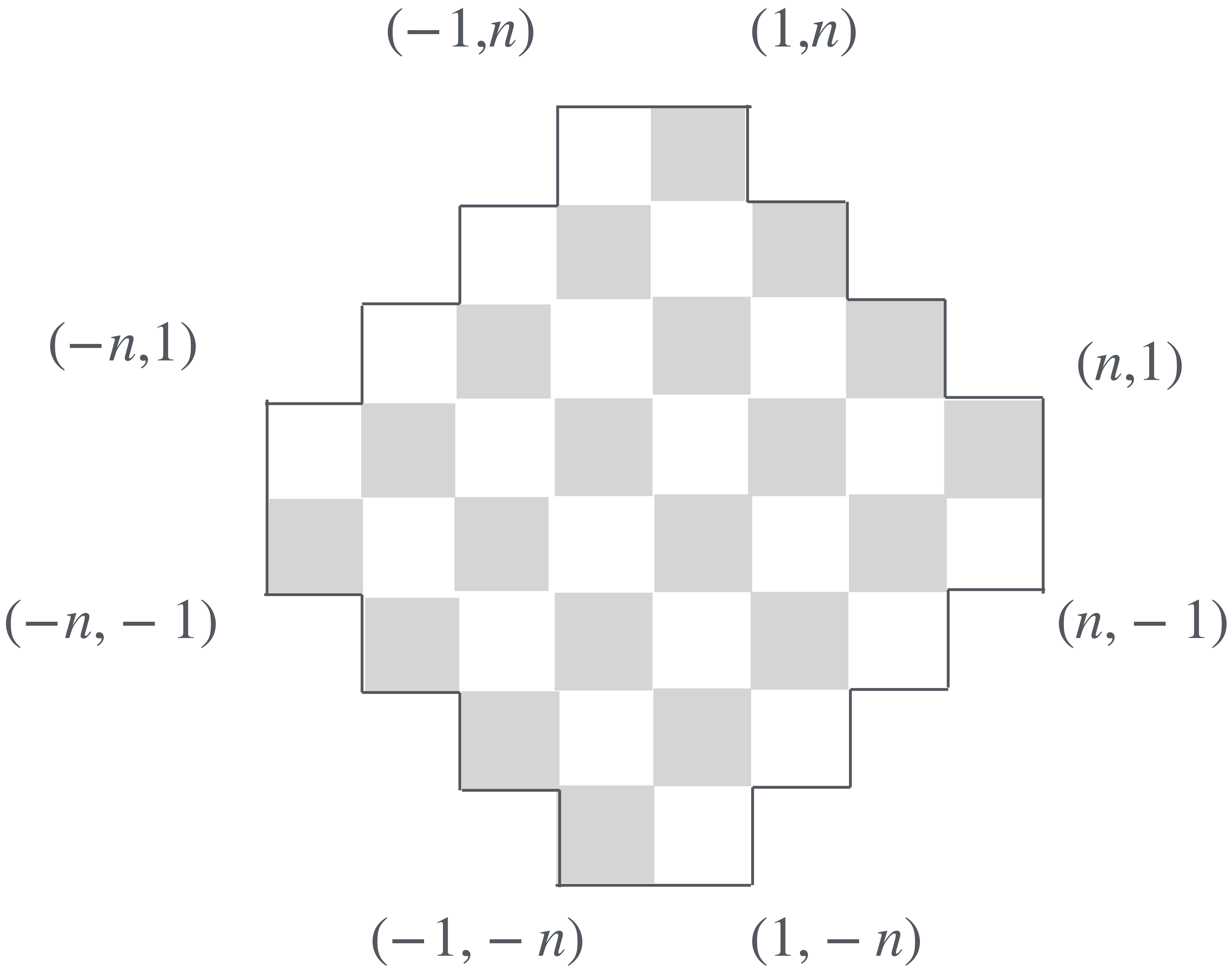
Lecture 2

Maurice Duits — January 22, Paris.

From last time….

Aztec diamond

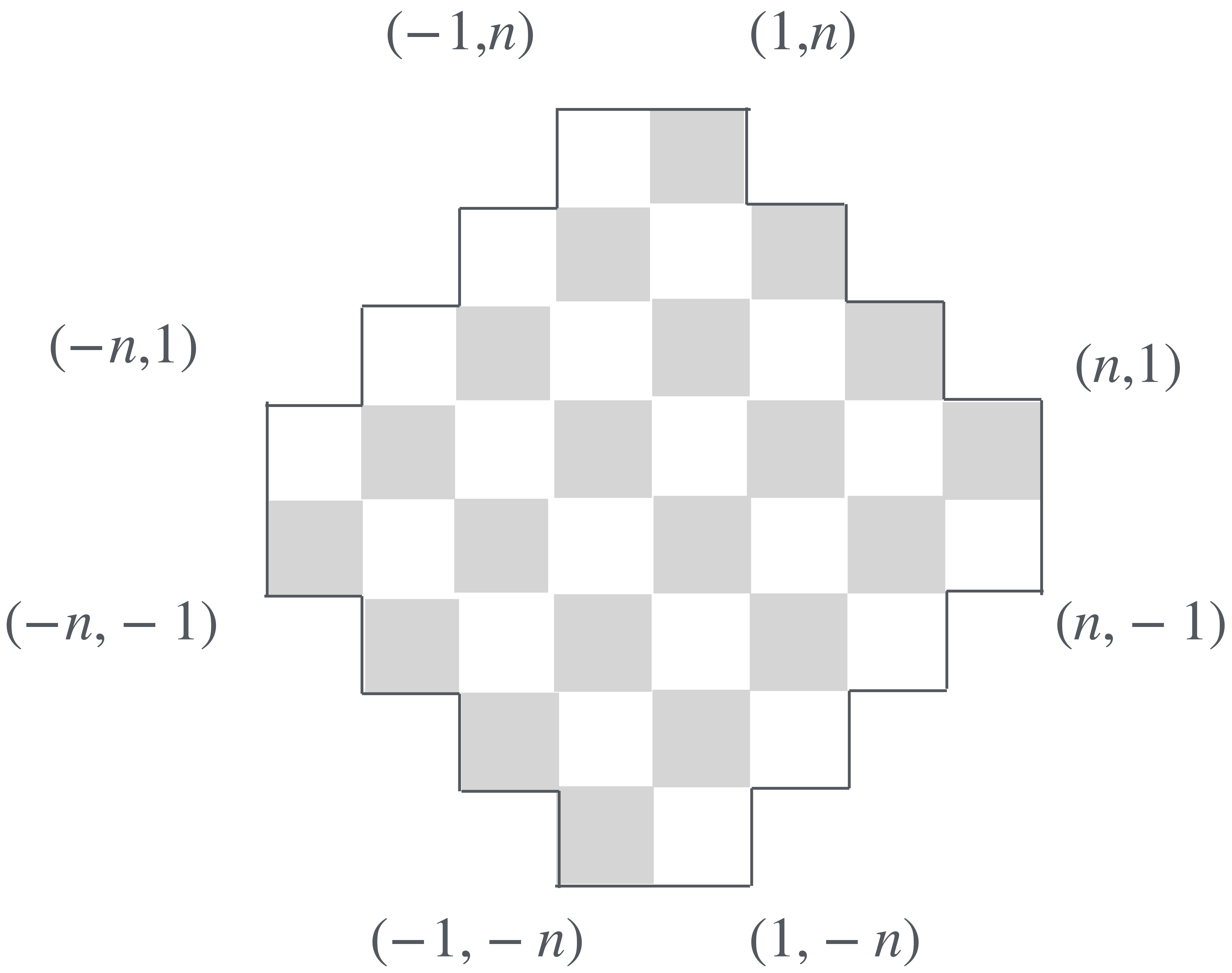
- The Aztec diamond of size n is the following domain on the right
- The sides of the domain have the shape of staircases of Maya temples, hence the name Aztec diamond (dont blame the lecturer….)
- This boundary may look strange at first, but will be very important.

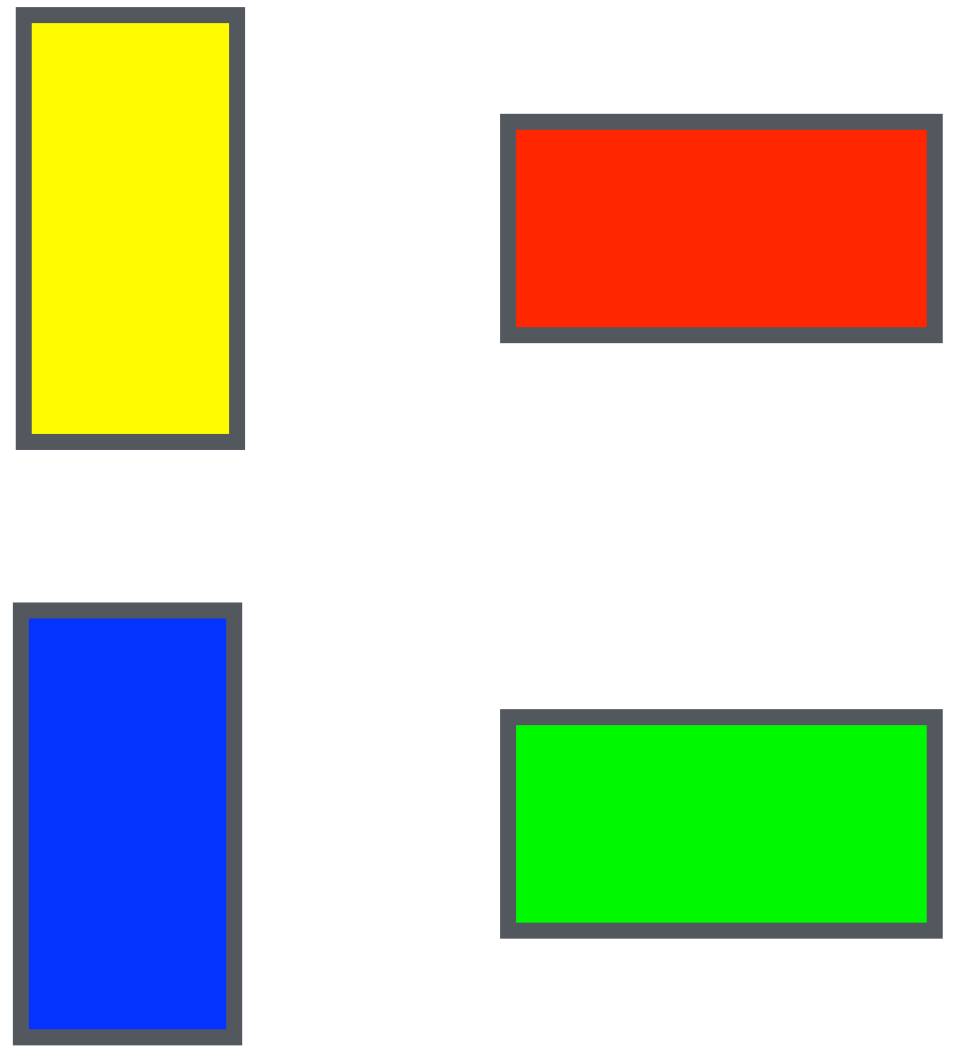


$n = 4$

Aztec diamond

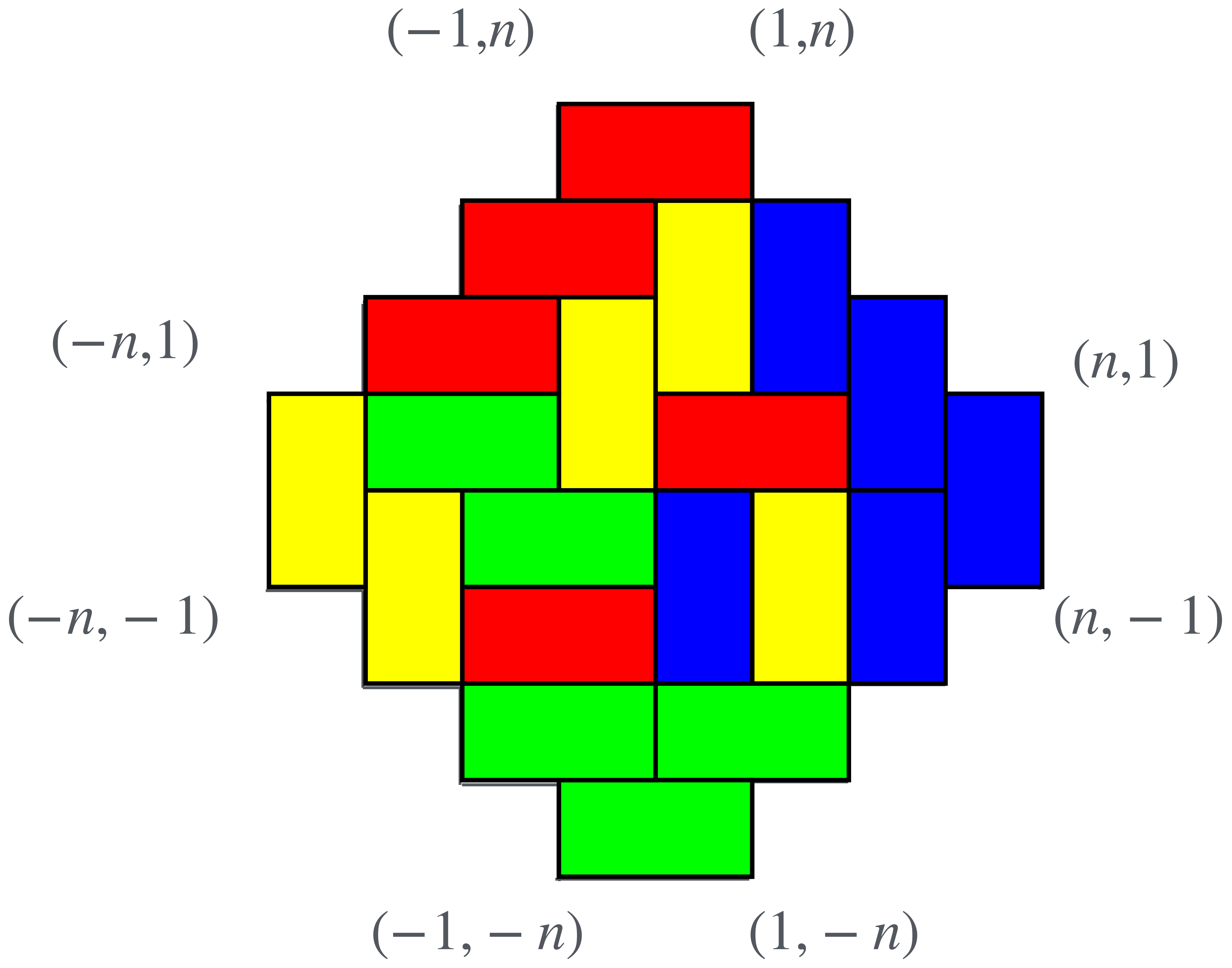
- We tile the Aztec diamond by dominos.





An example tiling

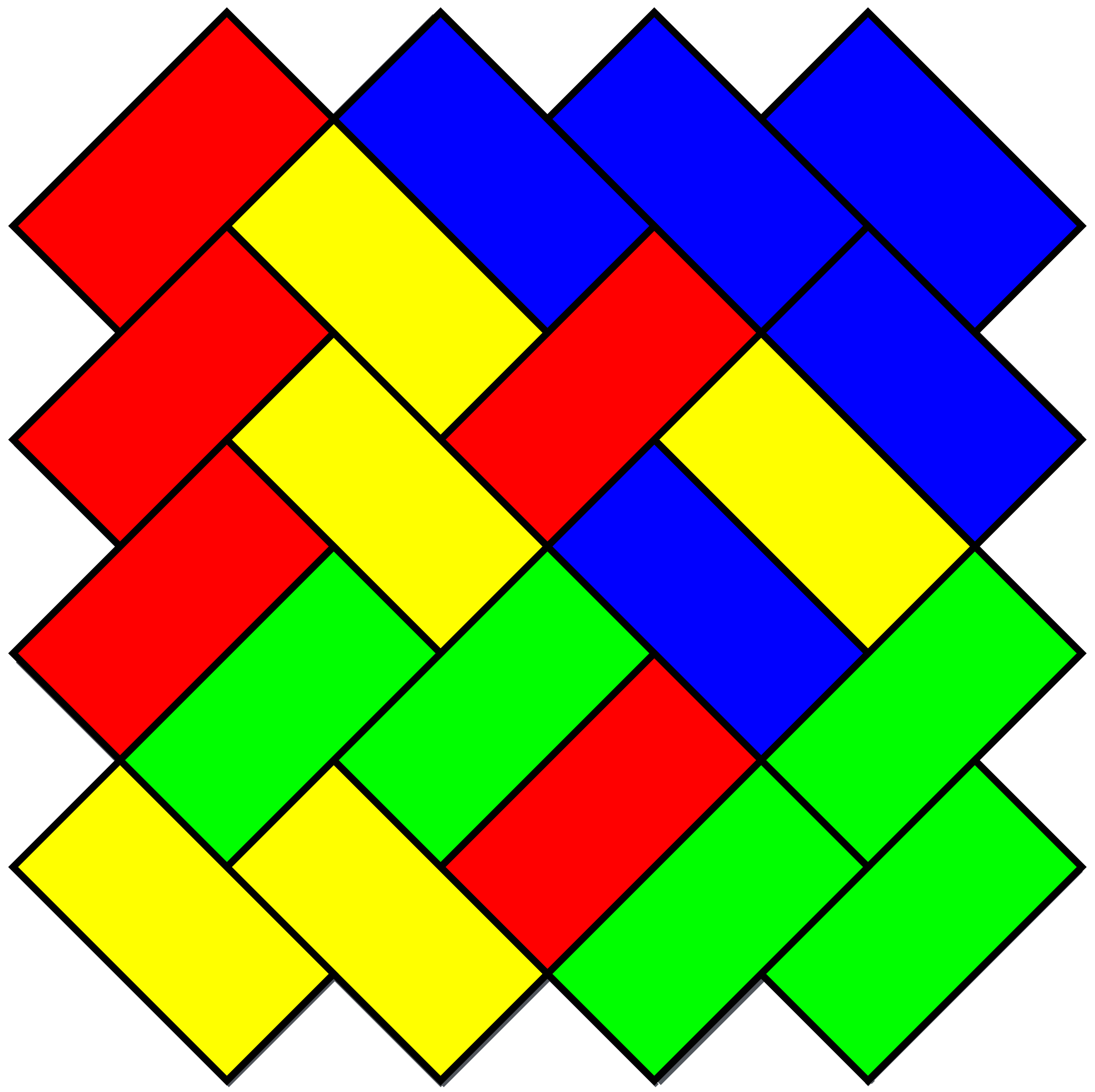
- There are two shapes, but four colors depending on the location of the white and black squares in the checkerboard.



$n = 4$

Aztec diamond (rotated)

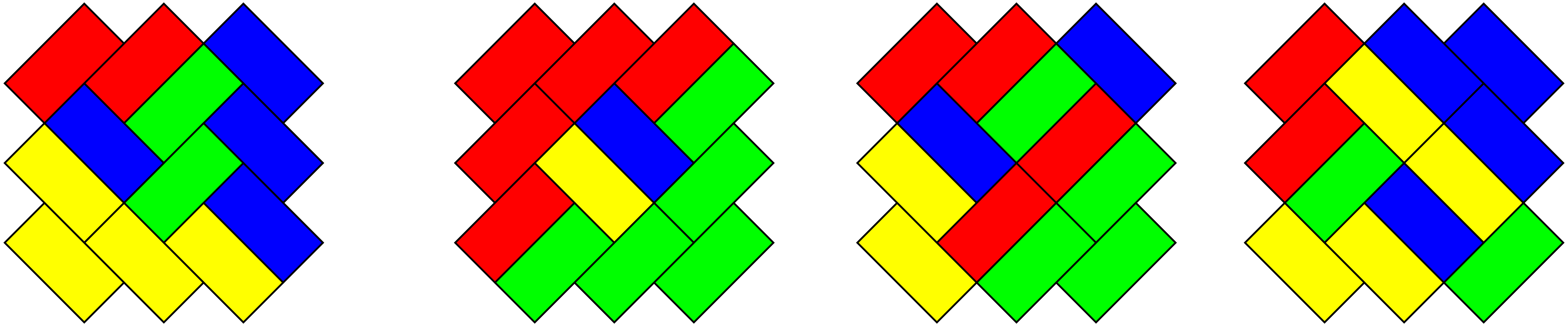
- We will often rotate the diamond by 45 degrees.
- This is not just to save space, but will also be natural.



The Aztec diamond

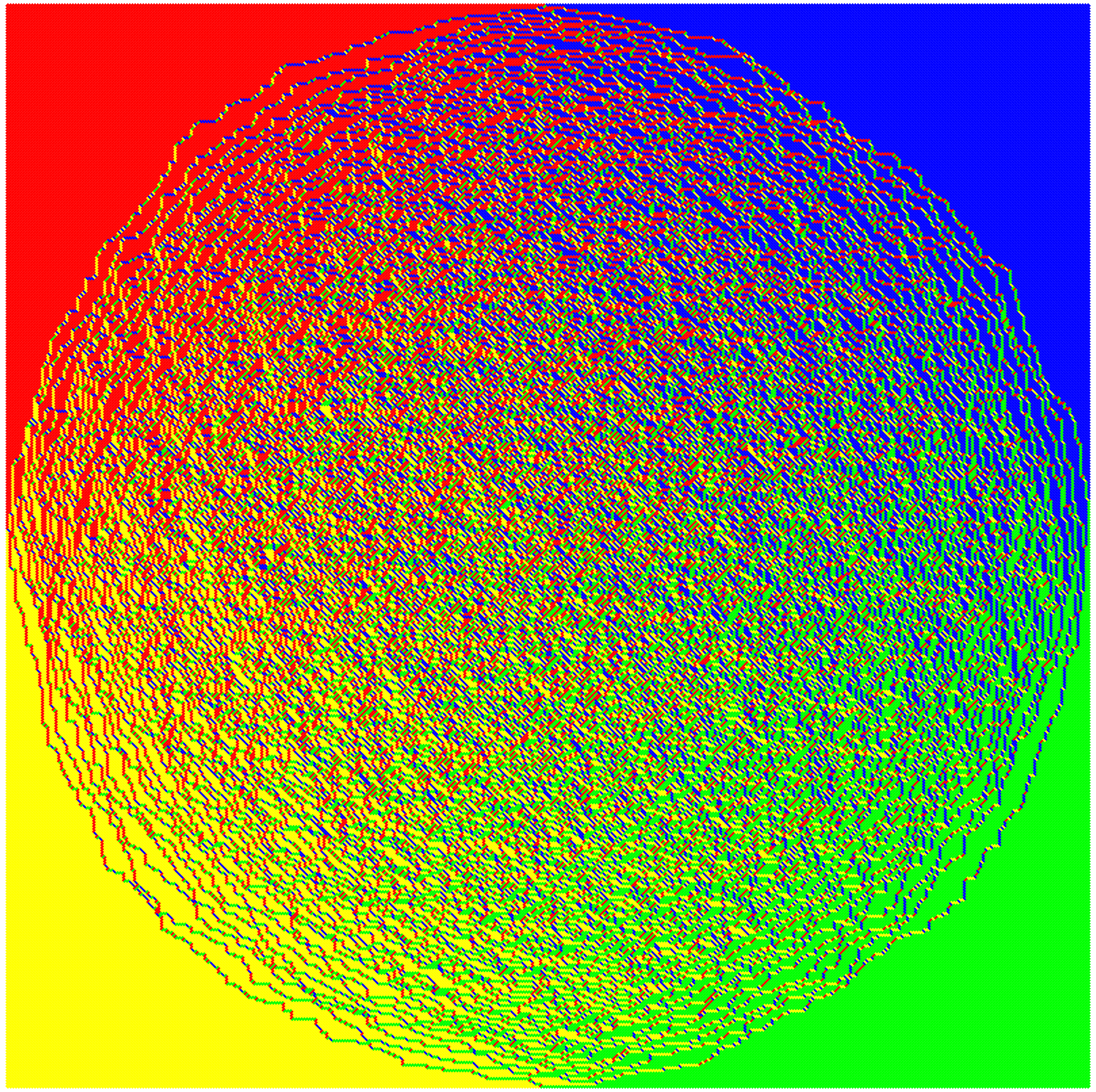
- There are many ways of tiling the Aztec diamond.
- In fact, one can prove that there are $2^{n(n+1)/2}$ ways of tiling the Aztec diamond.
- Let's take on uniformly at random and see how a typical tiling looks like.

Start with $n = 3$



The Aztec diamond

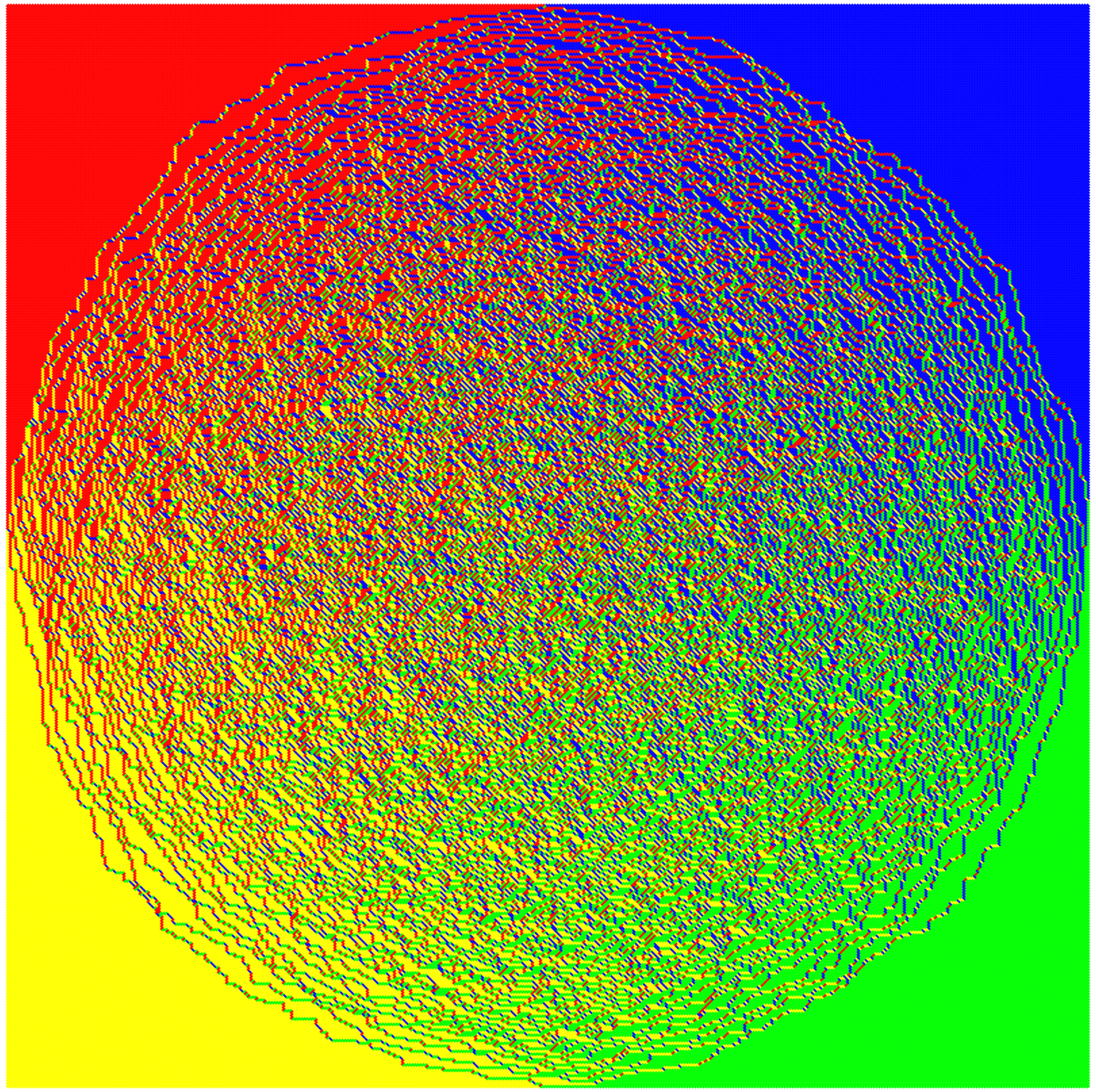
- And $n = 500$.......



The Aztec diamond

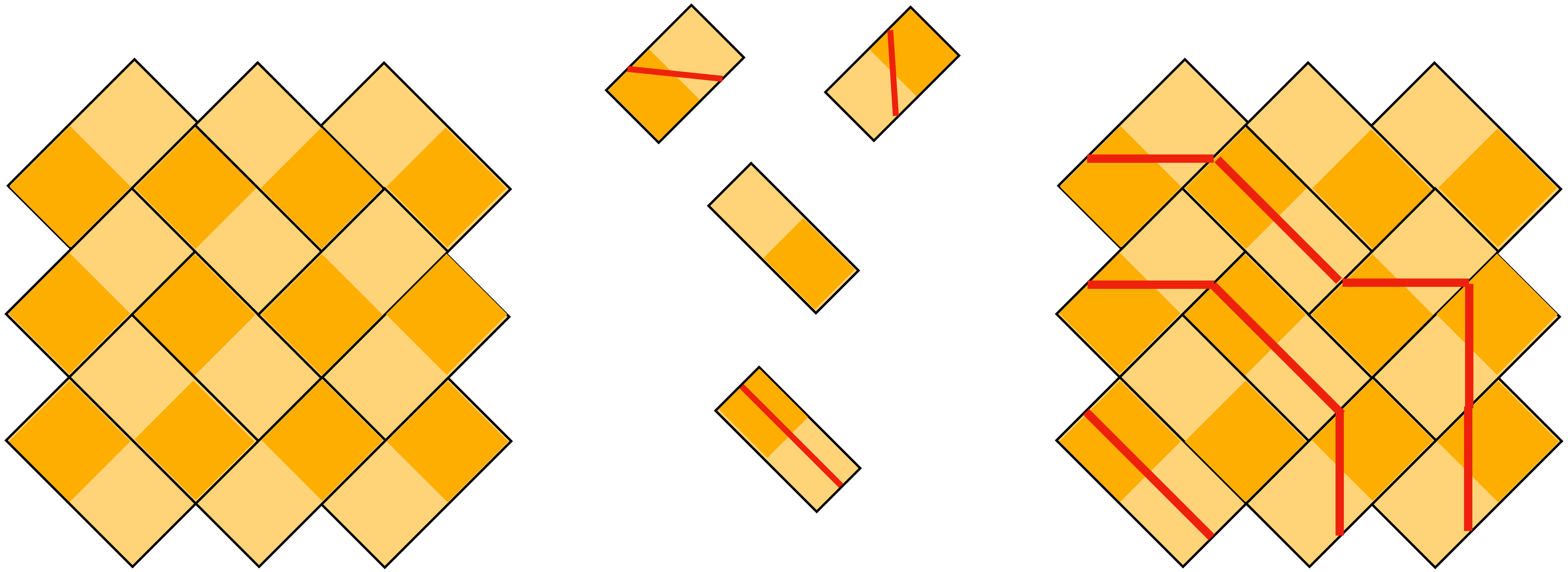
One **main goal of this course** is to understand this picture

- We will prove that the disordered region is a disk
- We will analyze the fluctuations at the interface between frozen and disordered region
- We will compute the microscopic processes in the bulk.
- We will show that there is a limit shape, and study its fluctuations.
- Along the way, we will see that this model has beautiful connections to different types of mathematics, which make the Aztec diamond a very rich model.



Tilings vs non-intersecting paths

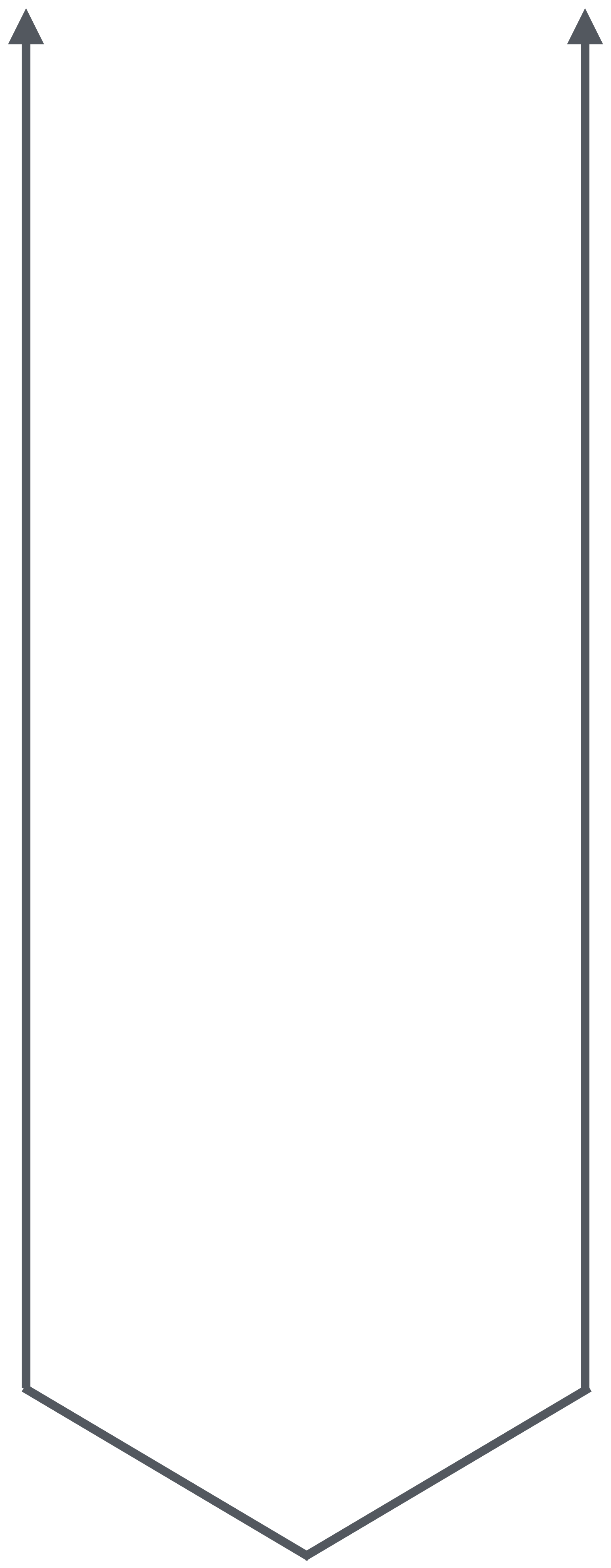
Domino tilings vs DR-paths

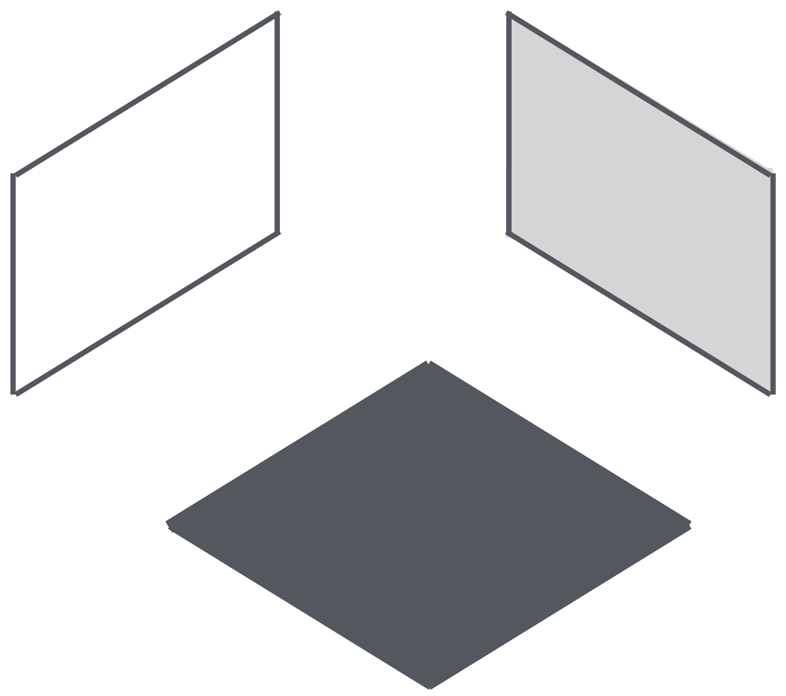


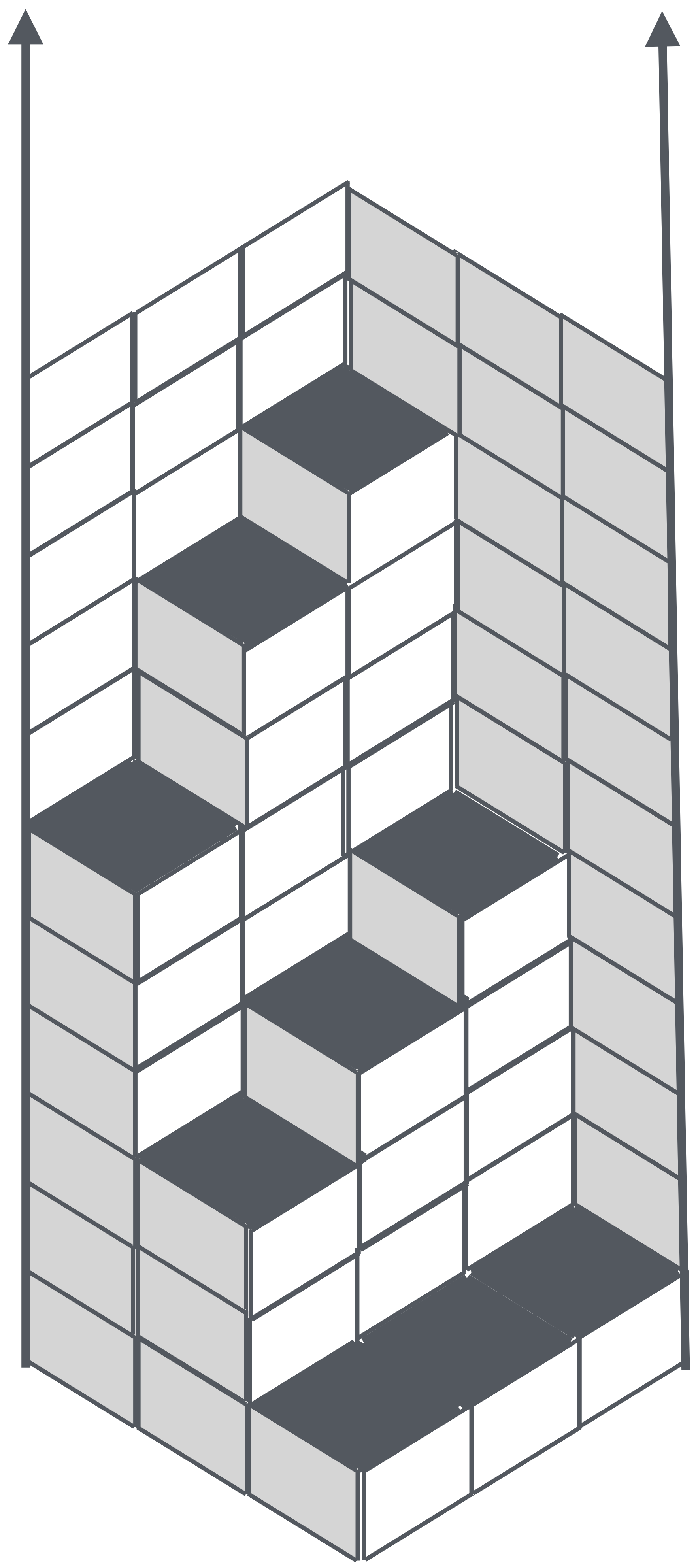
The tiling is completely determined by the configuration of non-intersecting paths, often referred to as DR paths.

Lozenge tilings of a semi-infinite hexagon

- Consider lozenge tilings of a semi-infinite hexagon

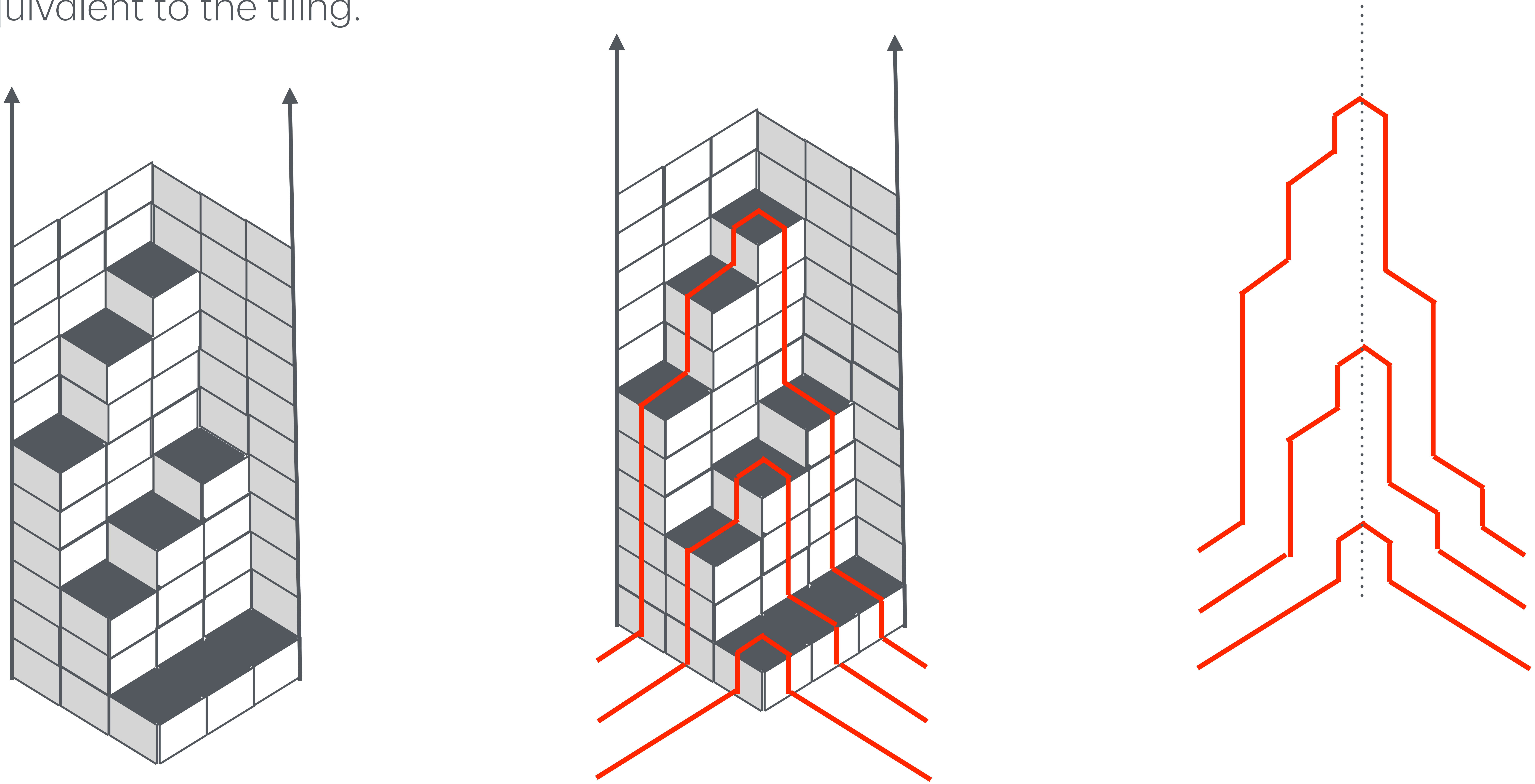






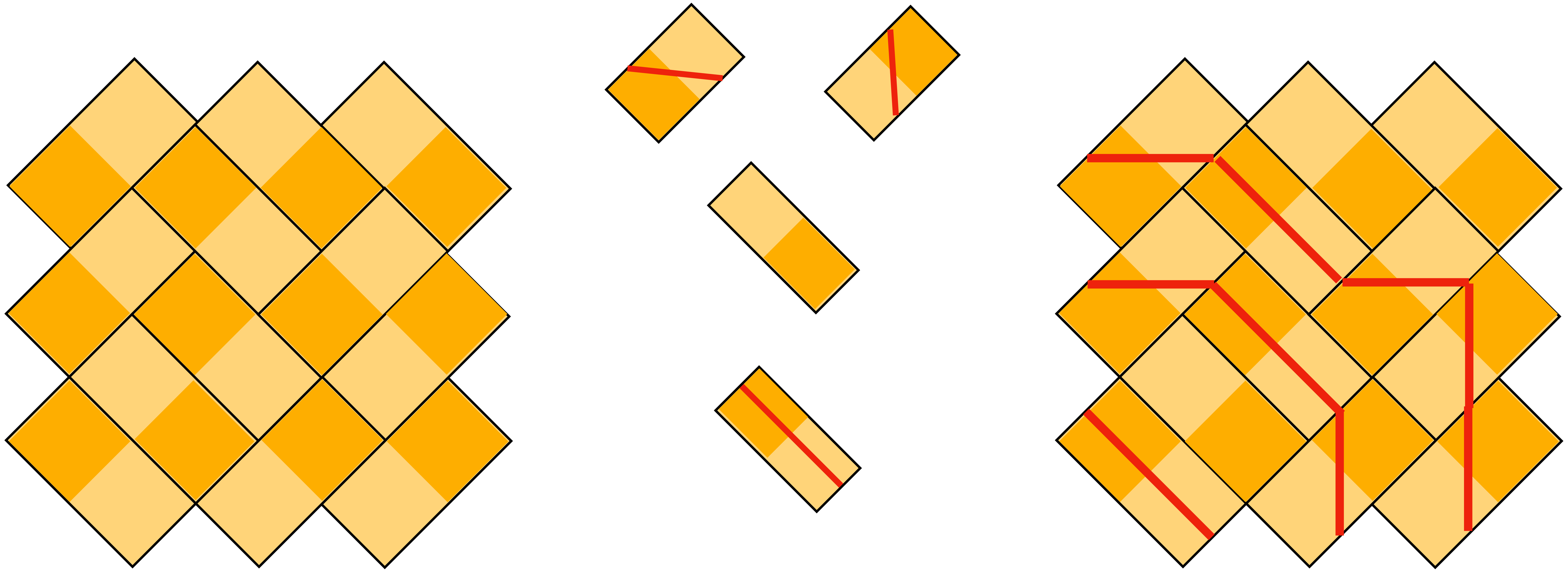
Tilings vs non-intersecting paths

- By drawing paths on the tilings we find a natural collection of non-intersecting paths that is equivalent to the tiling.



Infinite paths

Domino tilings vs DR-paths



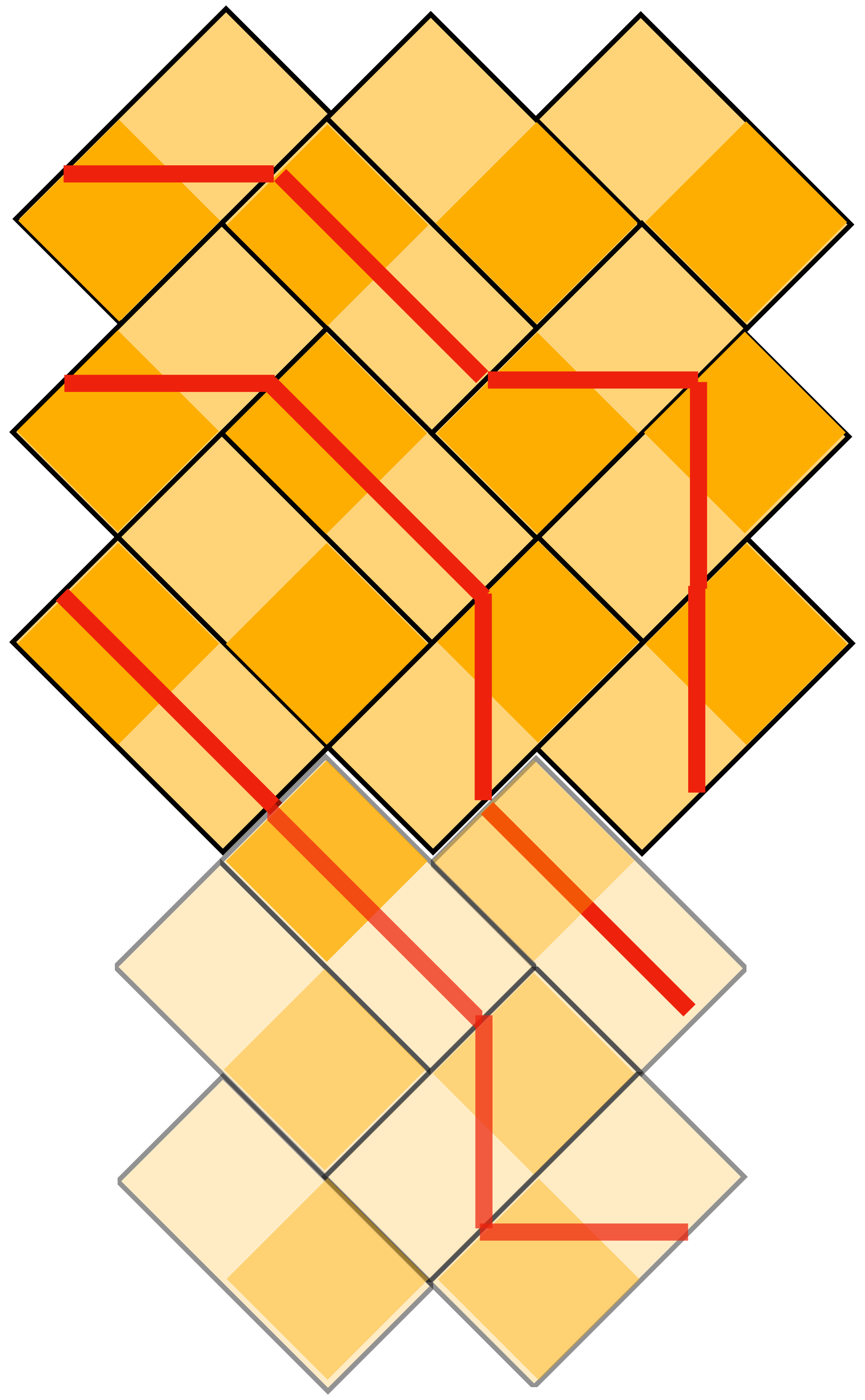
The tiling is completely determined by the configuration of non-intersecting paths, often referred to as DR paths.

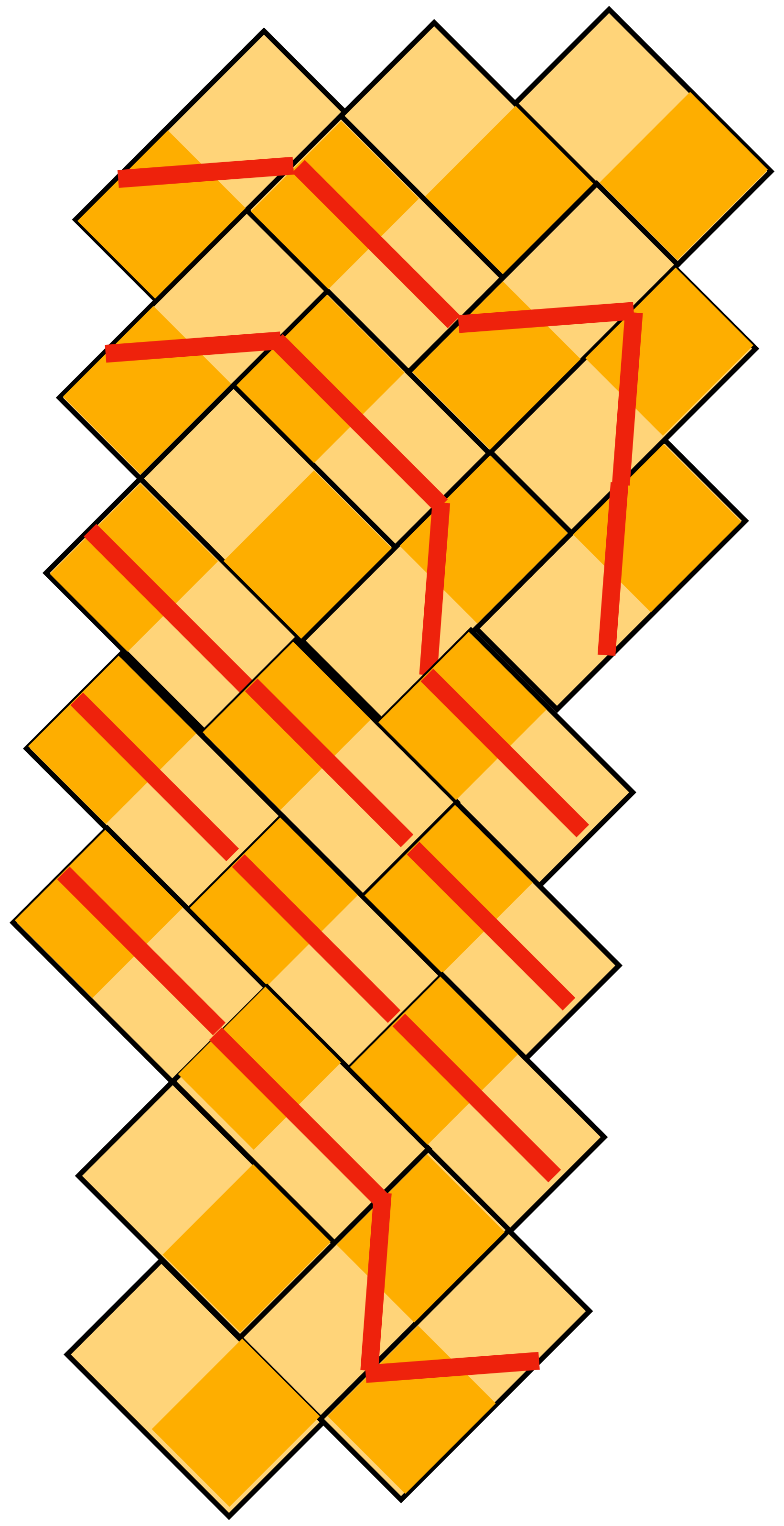
Domino tilings vs non-intersecting paths

- DR paths have the disadvantage of unequal length. Instead one can tile a larger domain.

Two Aztec diamond glued together.

It is not possible to tile the bigger domain such that there is a domino that has parts in both diamonds.





Two Aztec diamonds, with a corridor in between (of arbitrary length).

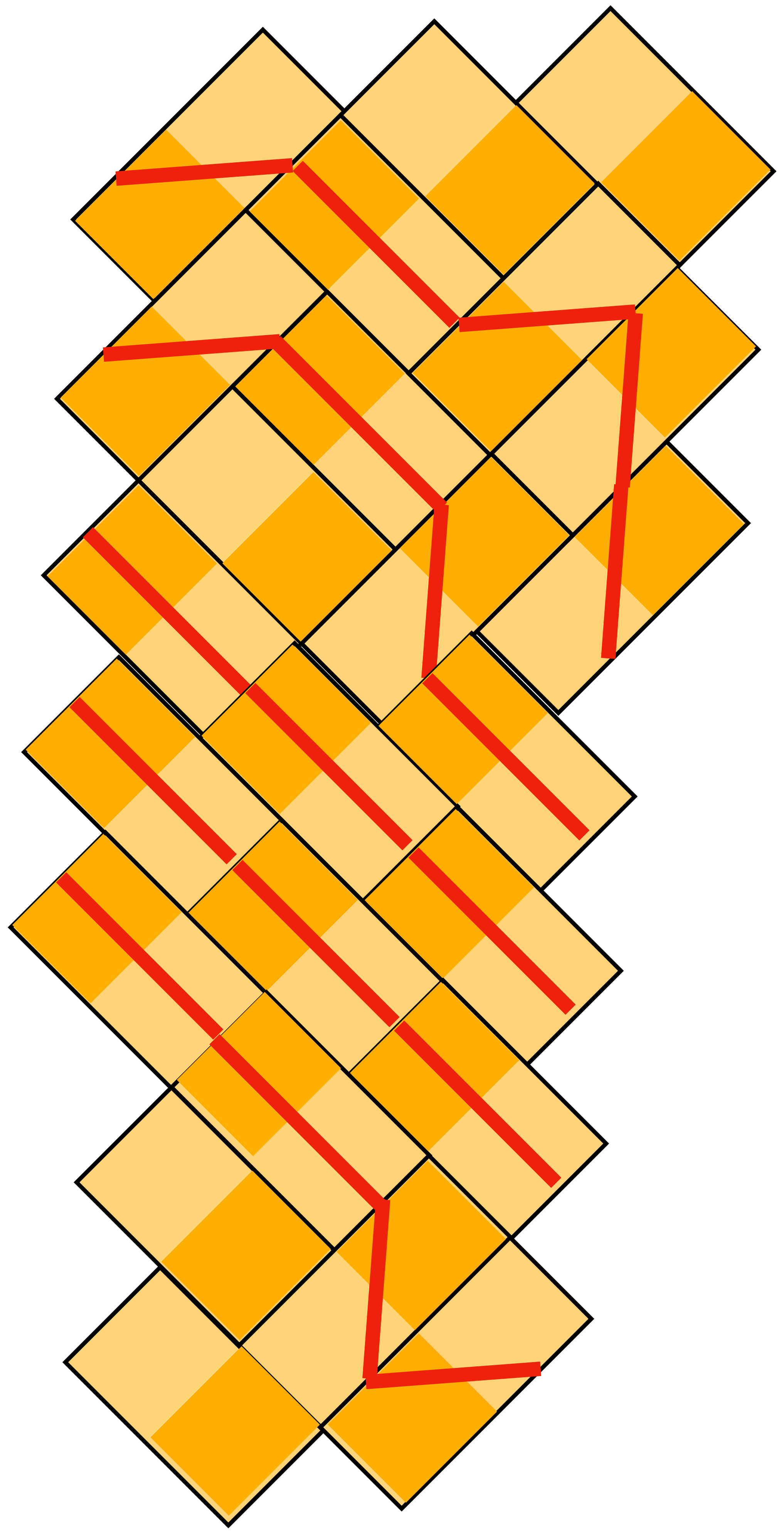
The tiling in the corridor must be trivial.

Taking the limit $n \to \infty$

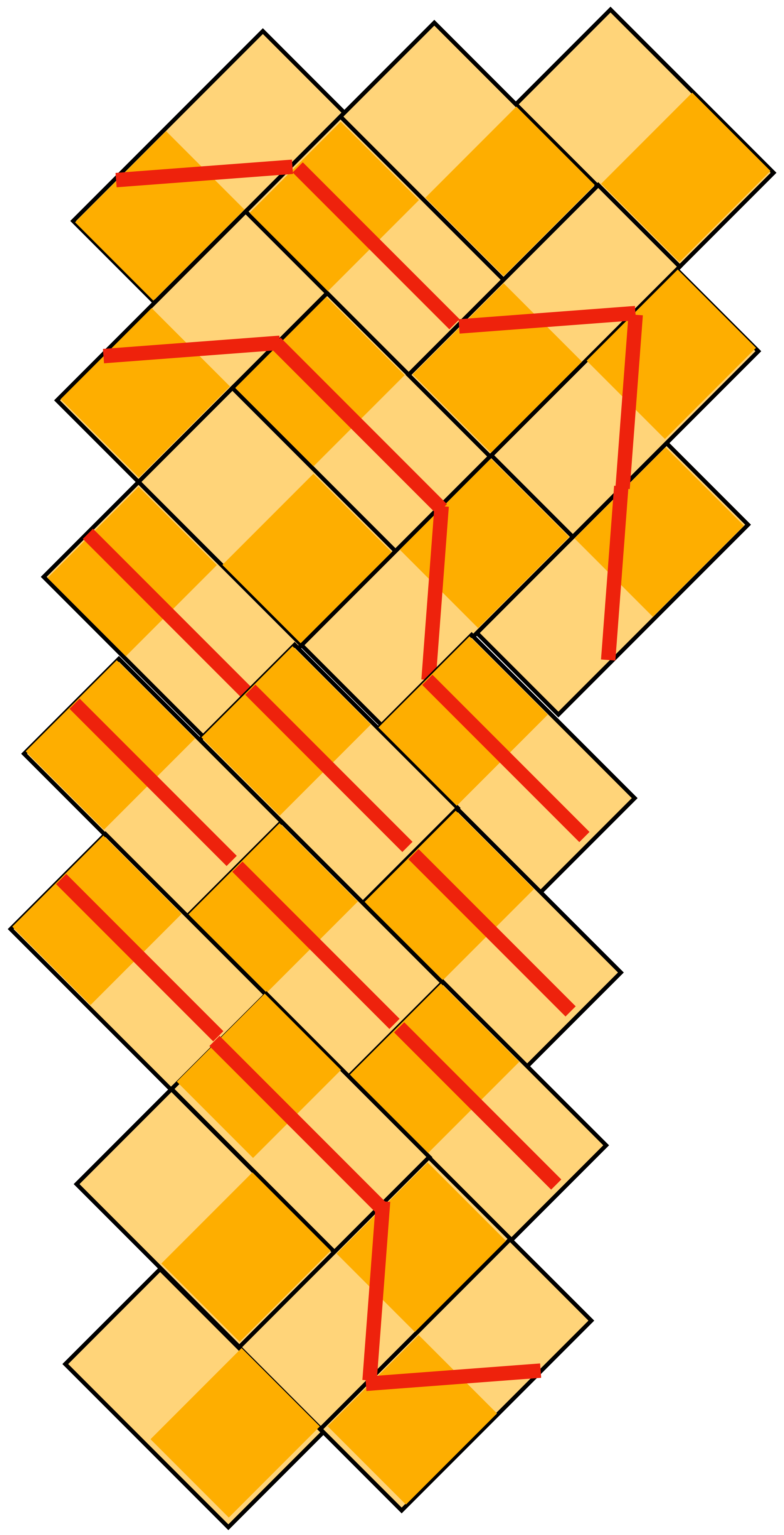
- Under some conditions, the process has a limit when the number of paths go to infinity, $n \to \infty$, keeping all other parameters fixed.

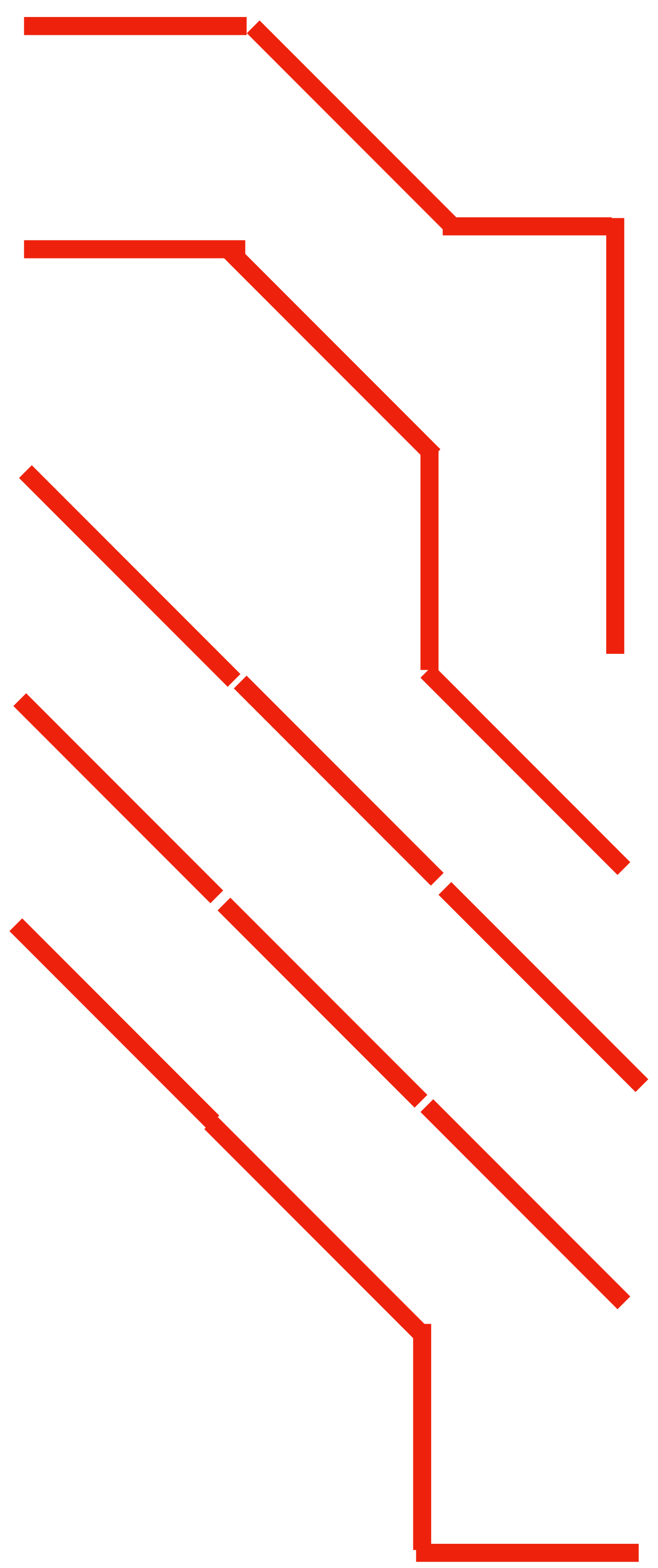
- The limit $n \to \infty$ greatly simplifies the structure!

- But first we will manipulate the paths.

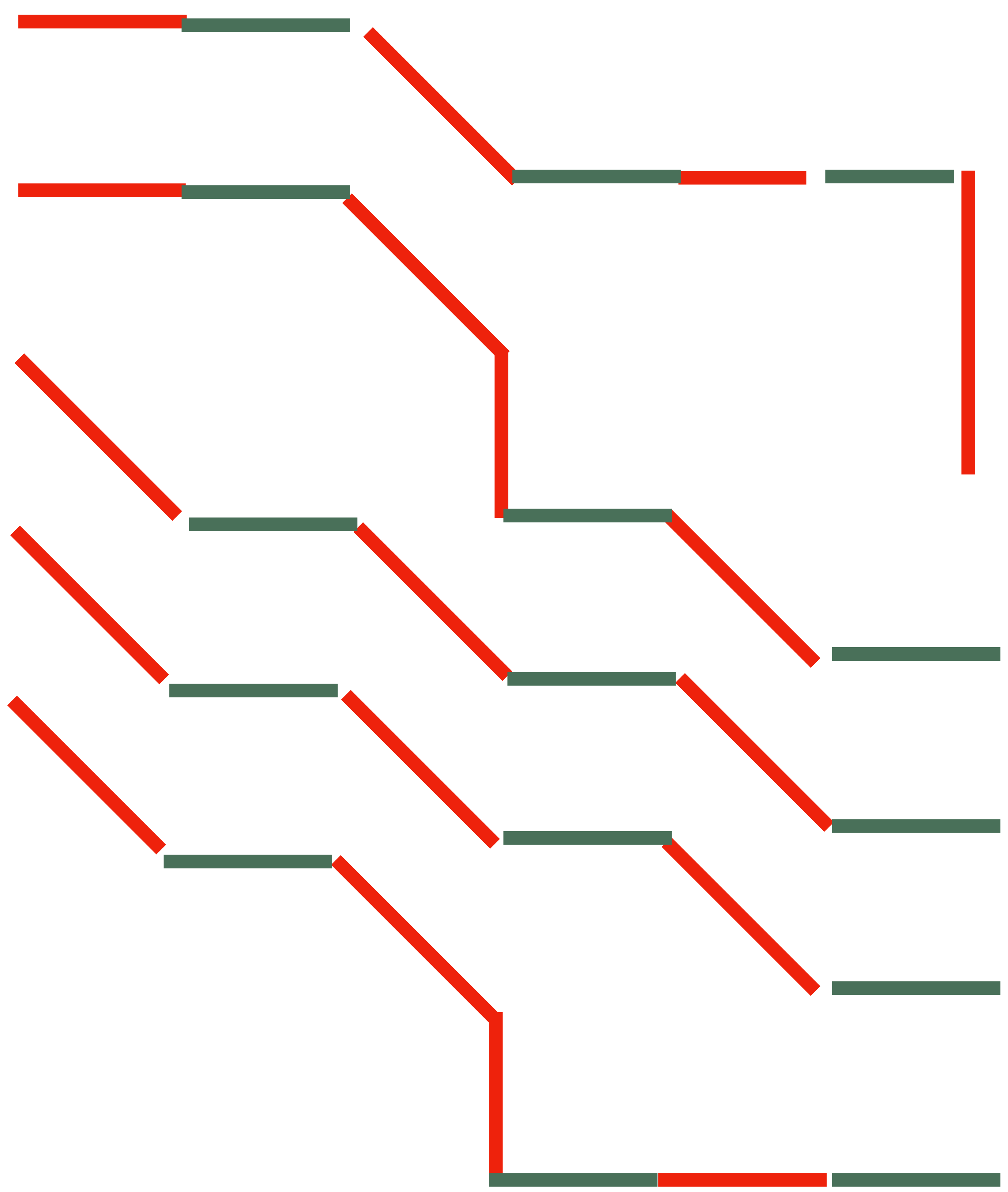


Manipulating paths



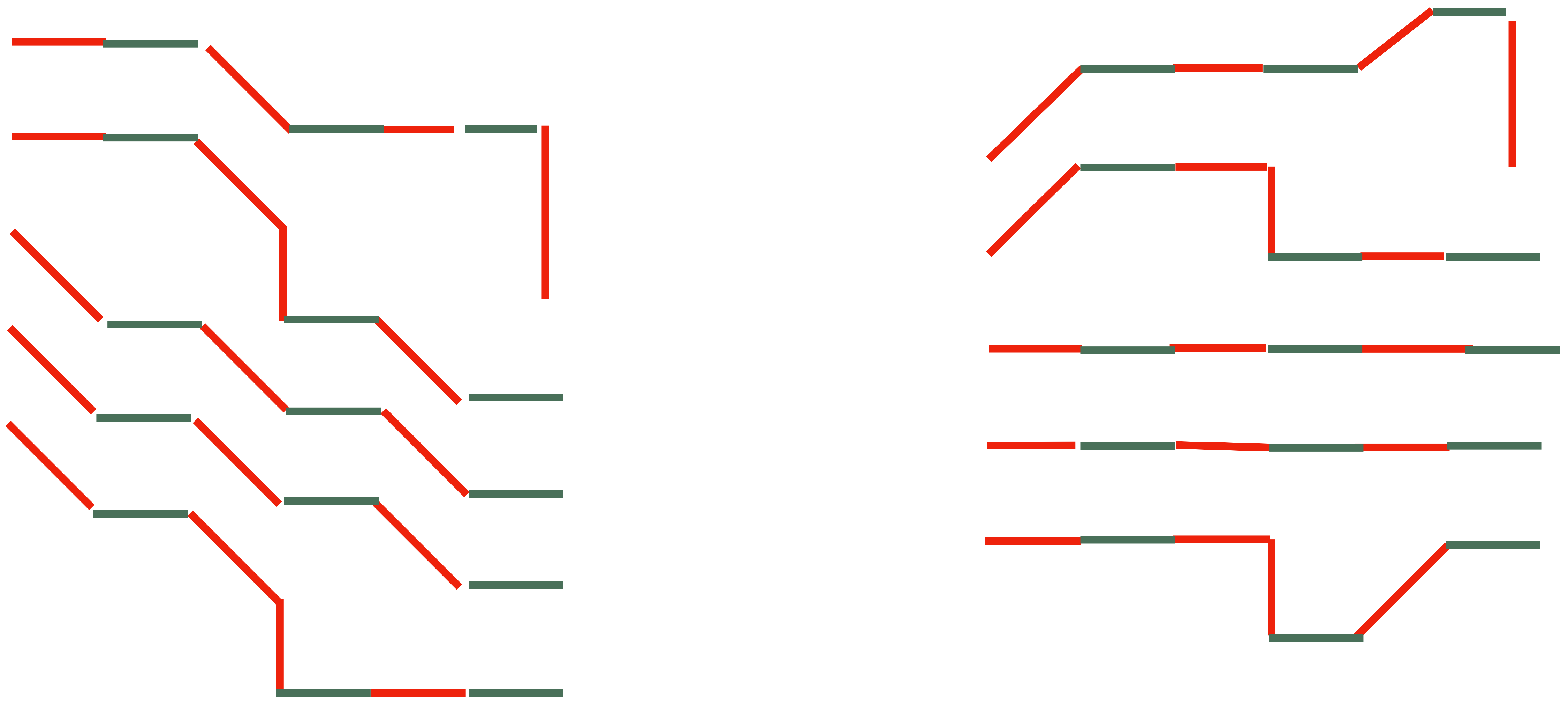


Remove the tilings



Insert auxillary green horizontal pieces

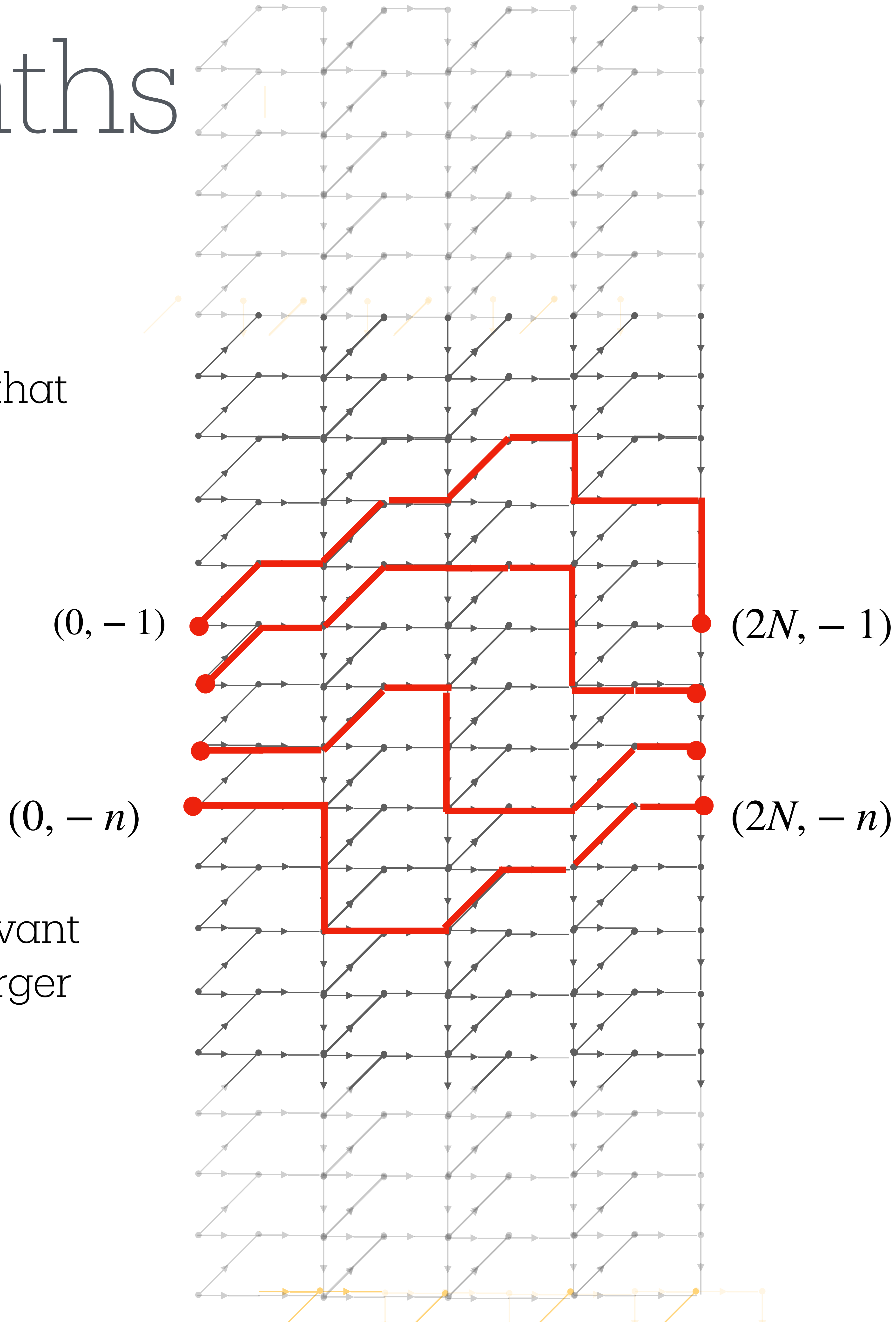
Manipulating paths



Perform a shear transform on the red parts.

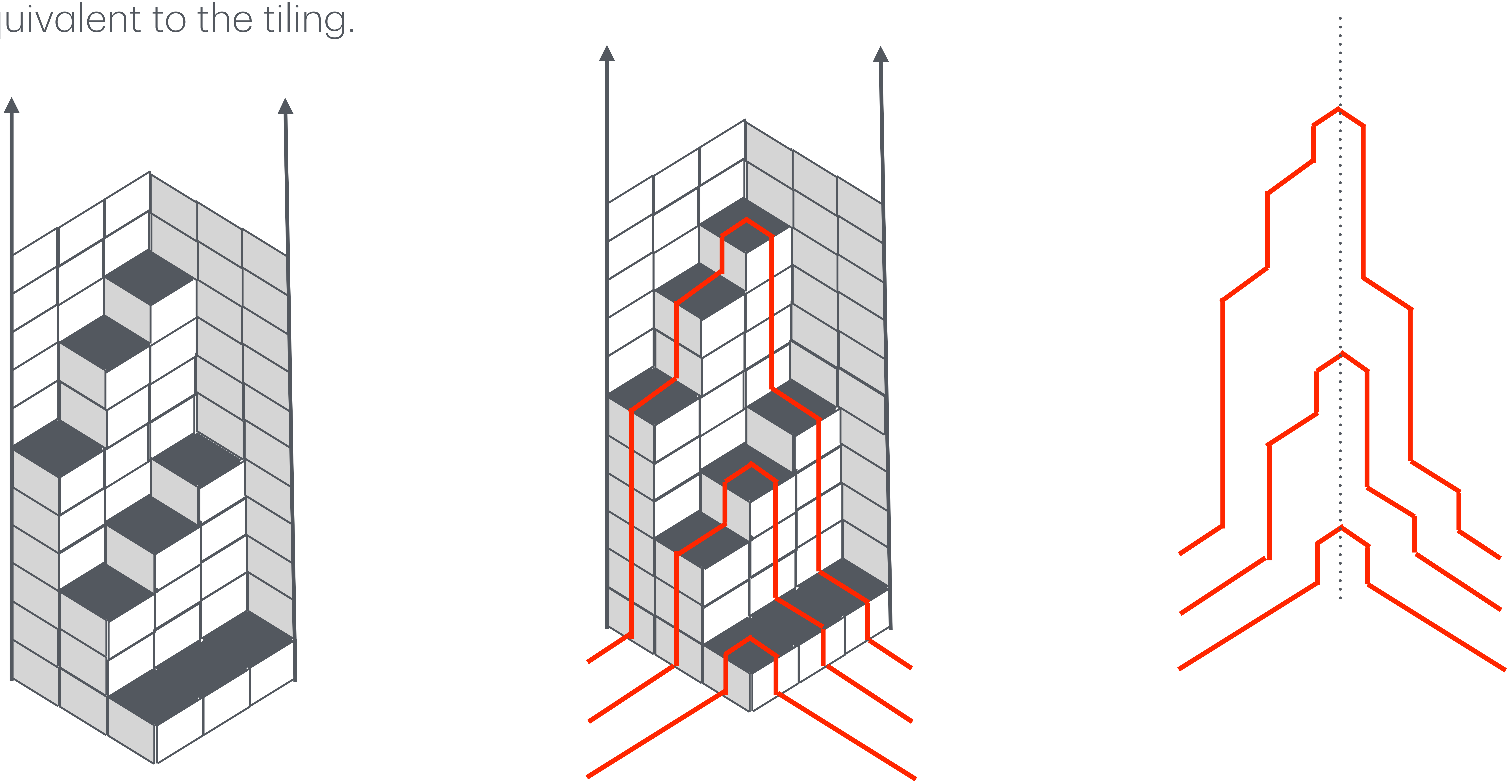
Non-intersecting paths

- Let $n, N \in \mathbb{N}$ with $n >> N$
- Define the set $\Pi_{n.i.}$ as the set of all collection of n paths that for $j = 1,\ldots,n$
  - start at $(0, -j)$
  - end at $(2N, -j)$
  - never intersect
- These paths are equivalent to the extended region. They decouple and only the top left part is actually relevant for the Aztec diamond. It is still useful to consider the larger region, as it helps with the technicalities
- We can take $n \to \infty$ first.



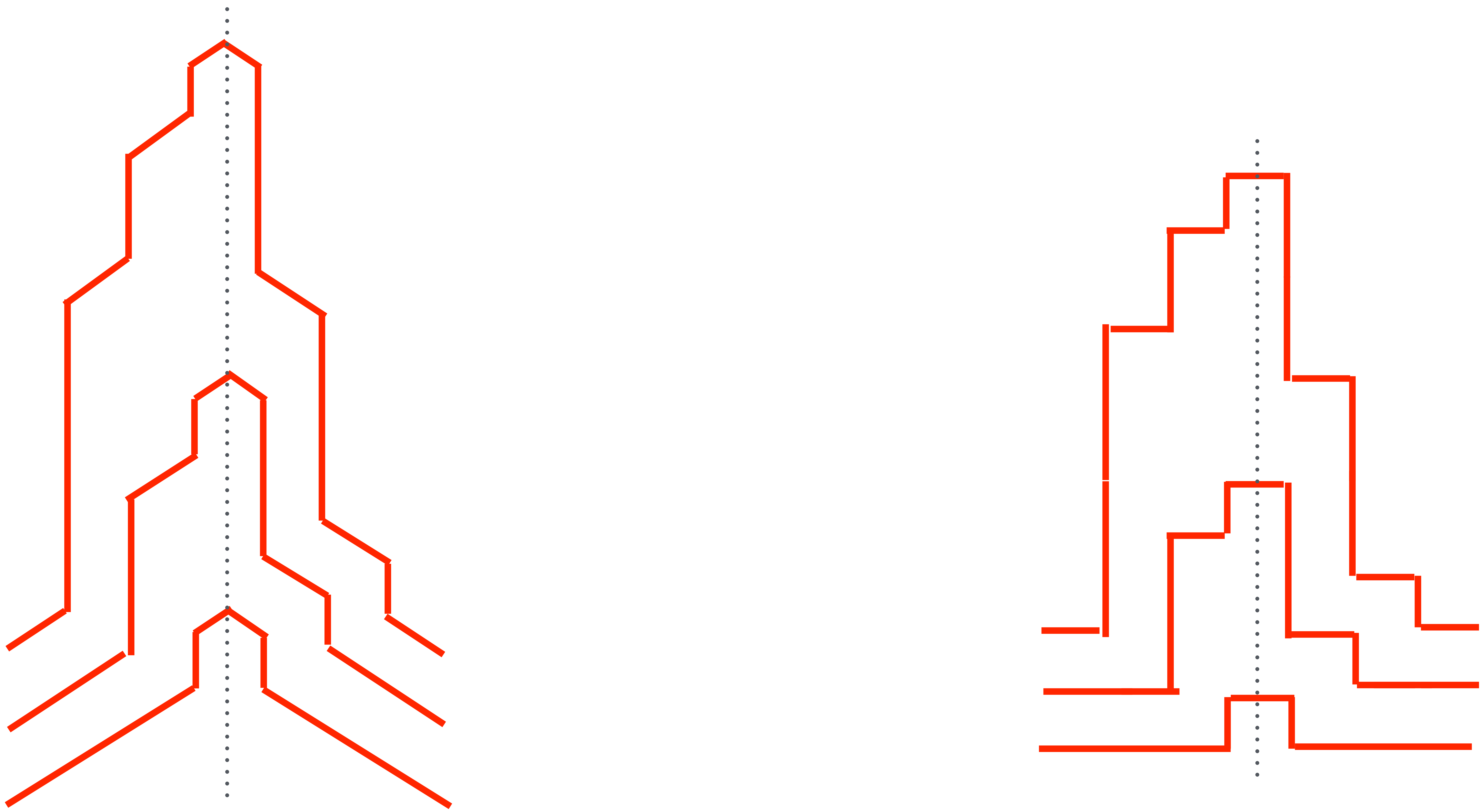
Lozenge tilings for the semi-infinite hexagon

- By drawing paths on the tilings we find a natural collection of non-intersecting paths that is equivalent to the tiling.



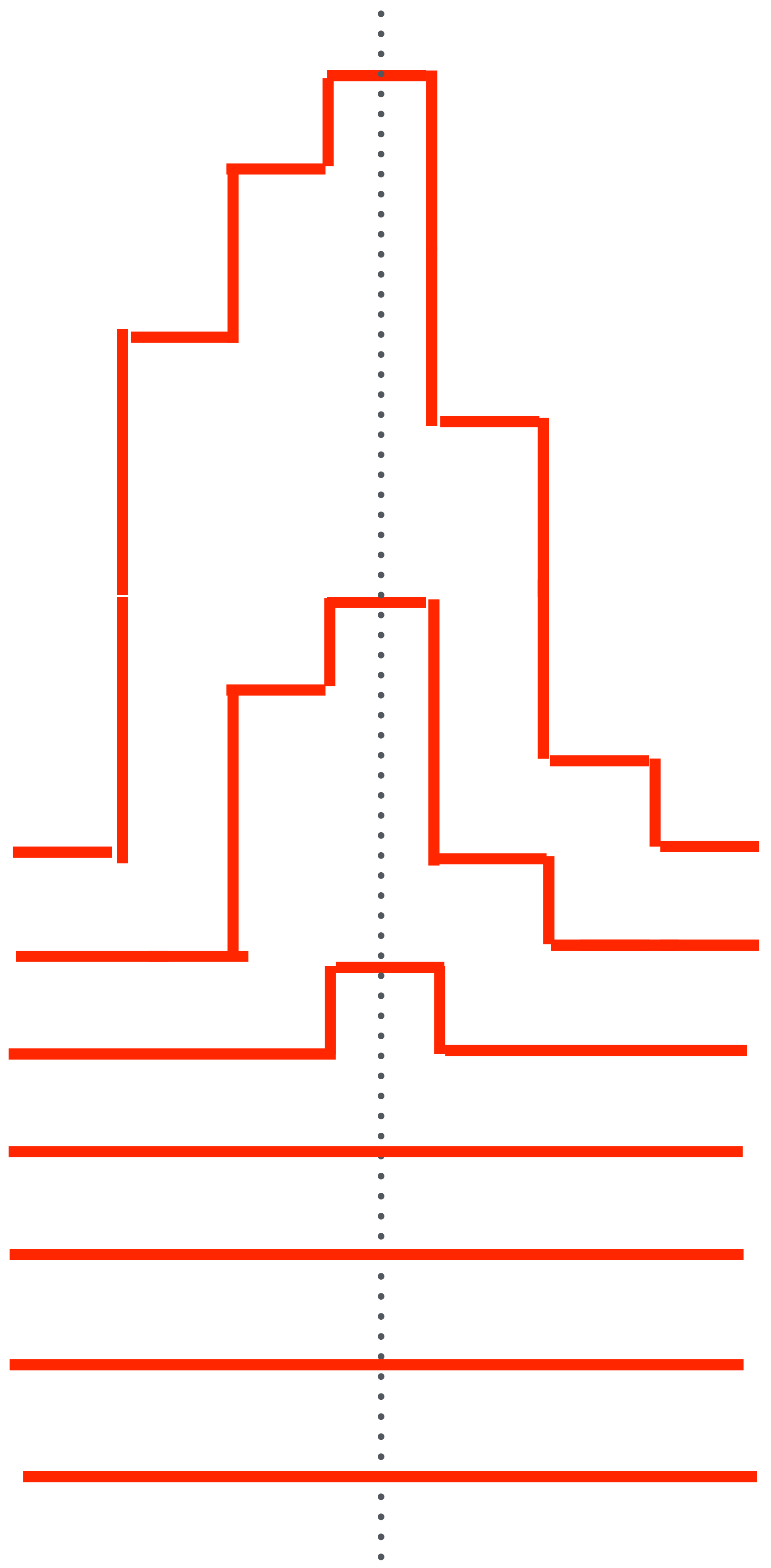
Shearing both sides

- A shear transform on the left and the right of the middles axis give the following collection of non intersecting paths.



Adding paths.

- We can add paths for free, since they naturally need to be straight lines.



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