The Aztec diamond

An analytic journey through random tilings

Maurice Duits — January 15, 2026 — Paris

About the course

Content

This is a 20h course with the following dates:

- Lecture 1: January 15, Thursday,
- Lecture 2: January 22, Thursday,
- Lecture 3: January 27, Tuesday
- Lecture 4: February 3, Tuesday
- Lecture 4: Tuesday,
- Lecture 6: Tuesday,
- Lecture 7: Tuesday

Each lecture will be self-contained

Lectures will be mostly on the blackboard (except for an introduction today)

Course has a homepage:

https://www.math.kth.se/~duits/#Aztec

Contains updated schedule, lecture notes and extra material.

Disclaimer: lecture notes are drafts and may be incomplete or contain errors.

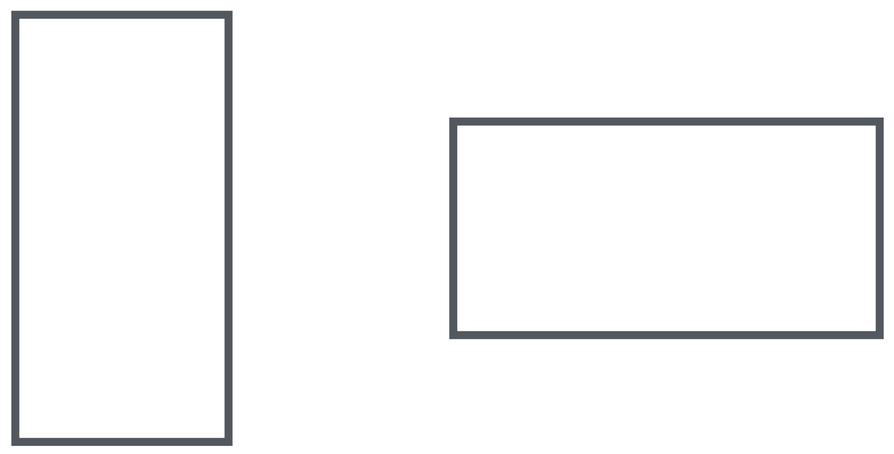
Tilings of planar domains

Tilings

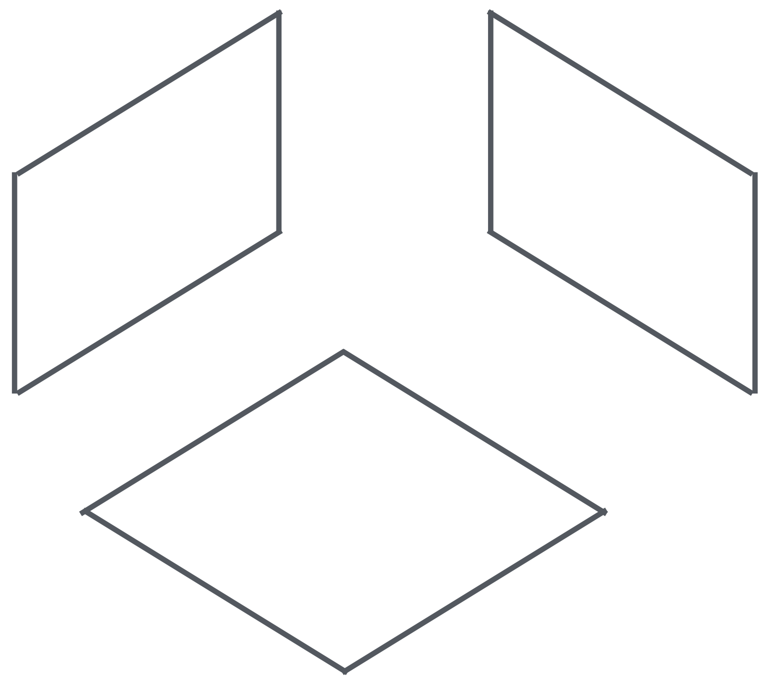
- A ***tiling of a planar domain*** is a complete covering of a domain such with a given set of shapes, such that no two shapes overlap.

- In this course, we will see mainly two different types of tilings.

  - Domino tilings

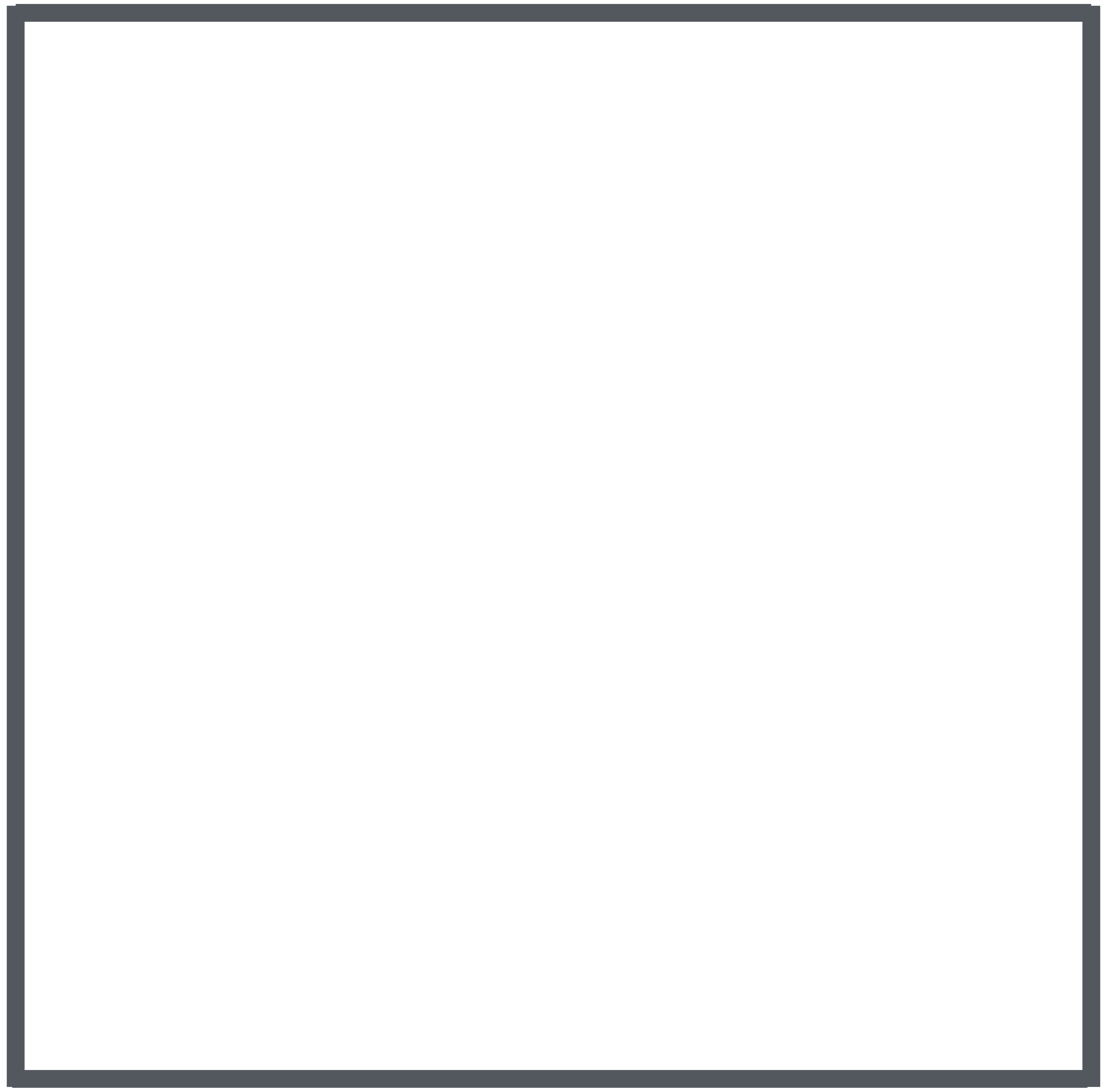


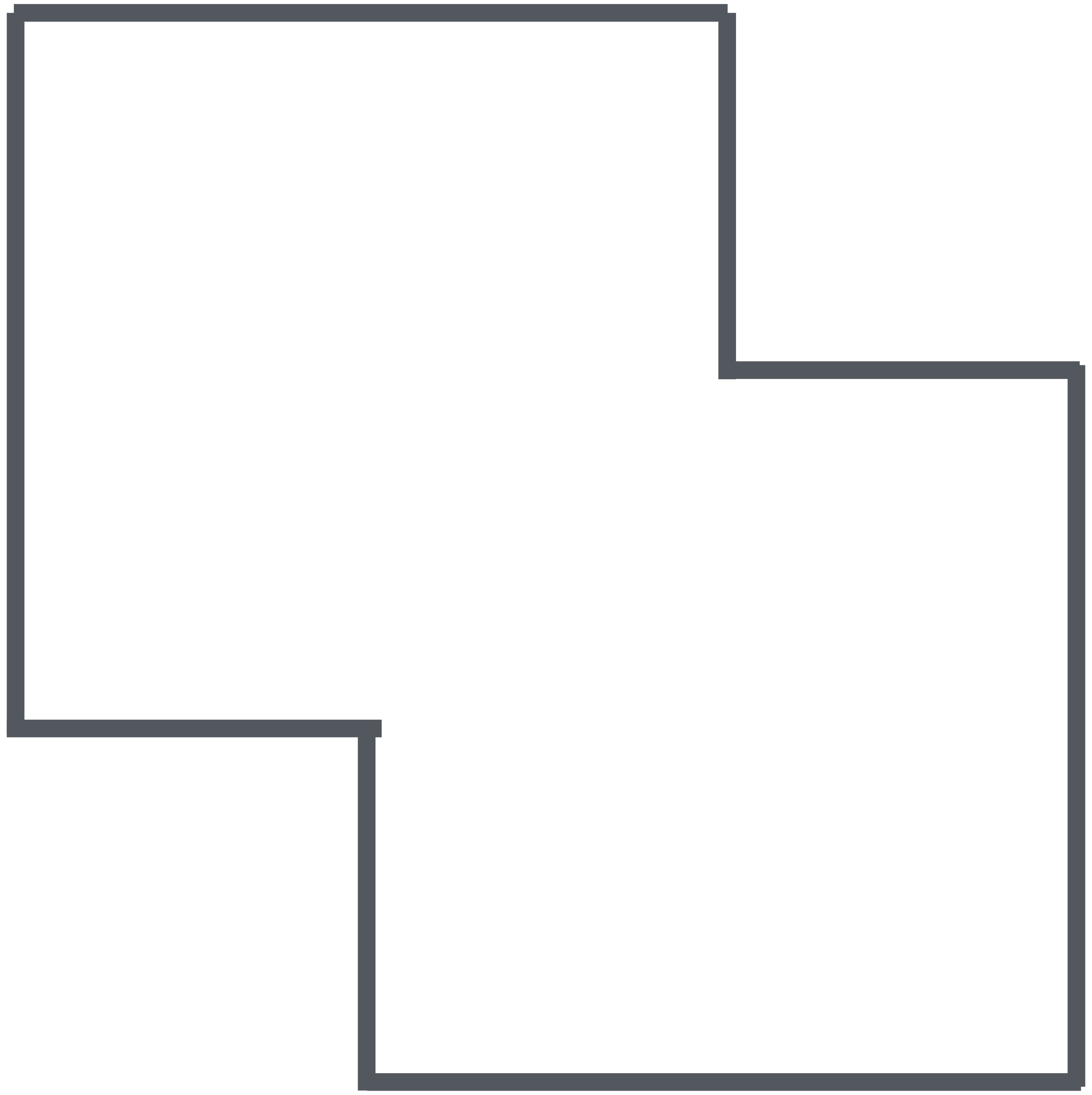
  - Lozenge tilings



Domino tilings

- Examples of domino tilings:





Domino tilings

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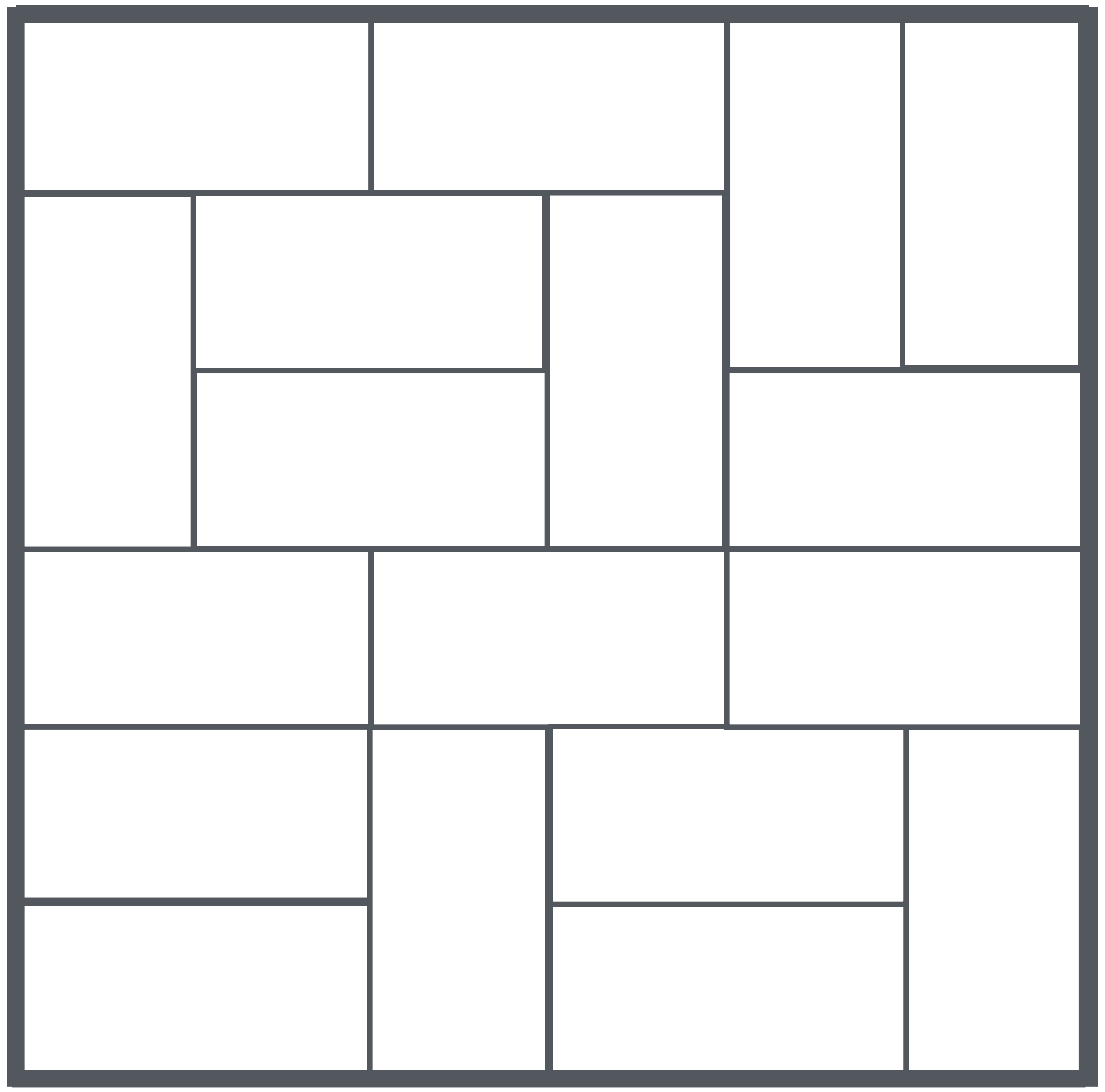
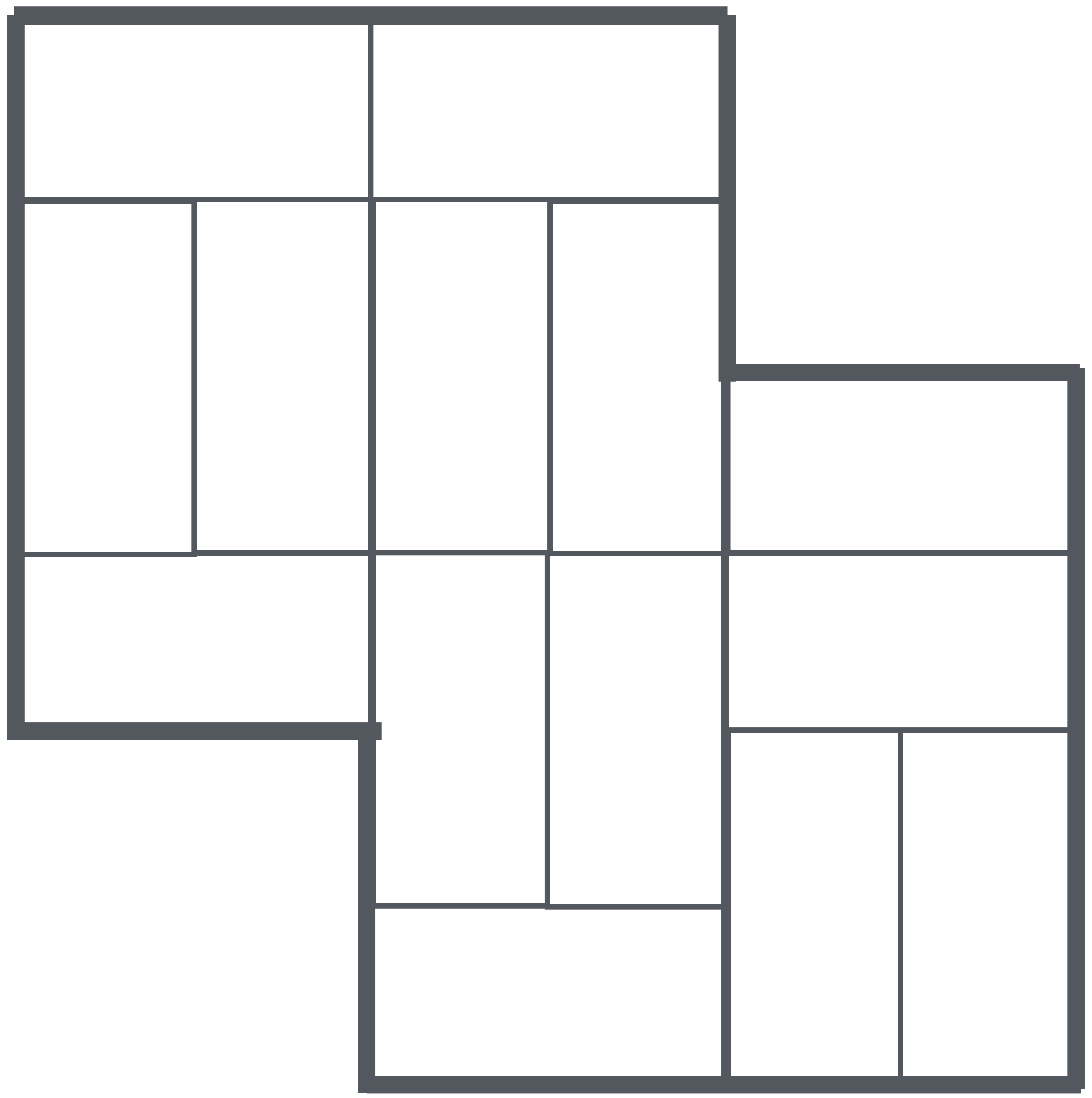
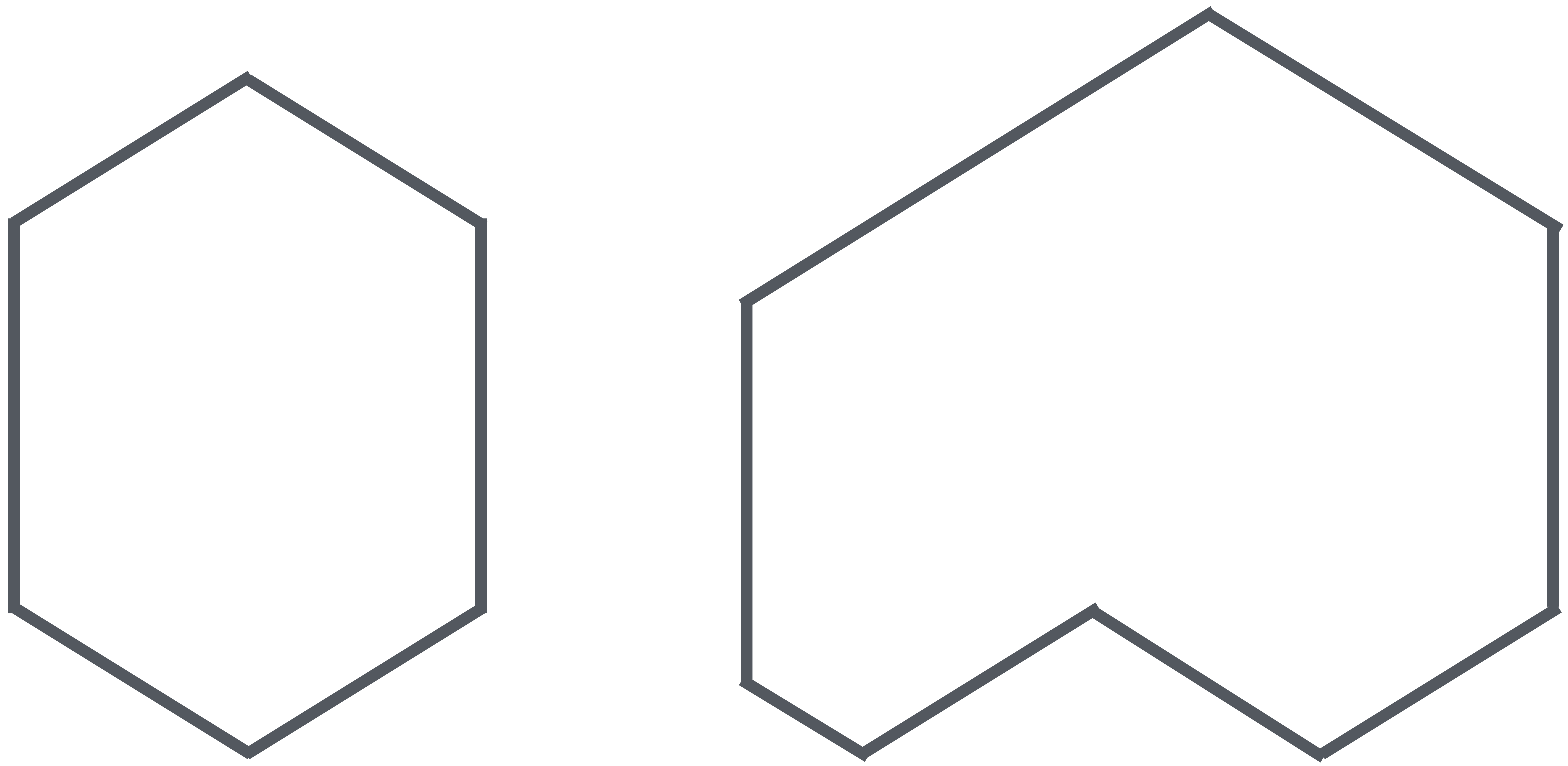




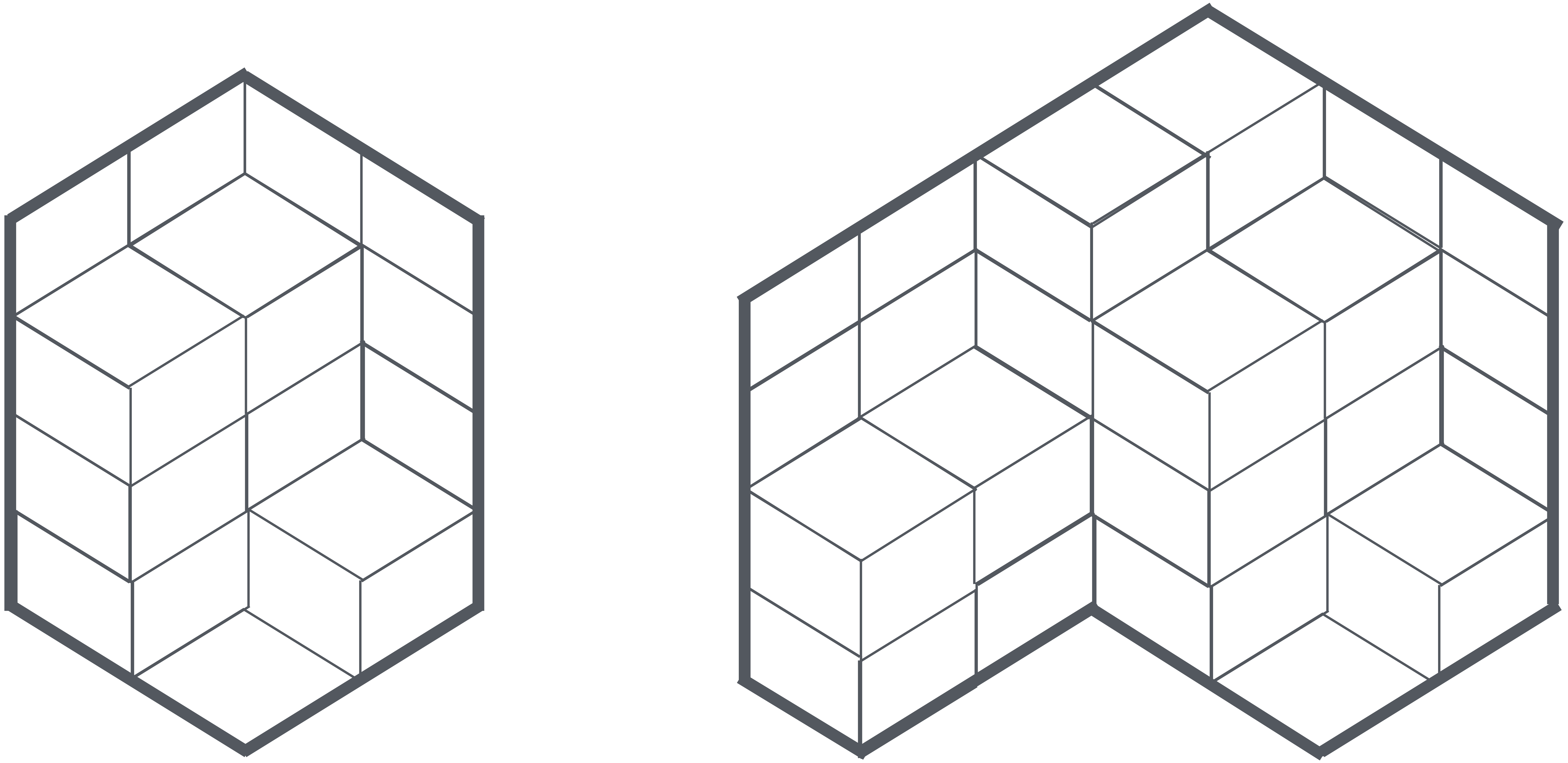
Lozenge tilings

- Examples of lozenge



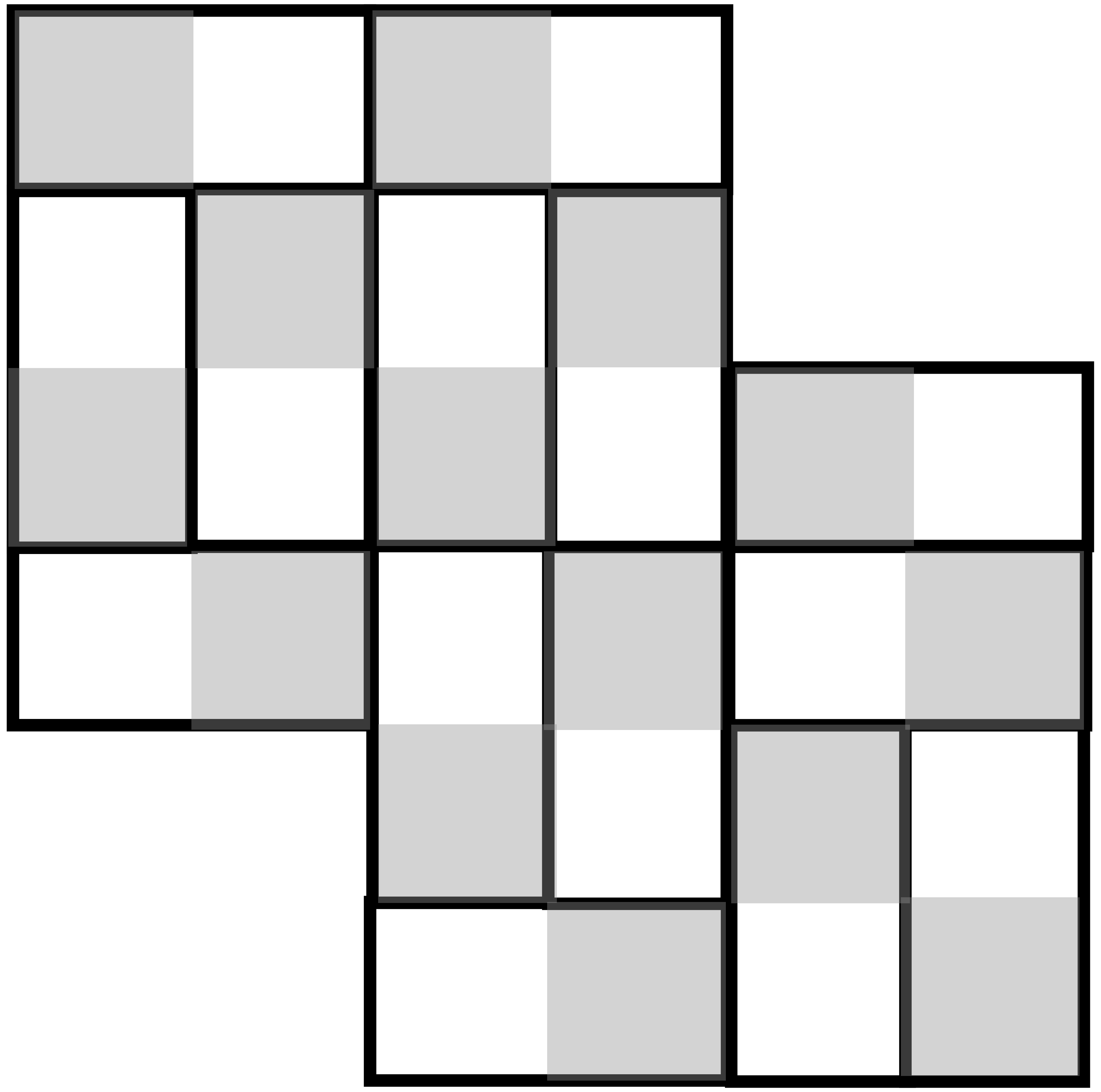
Lozenge tilings

- Examples of lozenge tilings



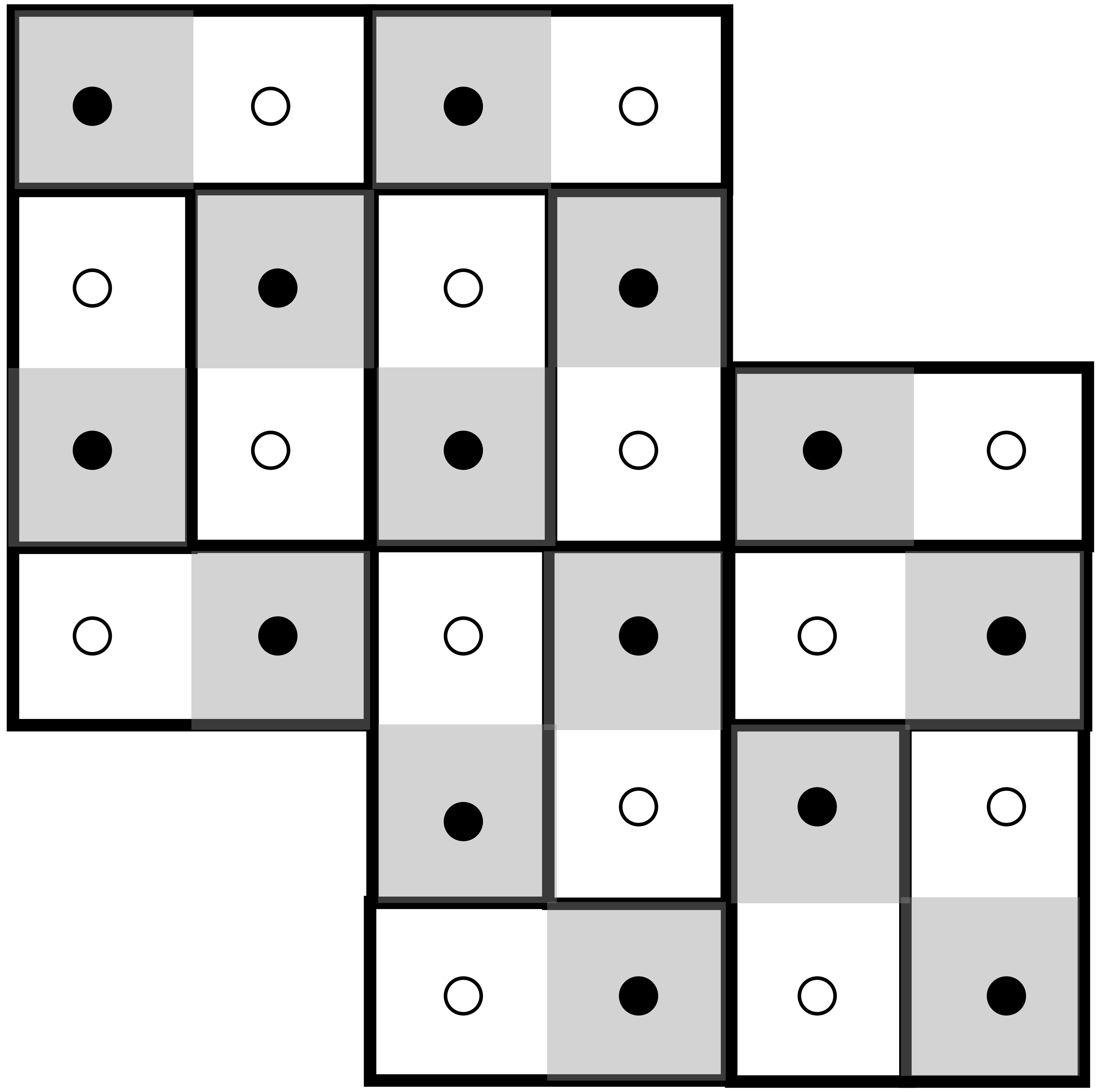
Domino tilings — checkerboard

- Draw a checkerboard on the domain:
- Each domino covers a black and white square.
- There need to be as many white squares as there are black squares. (Necessary but not sufficient.)



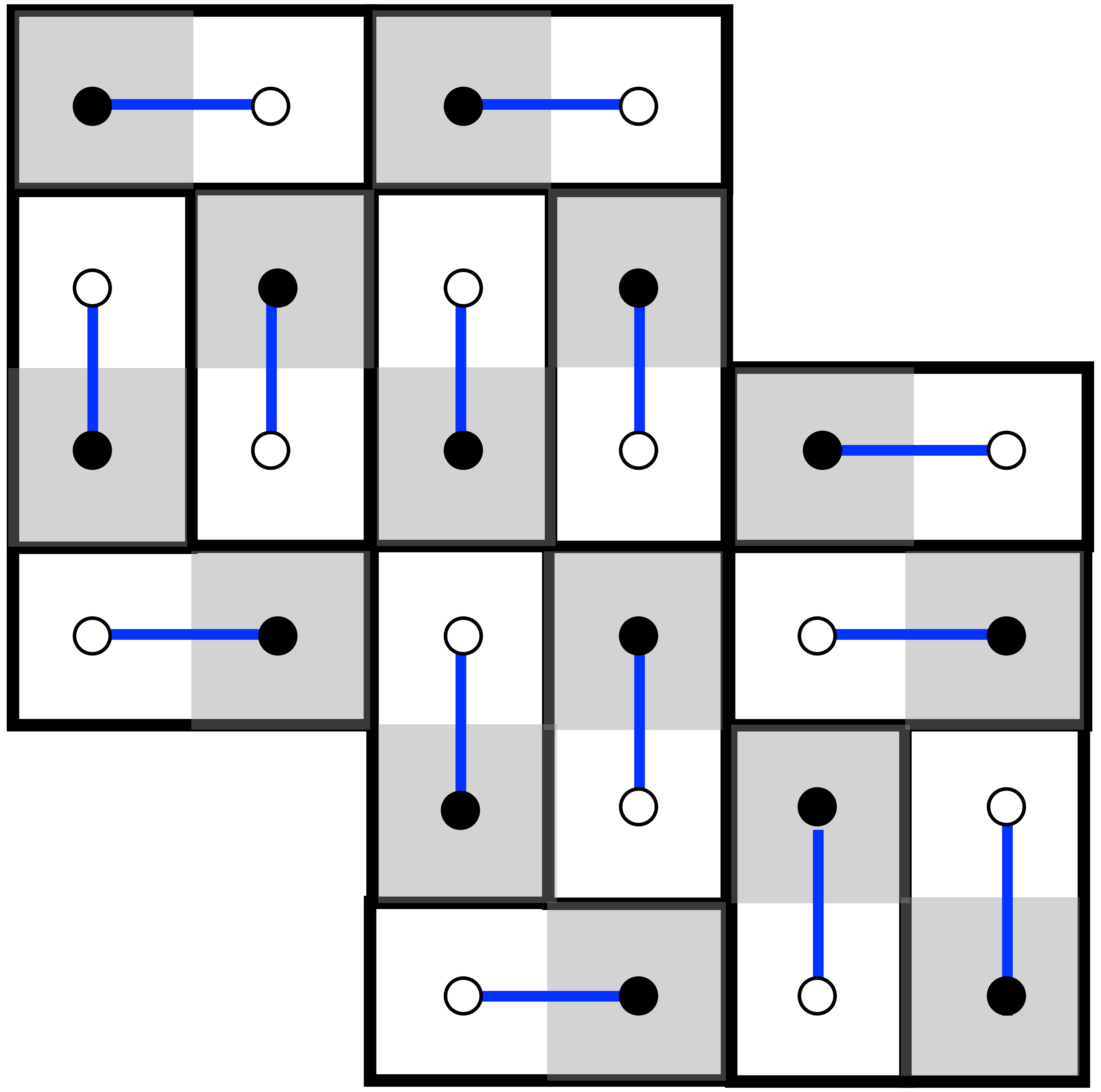
Domino tilings — checkerboard

- Replace the square by black and white dots.



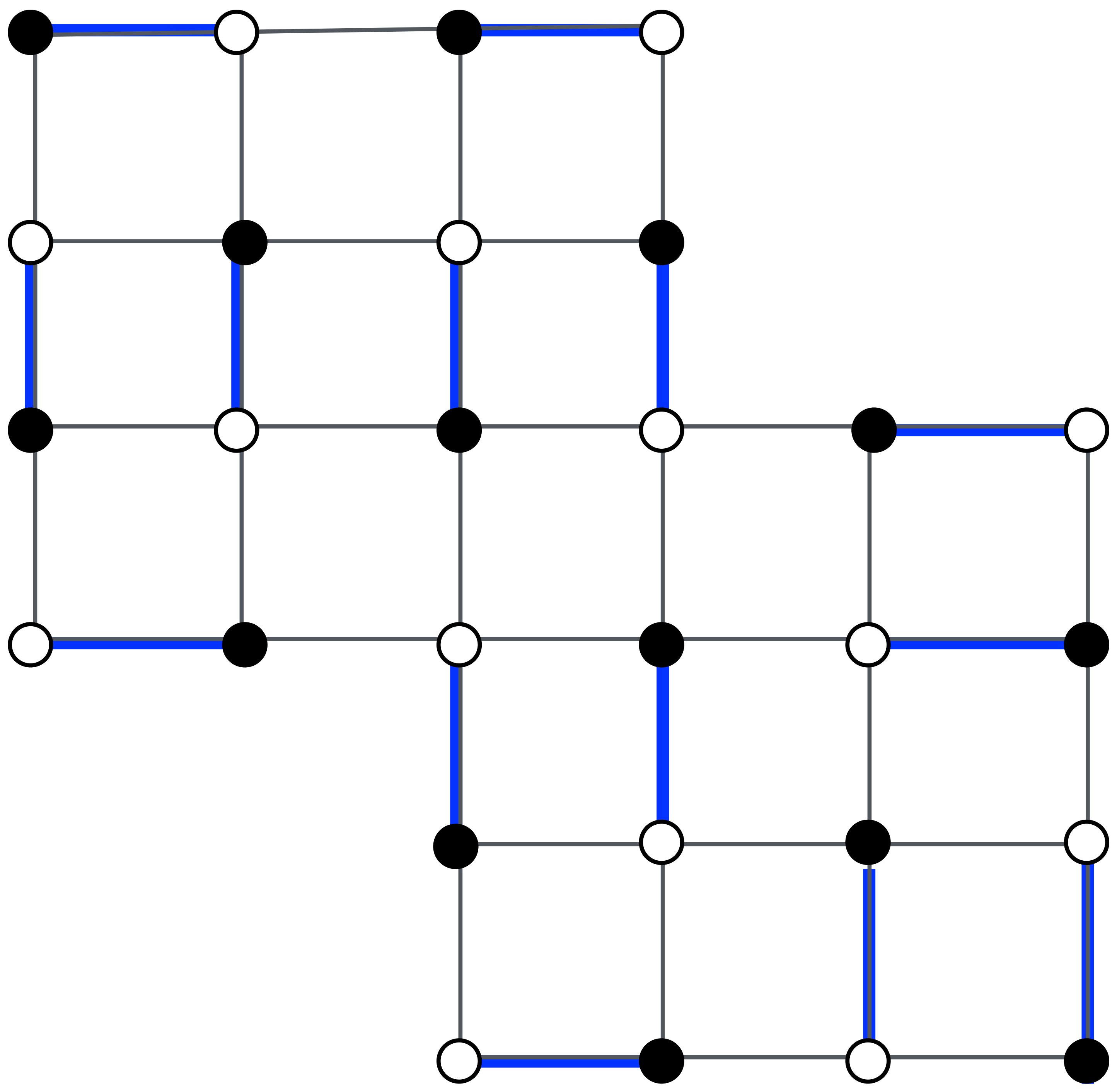
Domino tilings — checkerboard

- Replace the square by black and white dots.
- Then each domino provides a bond between a white and a black dot.



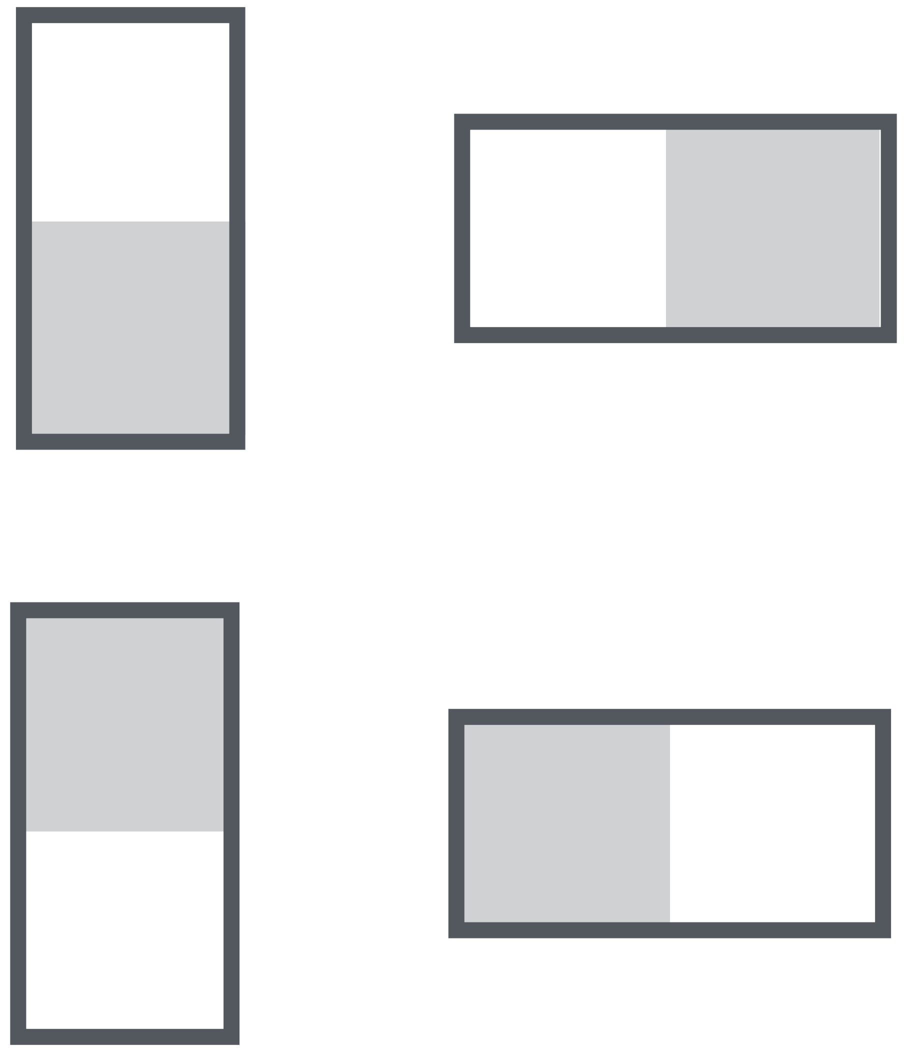
Domino tilings — checkerboard

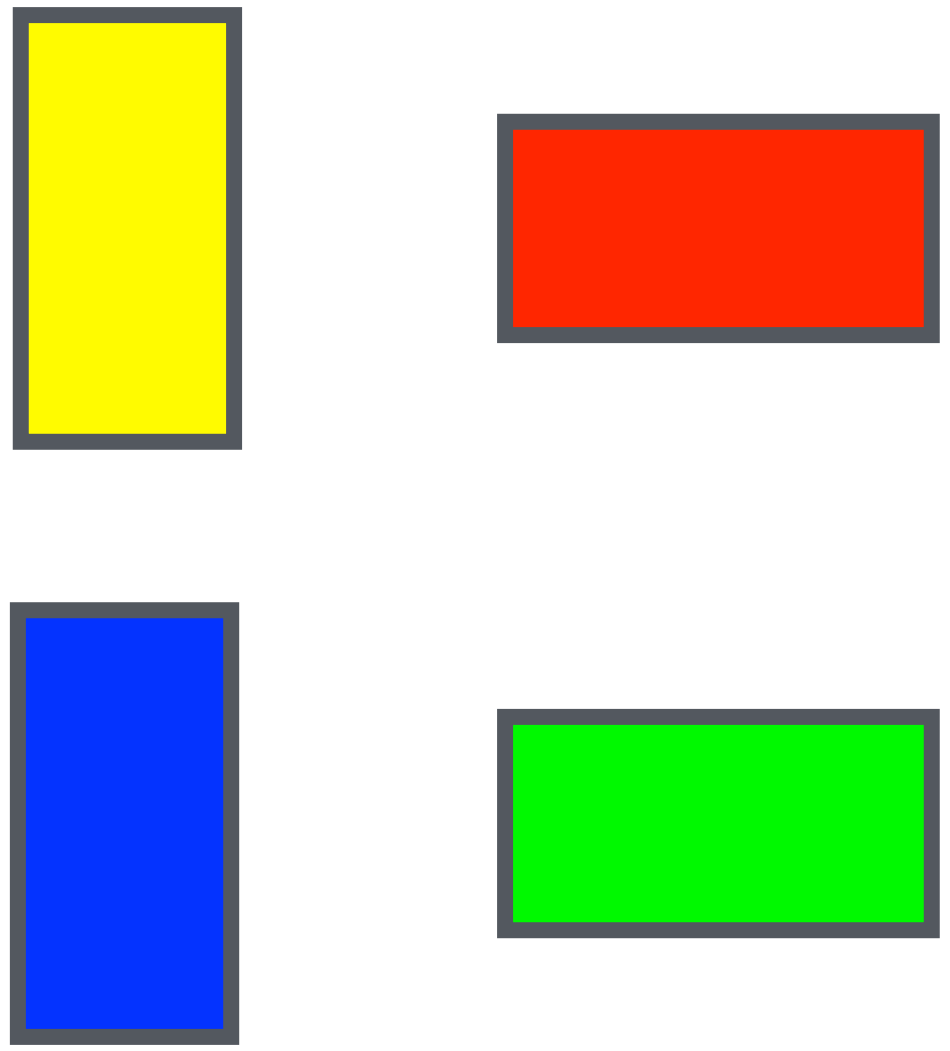
- Replace the square by black and white dots.
- Then each domino provides a bond between a white and a black dot.
- A domino tiling of a planar domain, is equivalent to a **perfect matching for a bipartite graph** where the graph is subset of the square lattice.
- This is called a ***dimer*** configuration.
- Lozenge tilings correspond to hexagonal lattice.
- *Kasteleyn's theory will not be part of the course*



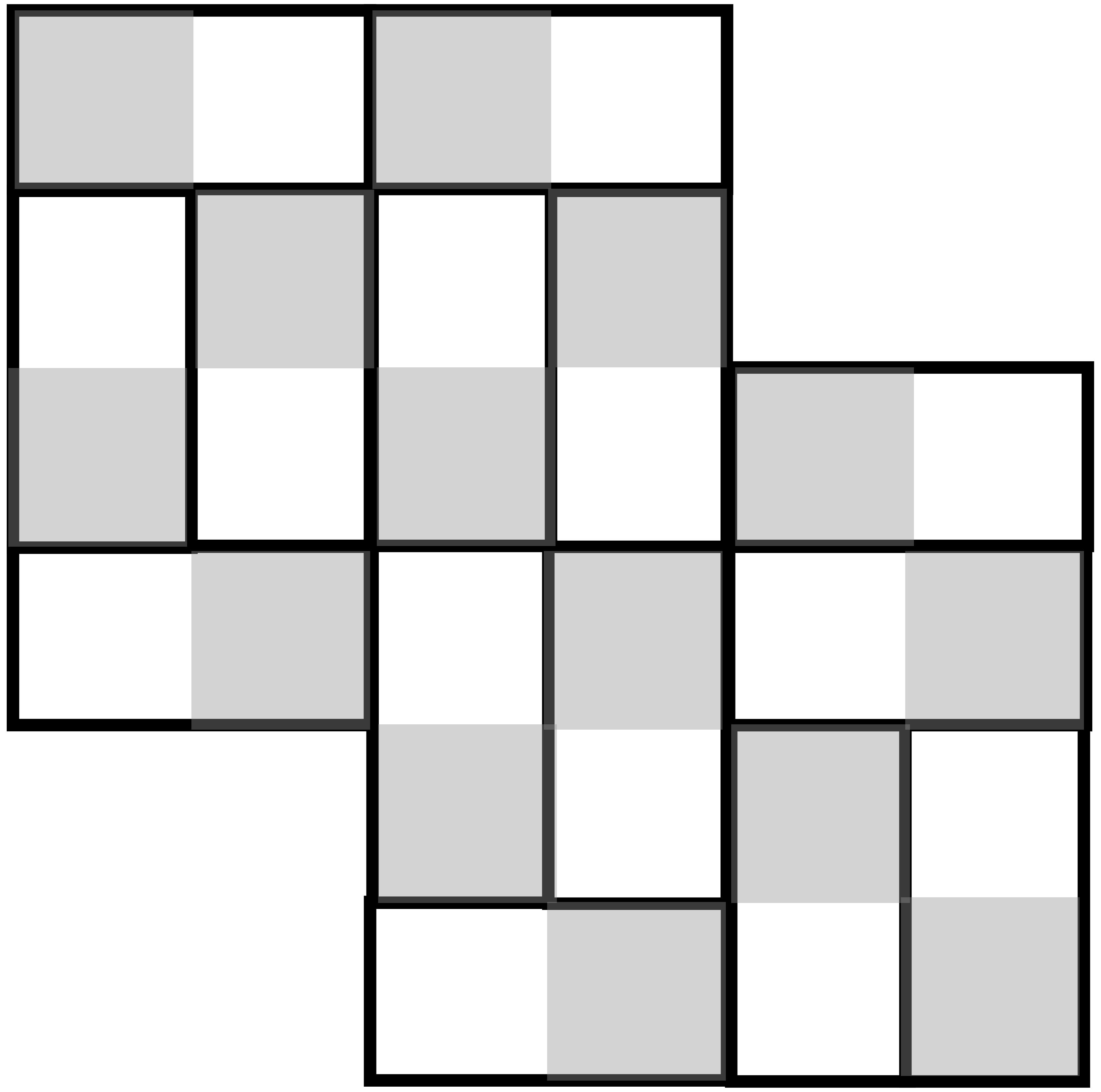
Domino tilings — checkerboard

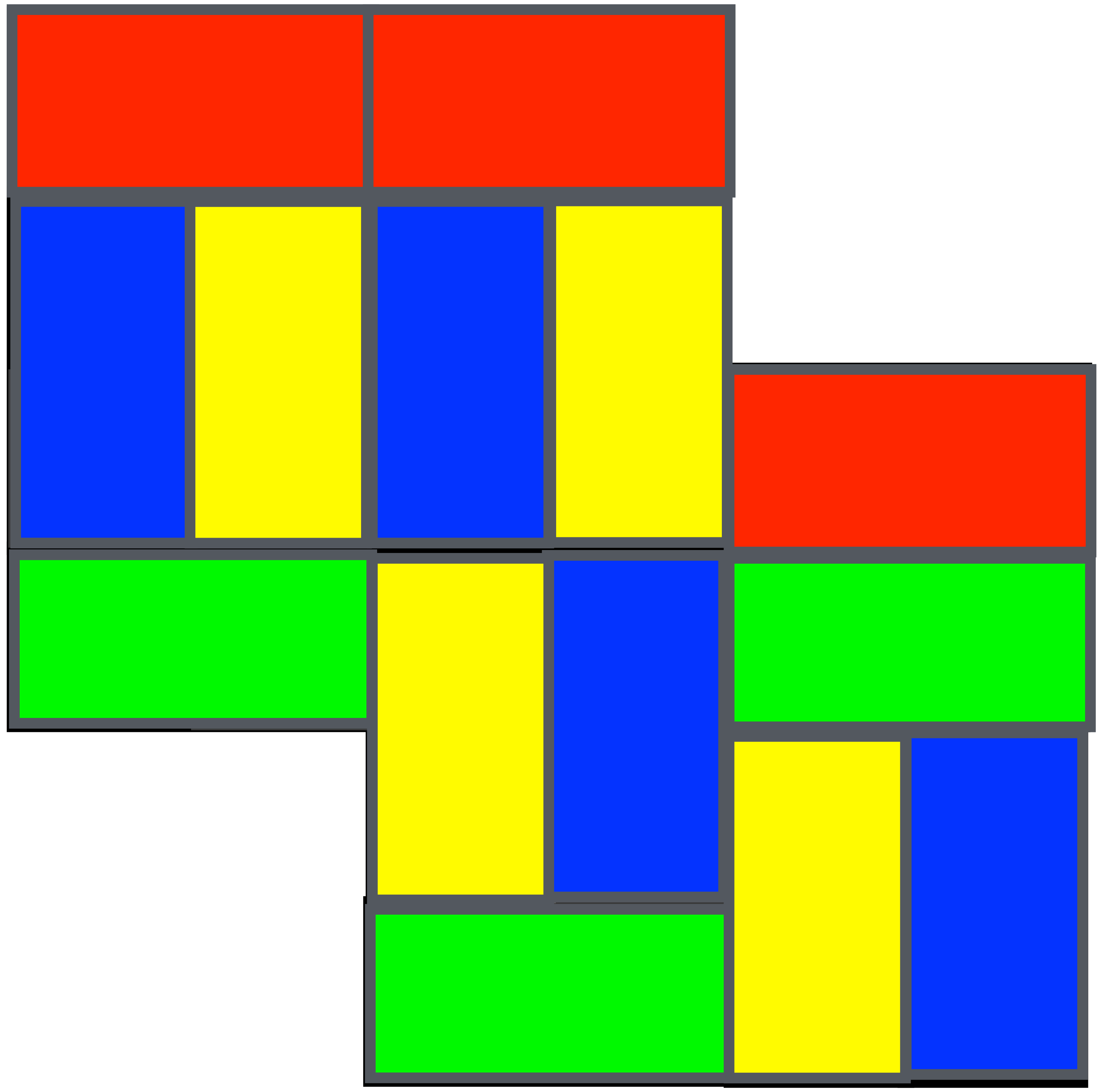
- This also shows we have in fact four different type of dominos, depending on the position of the white and black squares
- It will be convenient, and visually appealing, to give these four domino different colors.





Domino tilings — Colors



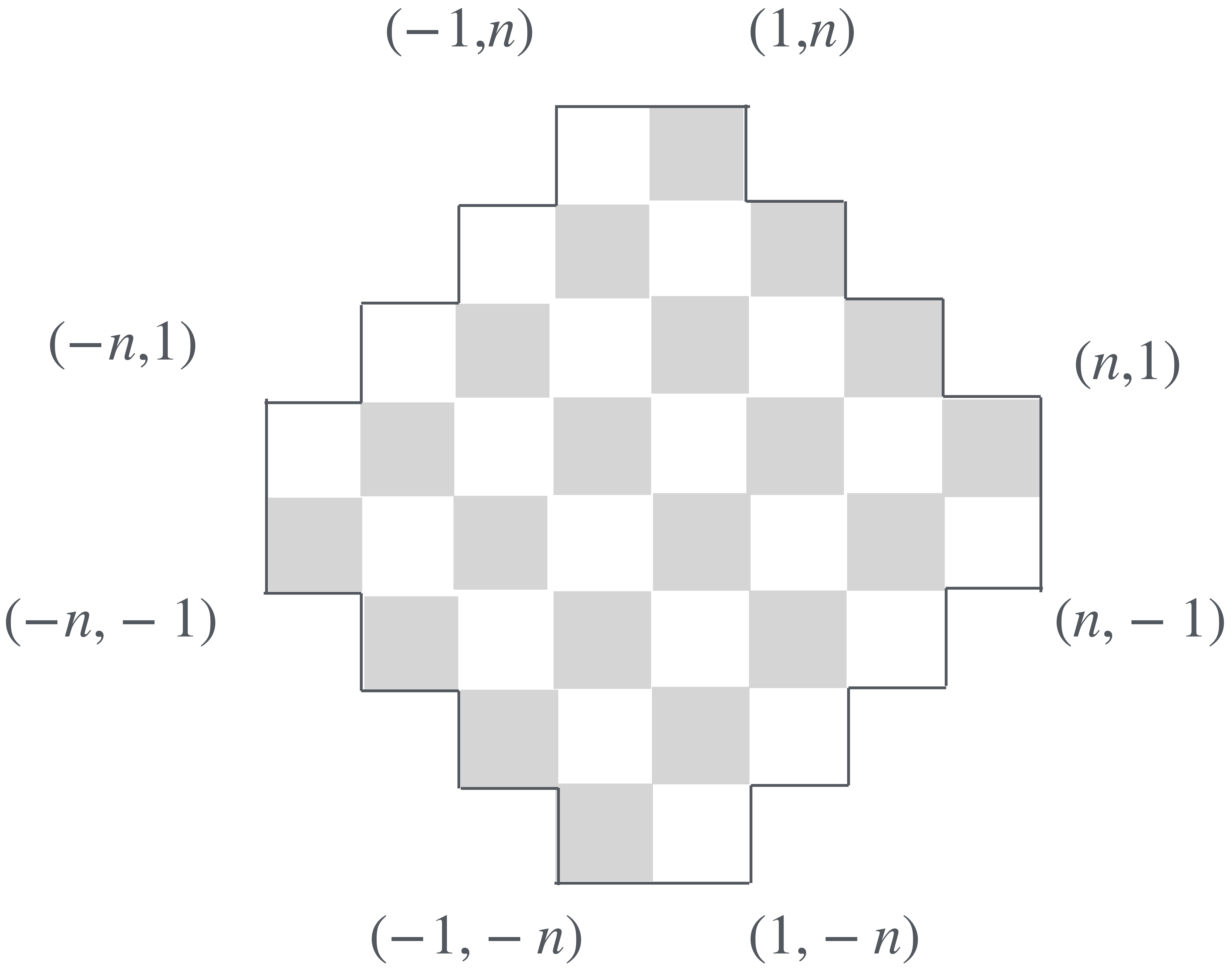


The Aztec Diamond

(The running example in the course)

Aztec diamond

- Although the methods in this course apply to broader classes of tiling models, the course will be centered around the **Aztec diamond.**
- The Aztec diamond of size n is the following domain on the right
- The sides of the domain have the shape of staircases of Maya temples, hence the name Aztec diamond (dont blame the lecturer….)
- This boundary may look strange at first, but will be very important.



$n = 4$

Aztec diamond

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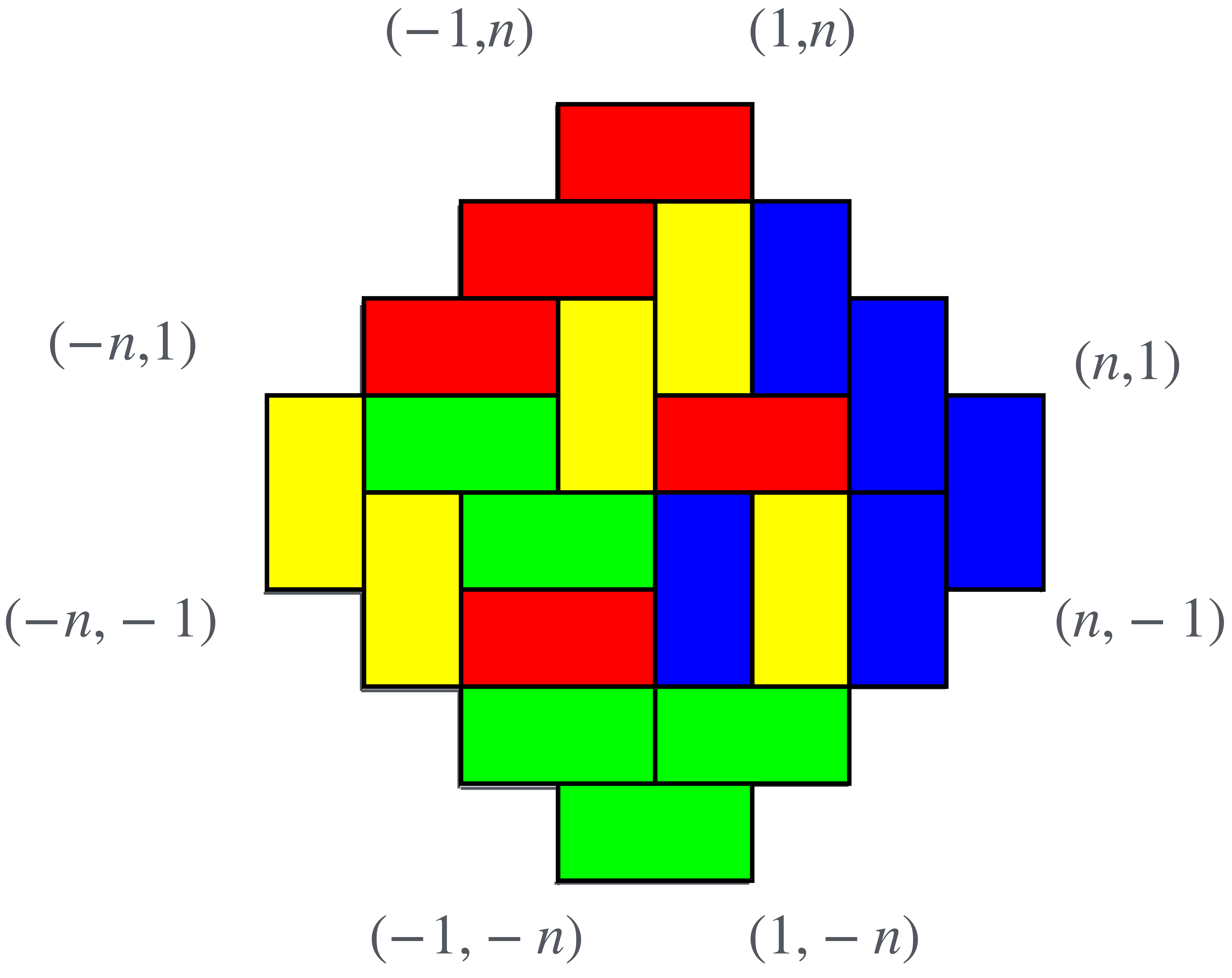
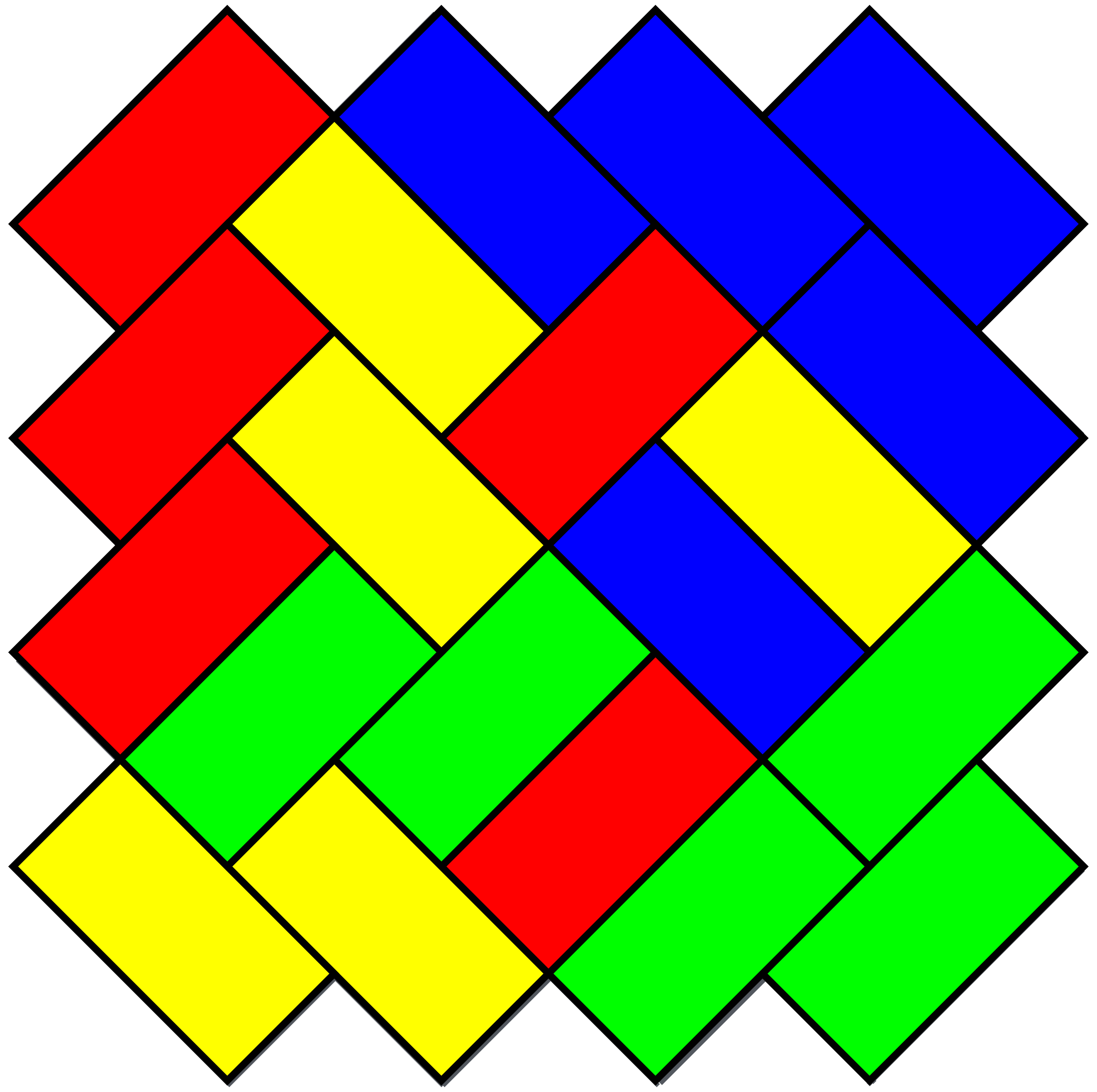


$n = 4$

Aztec diamond (rotated)

- We will often rotate the diamond by 45 degrees.
- This is not just to save space, but will also be natural.

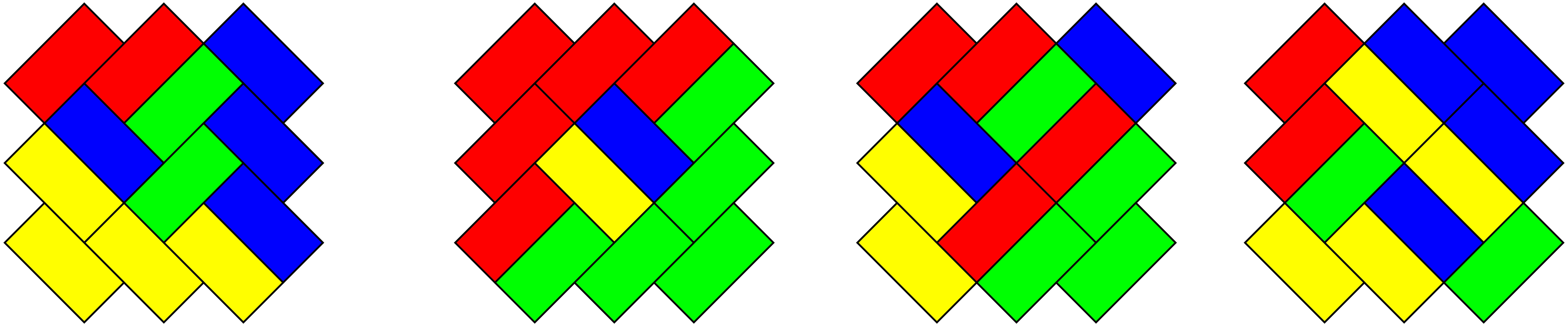


Random tilings

The Aztec diamond

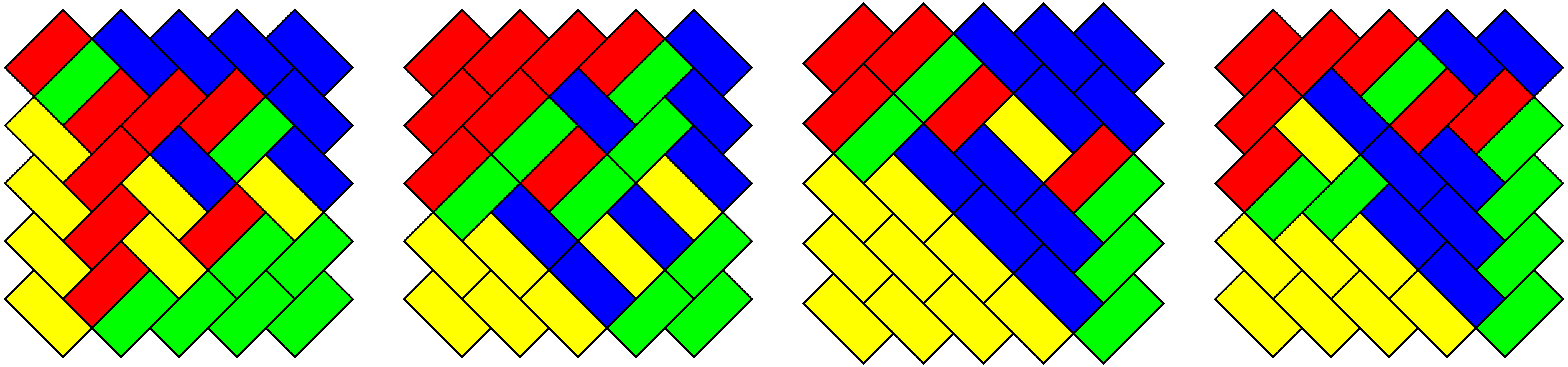
- There are many ways of tiling the Aztec diamond.
- In fact, one can prove that there are $2^{n(n+1)/2}$ ways of tiling the Aztec diamond.
- Let’s take on uniformly at random and see how a typical tiling looks like.

Start with $n = 3$



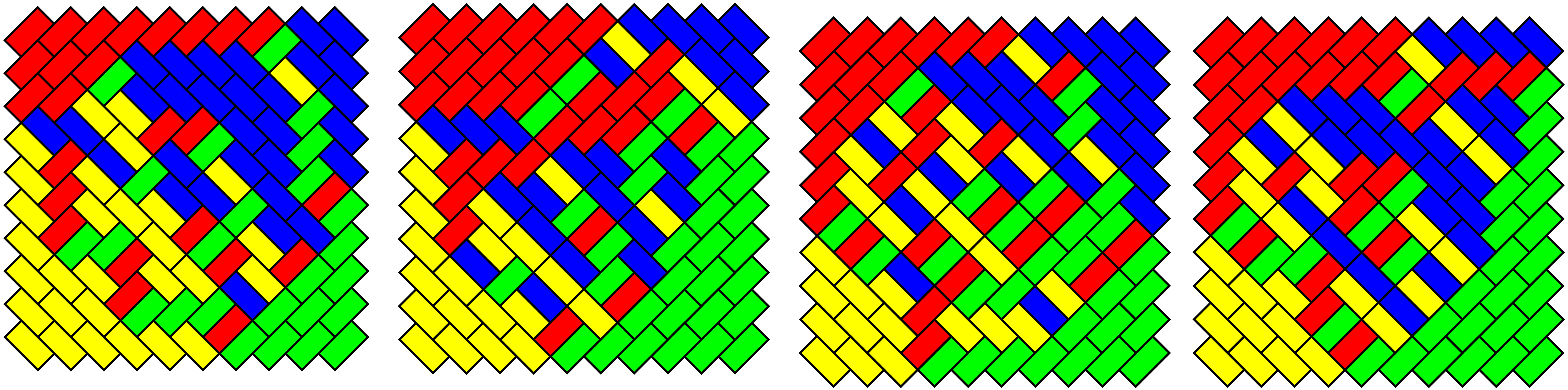
The Aztec diamond

- Then four sample for $n = 5$



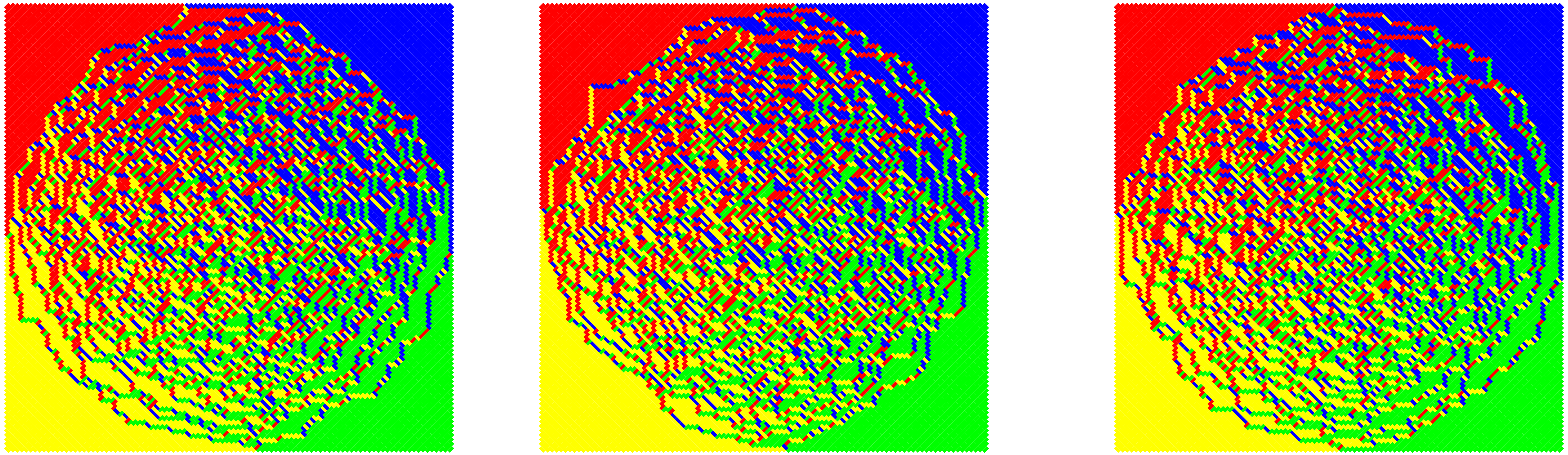
The Aztec diamond

- Then four samples with $n = 10$



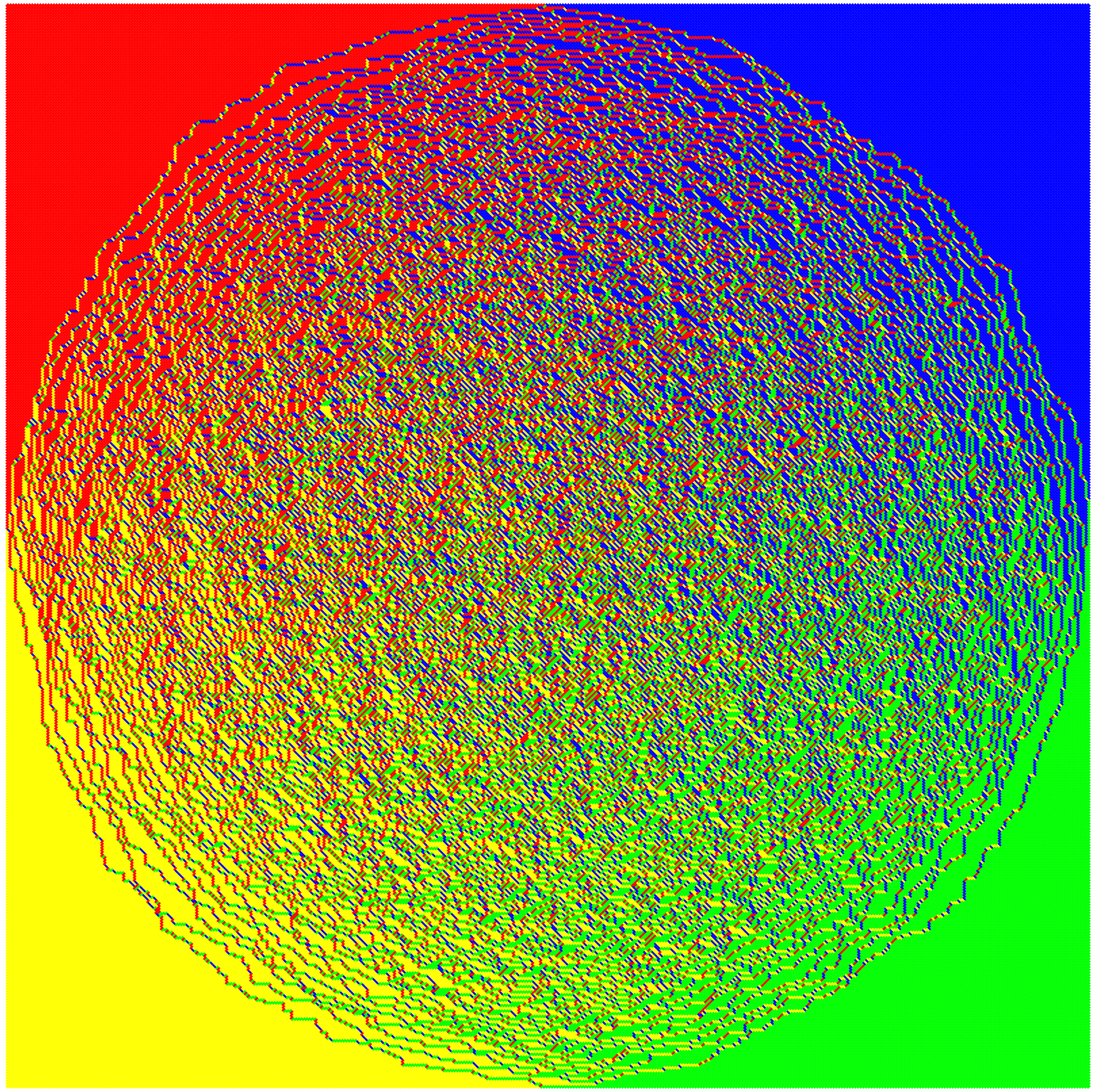
The Aztec diamond

- And $n = 100$.......



The Aztec diamond

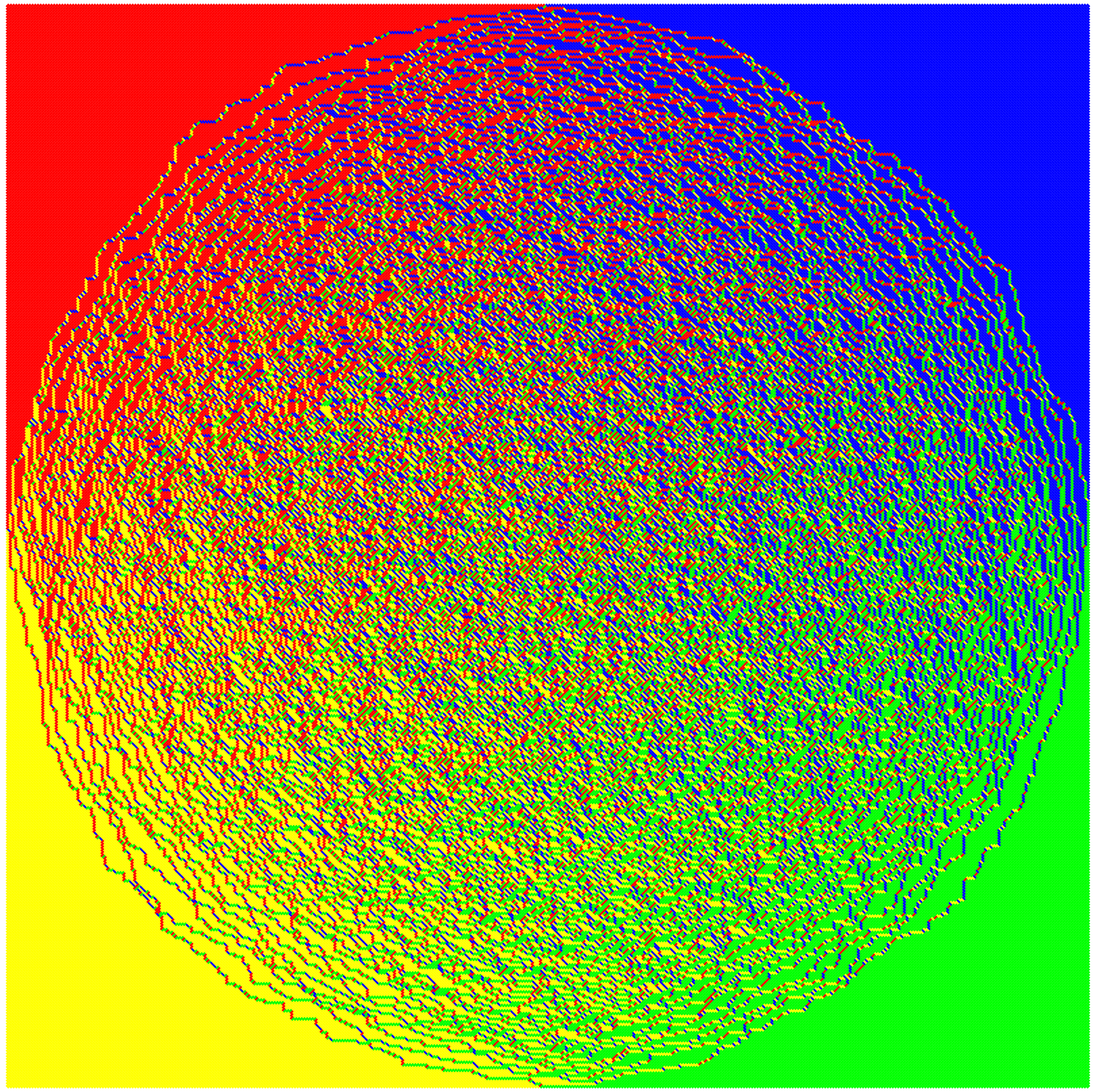
- And $n = 500$.......



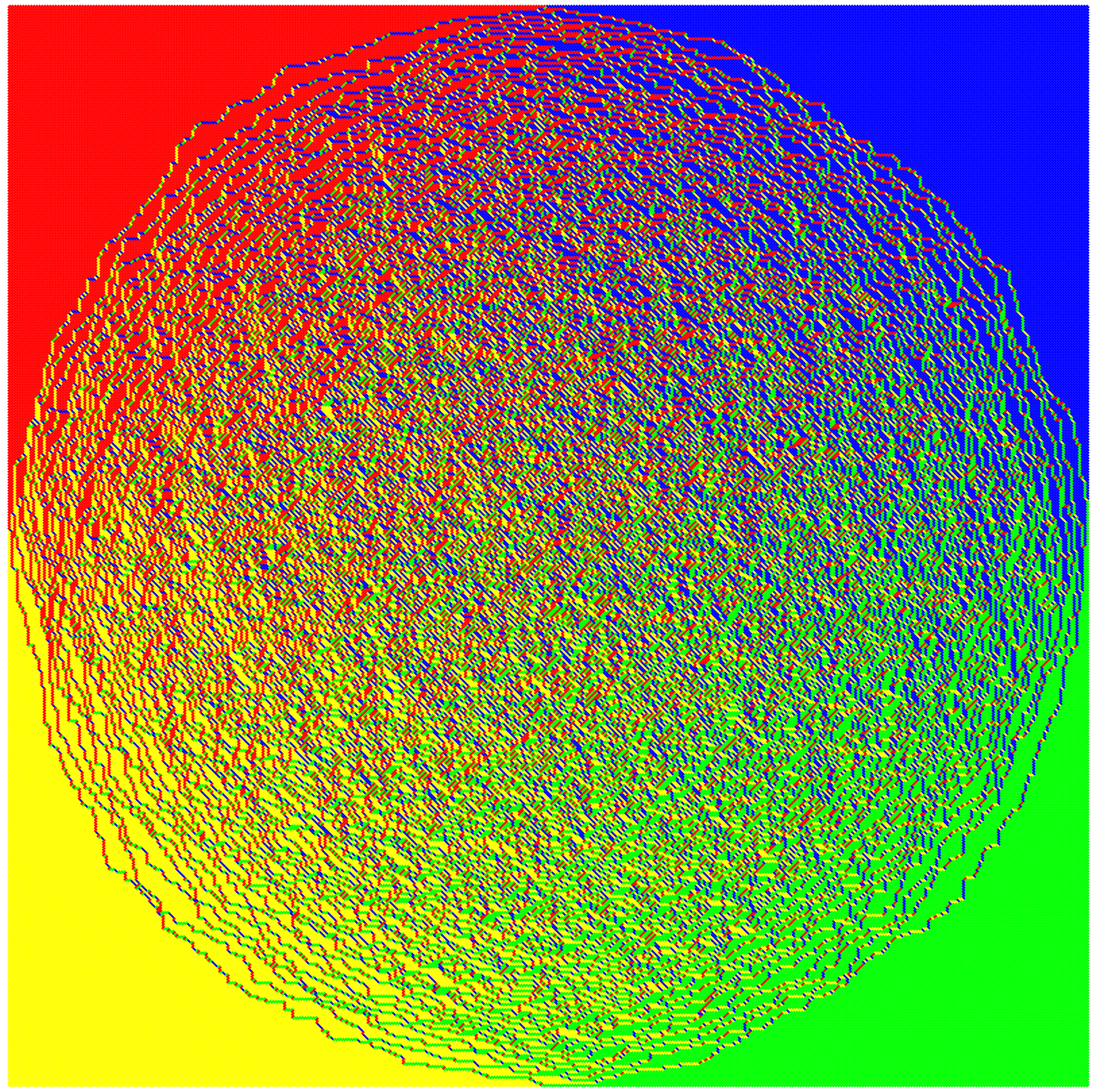
The Aztec diamond

One **main goal of this course** is to understand this picture

- We will prove that the disordered region is a disk
- We will analyze the fluctuations at the interface between frozen and disordered region
- We will compute the microscopic processes in the bulk.
- We will show that there is a limit shape, and study its fluctuations.
- Along the way, we will see that this model has beautiful connections to different types of mathematics, which make the Aztec diamond a very rich model.



The Aztec diamond



Why? Why is this happening and why should we care?

Universality

Classical probability

- The phenomena that large random systems concentrate around a deterministic limit and have universal fluctuations, goes back to de Moivre-Laplace.

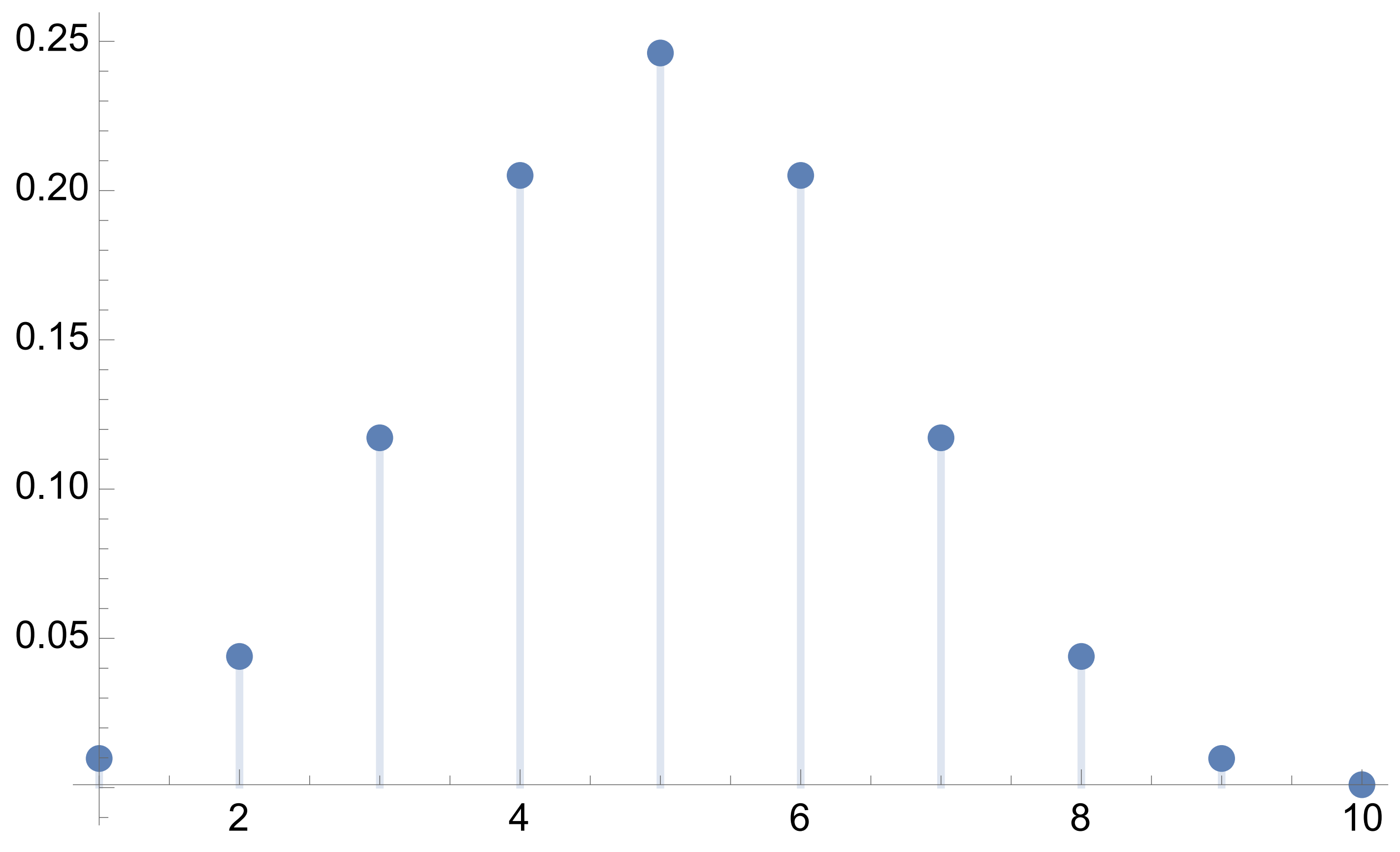
- If you flip a coin n times the number of times you get head follows a binomial distribution.

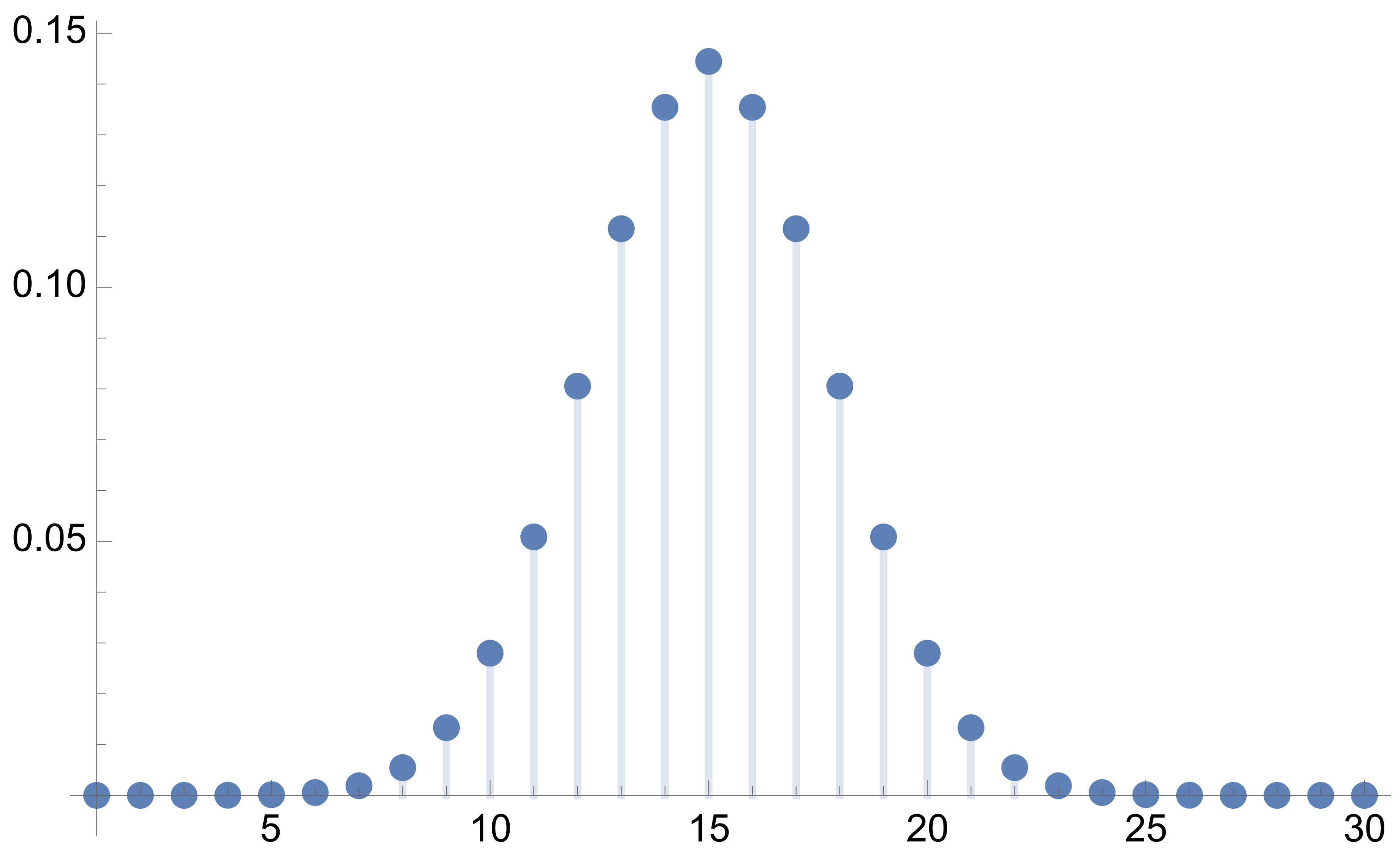
$$\mathbb{P}\left(k \text{ head after } n \text{ tosses}\right) = \binom{n}{k} p^k (1-p)^{n-k}$$

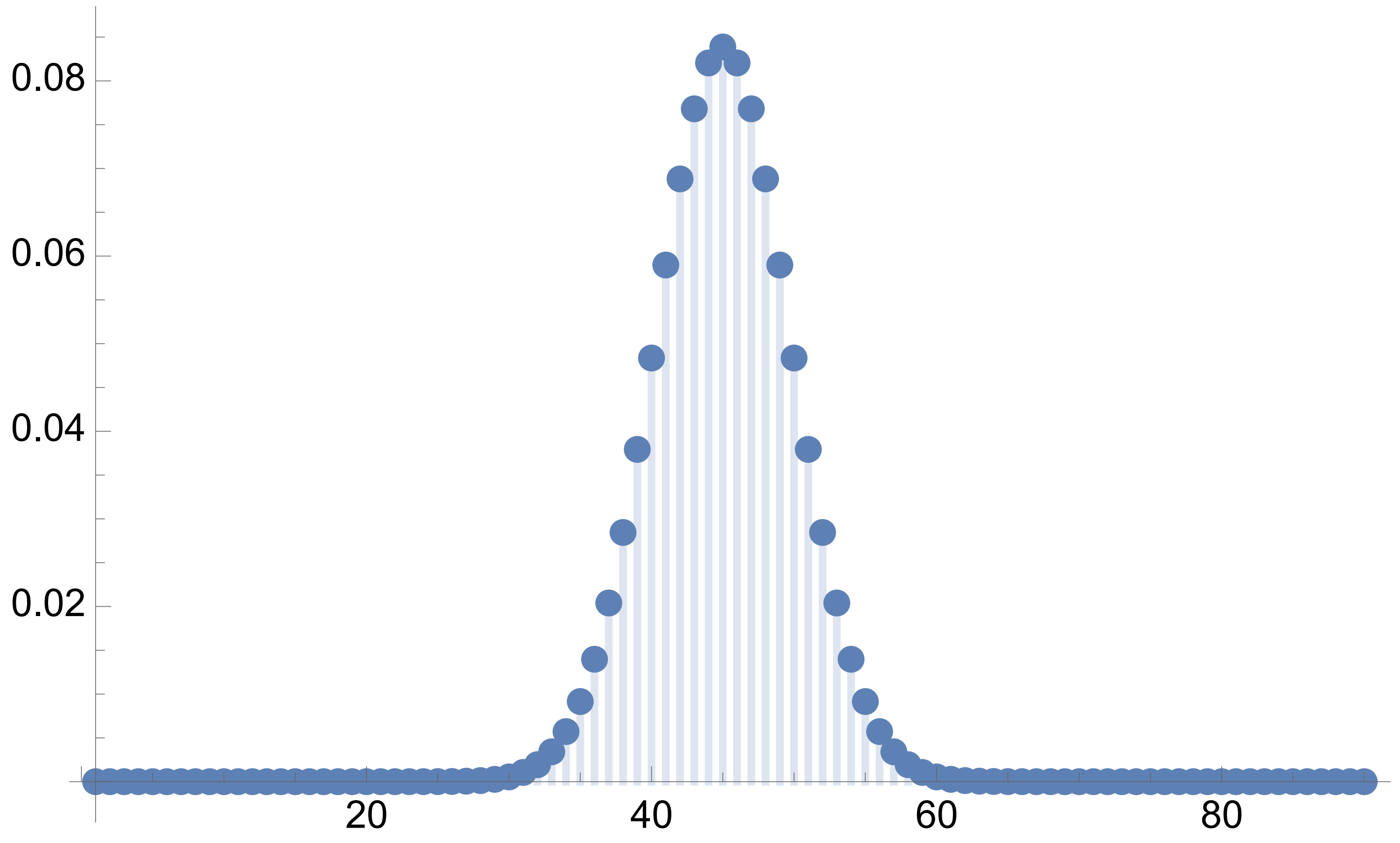
- What happens when n is large?

Classical probability

- **De Moivre- Laplace:** When n grows large the distribution will look like a Gaussian







$$\binom{n}{k} p^k(1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{(k-np)^2}{2np(1-p)}\right)$$

Classical probability

- Let $X_1, \ldots, X_n$ be independent and identical distributed random variables with mean μ and variance σ^2. Then:
  - Law of large numbers:

$$\frac{1}{n}\sum_{j=1}^{n} X_j \to \mu, \quad \text{almost surely as } n \to \infty$$

  - Central Limit Theorem:

$$\frac{\sum_{j=1}^{n} X_j - n\mu}{\sqrt{n}} \to X \sim N(0,\sigma^2), \quad \text{in distribution as } n \to \infty$$

- Independent of the precise distribution, and dont need identical distributions.....

Classical probability

- The principle that large systems concentrate around a deterministic limit and have fluctuations that give rise to universal probabilistic laws is typical.
- In strongly correlated systems new type of universal behaviors appear.

  Finding these behaviors, studying their properties and proving their universal nature is one of the key goals in modern probability and mathematical physics.

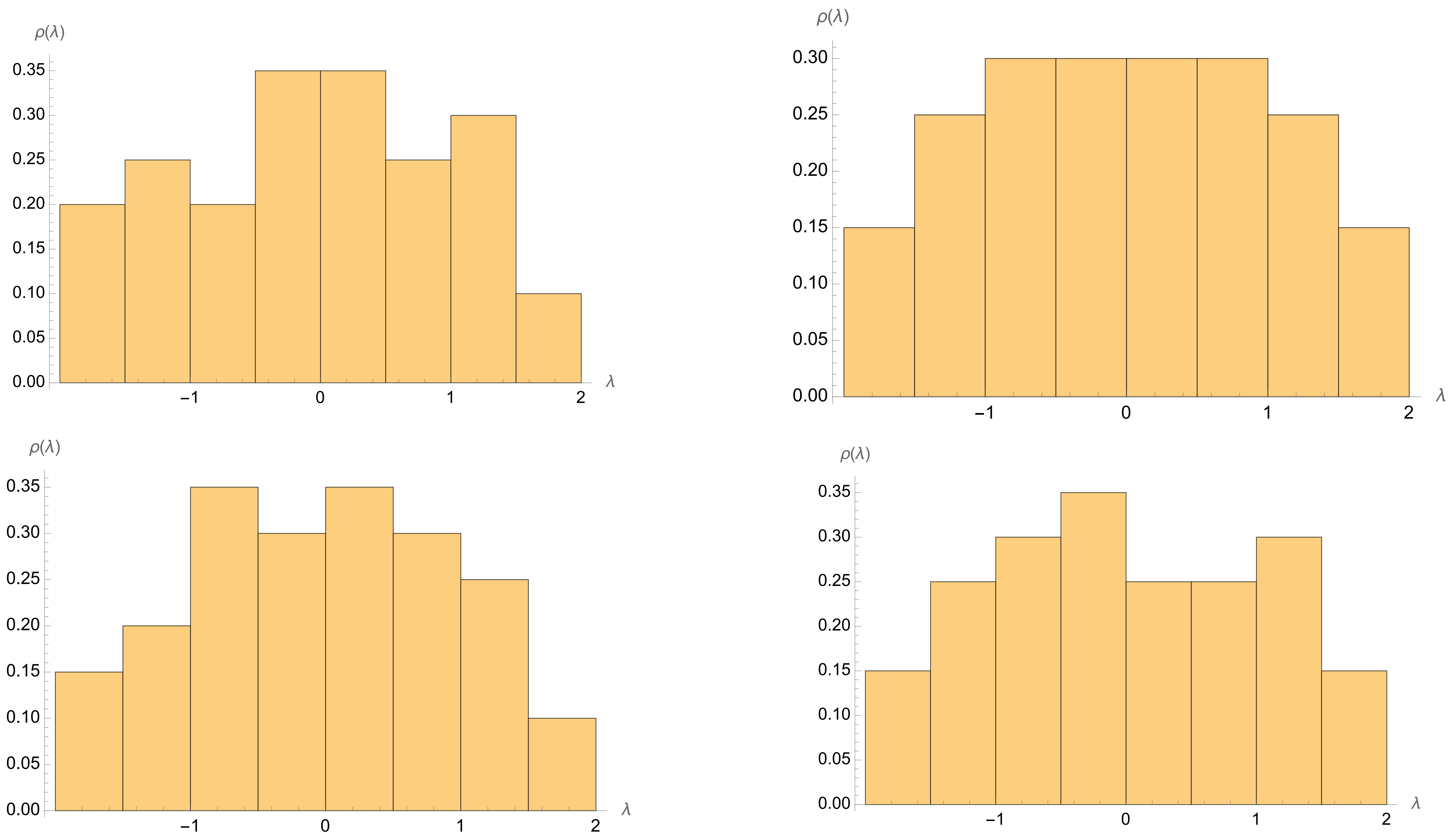
- Roughly, there are three questions:
  - Study a particular model that is sufficiently rich and generic, but also tractable.
  - Prove that the probabilistic objects that emerged, are universal.
  - Study the properties of these universal objects.

Random matrix theory

- Let $n \in \mathbb{N}$ and consider a random $n \times n$ Hermitian matrix M defined by
  - $M_{ij} \sim N_{\mathbb{C}}(0,1/n)$ for $1 \leq i < j \leq n$
  - $M_{jj} \sim N_{\mathbb{R}}(0,1/n)$
  - $M_{ji} = \overline{M}_{ij}$ for $1 \leq i < j \leq n$
- Since M is Hermitian it has n eigenvalues, $\lambda_1, \ldots, \lambda_n$
- These eigenvalues will not be independent, but strongly correlated.

Random matrix theory

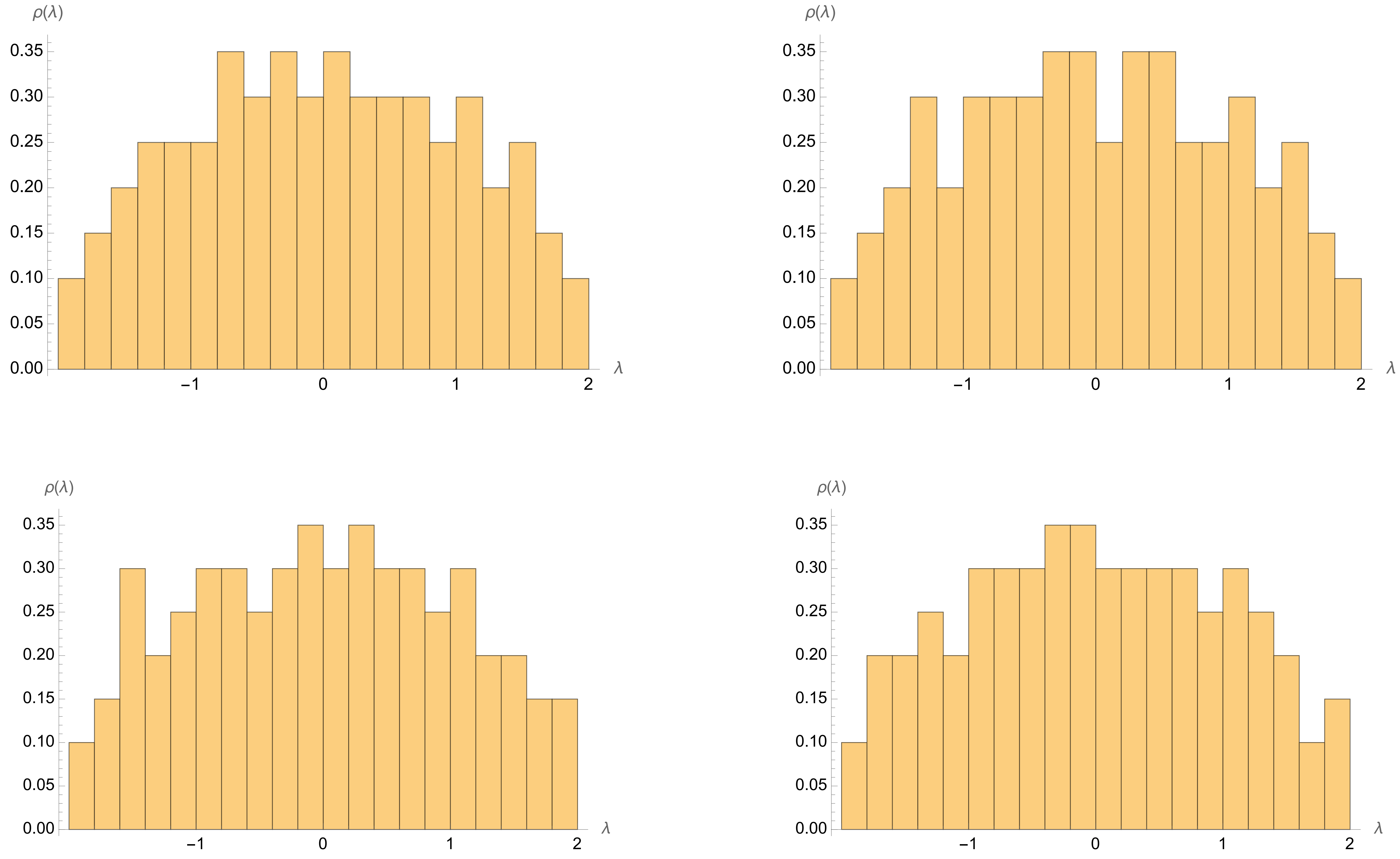
- We draw a histogram that counts the number of eigenvalues in intervals.



$n = 40$

Random matrix theory

- We draw a histogram that counts the number of eigenvalues in intervals.



$n = 100$

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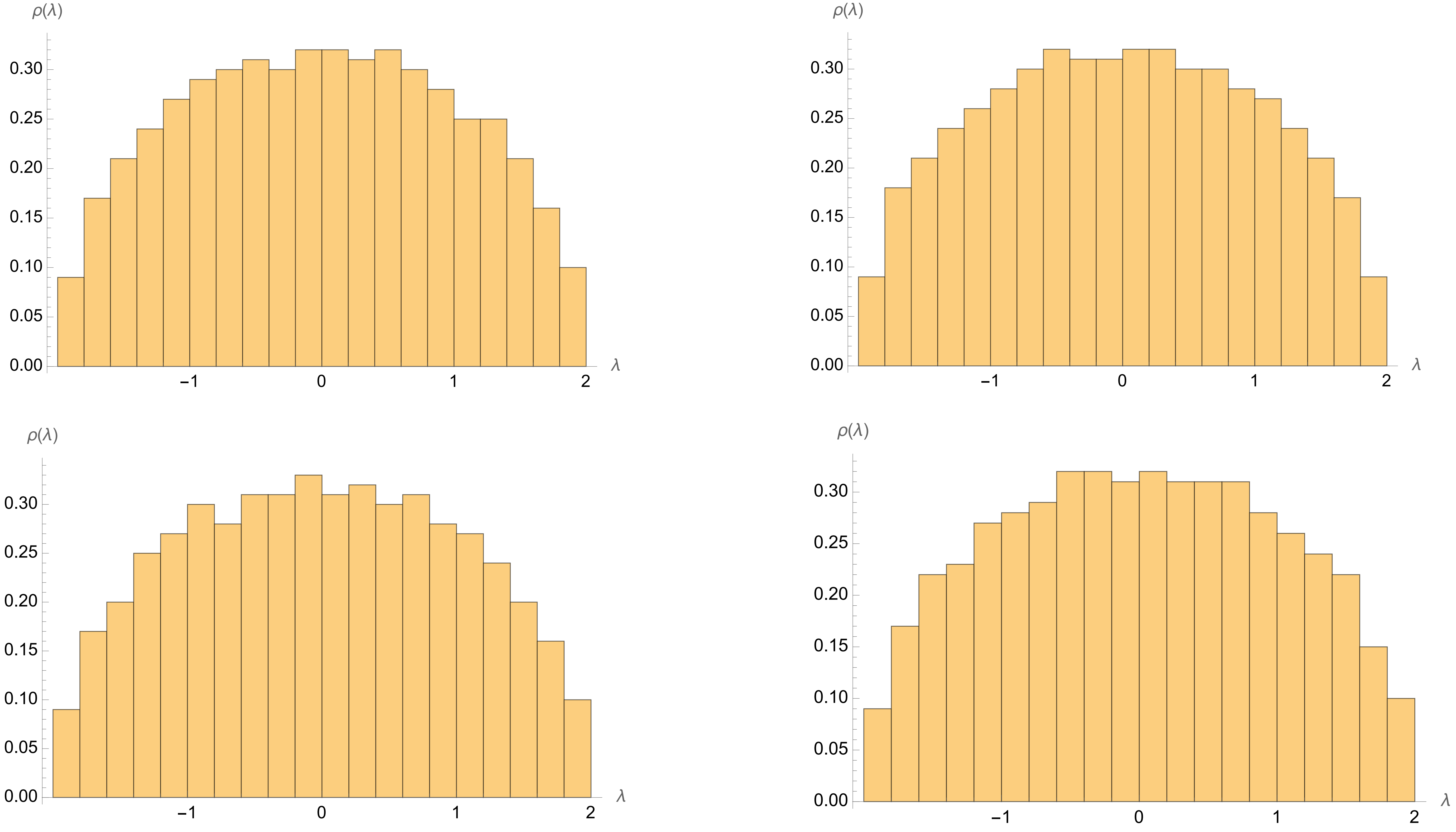

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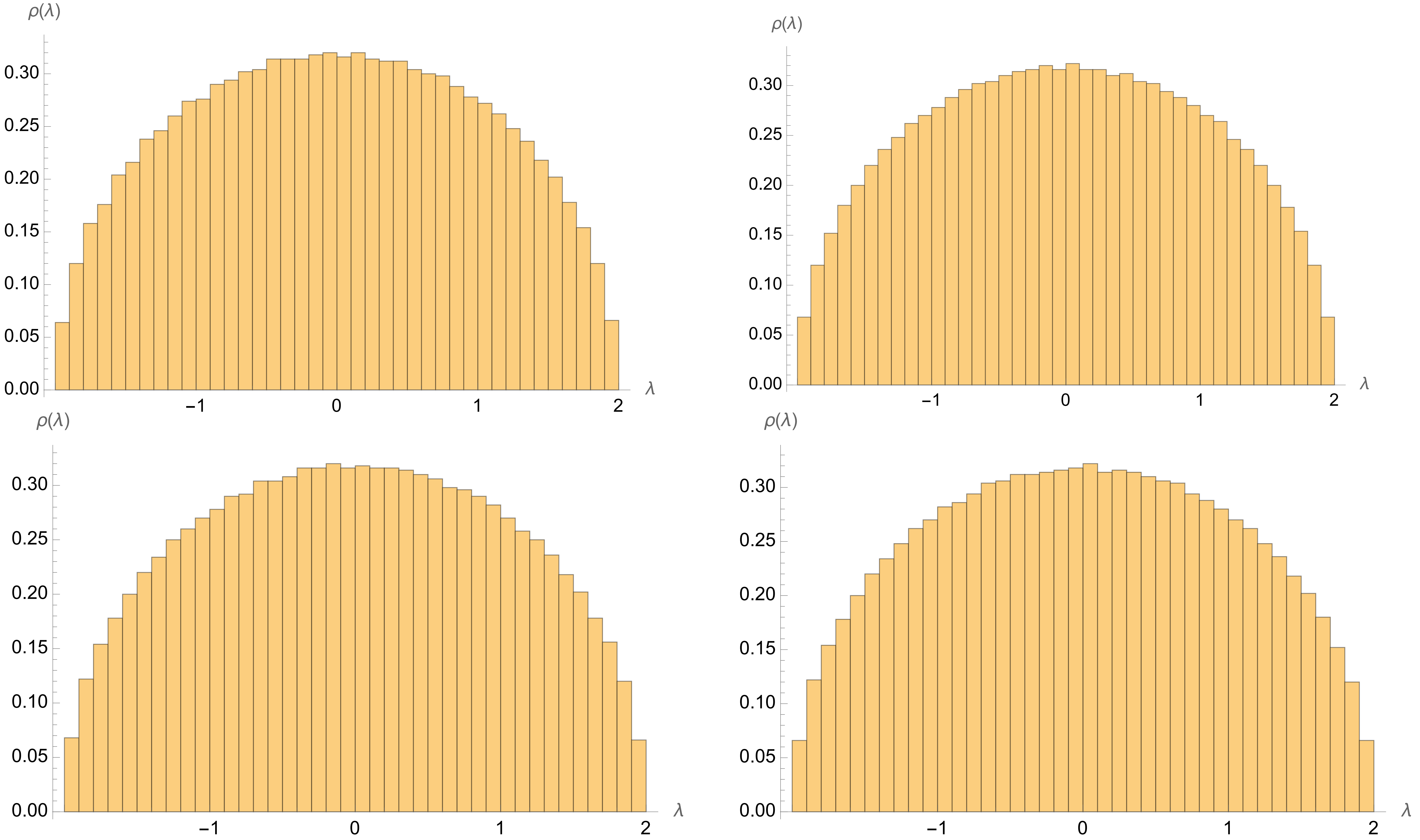


$n = 5000$

Random matrix theory

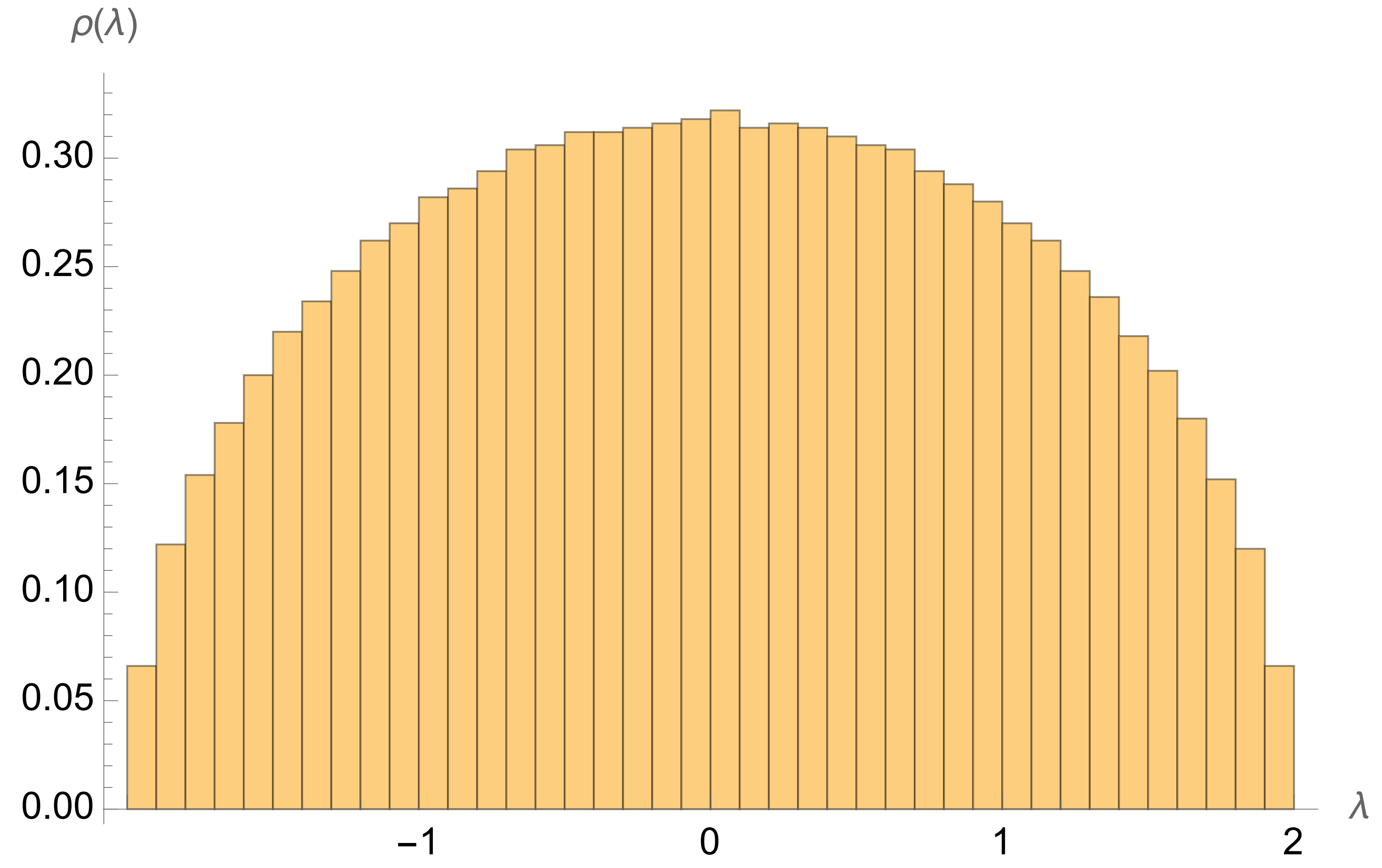
- Law of large numbers:

Wigners semi-circle law:

For a subset $E \subset \mathbb{R}$ the fraction of points in E is, in the limit, given by the semi-circle law:

$$\frac{\#\{\lambda_j \in E\}}{n} \to \frac{1}{2\pi} \int_E \sqrt{4 - x^2} dx$$

almost surely as $n \to \infty$



Random matrix theory

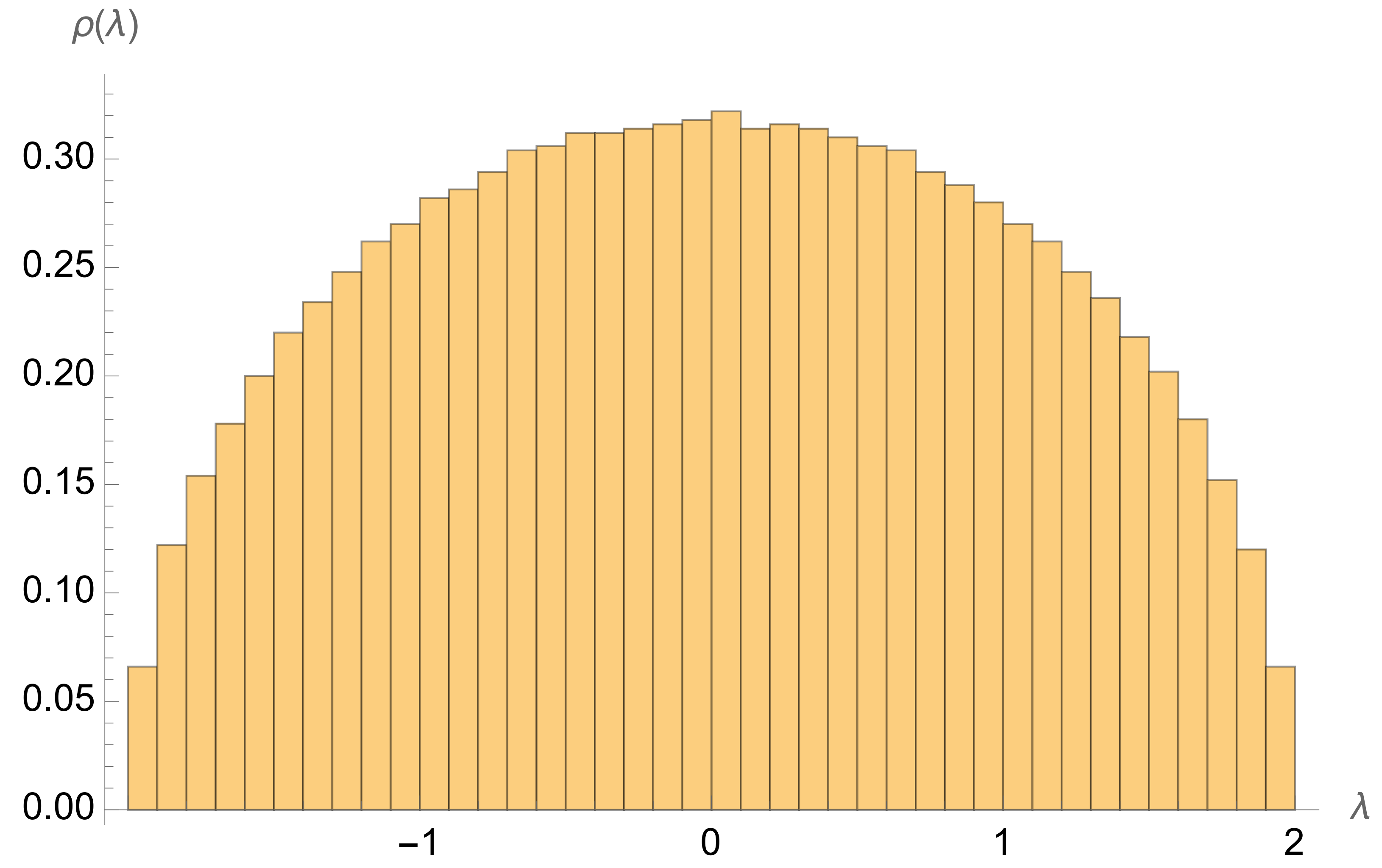
- One can explain the semi-circle law as follows:

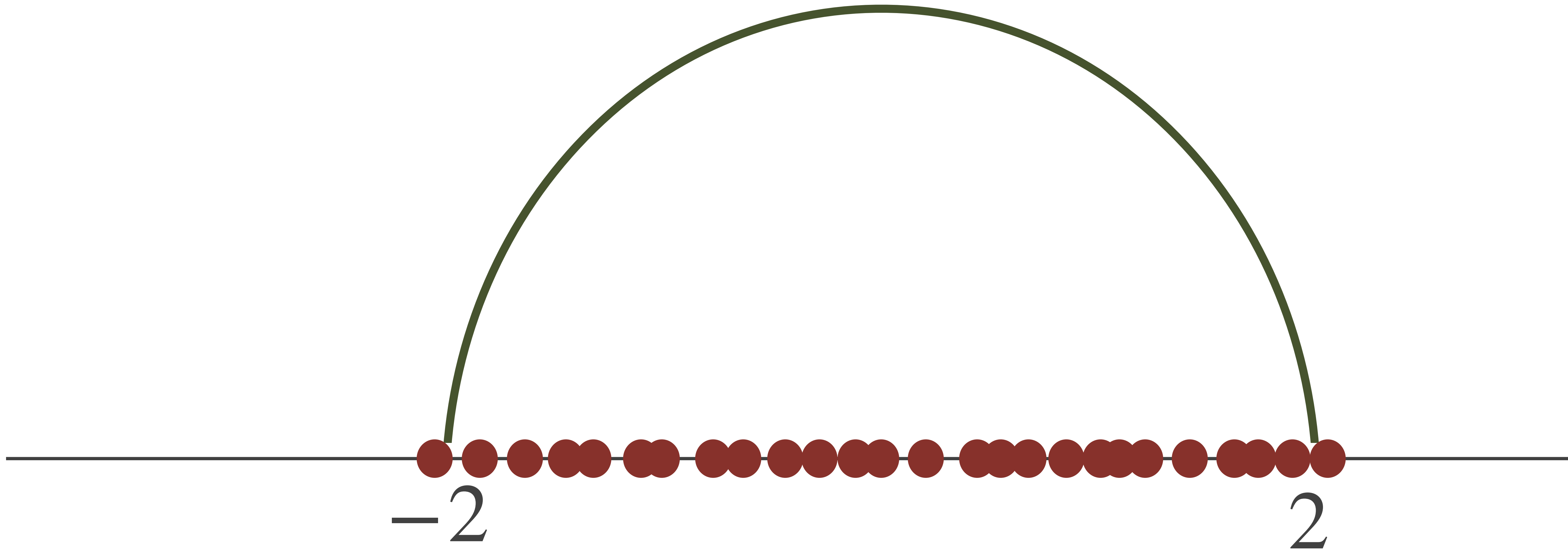
- First one proves that the eigenvalues have the jpdf

$$\sim \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2 e^{-\frac{n}{2}\sum_{j=1}^{n} \lambda_j^2} d\lambda_1 \cdots d\lambda_n$$

Repulsion   Confinement

- The semi-circle law is the equilbrium between these terms.





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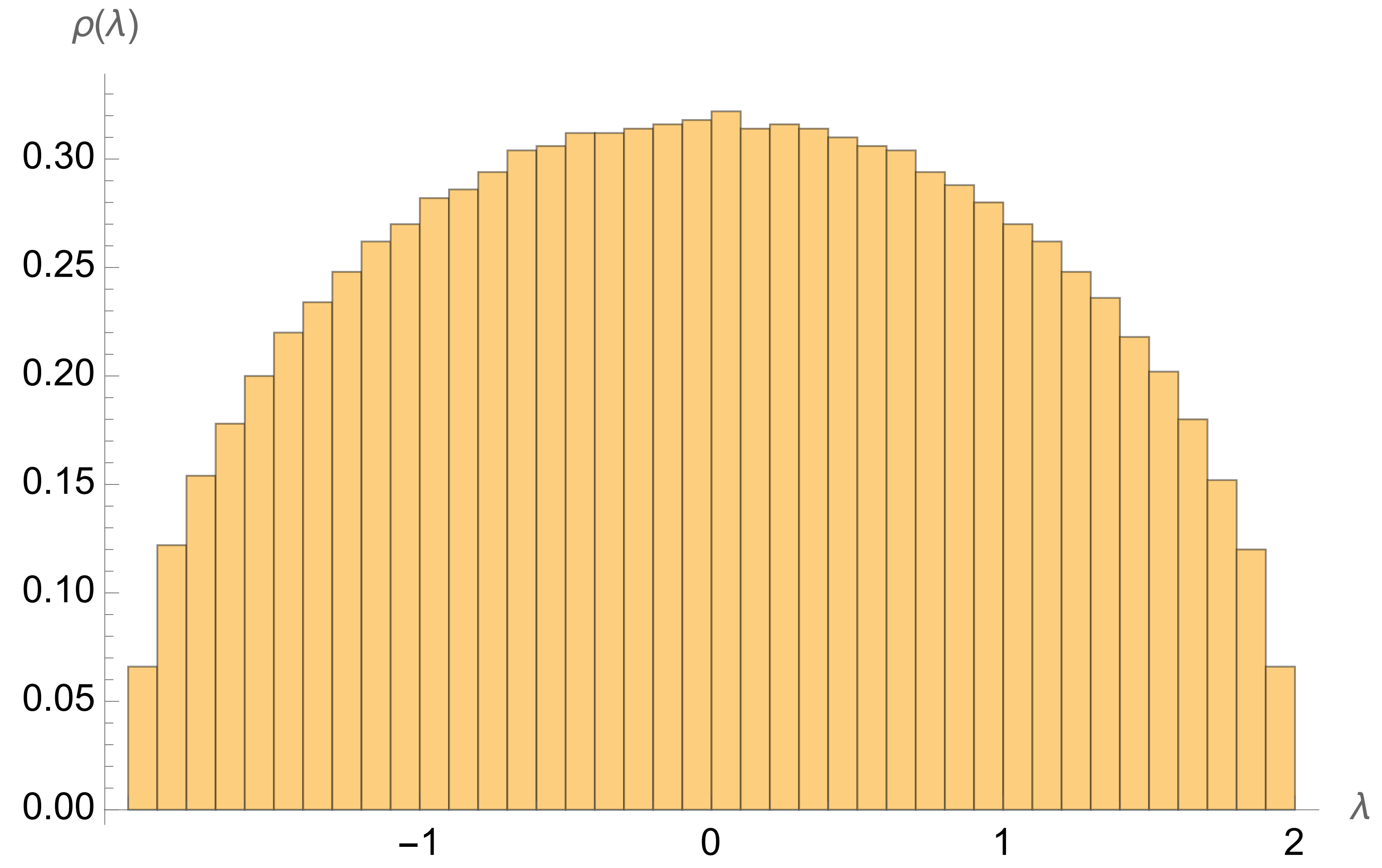
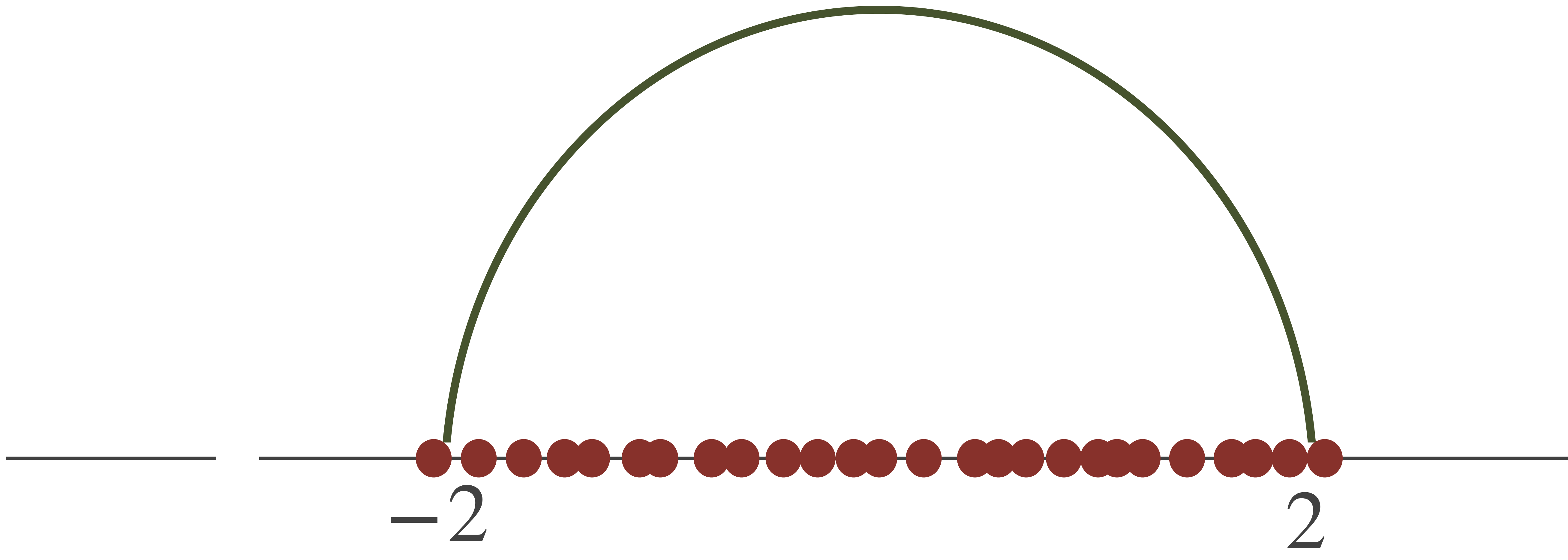
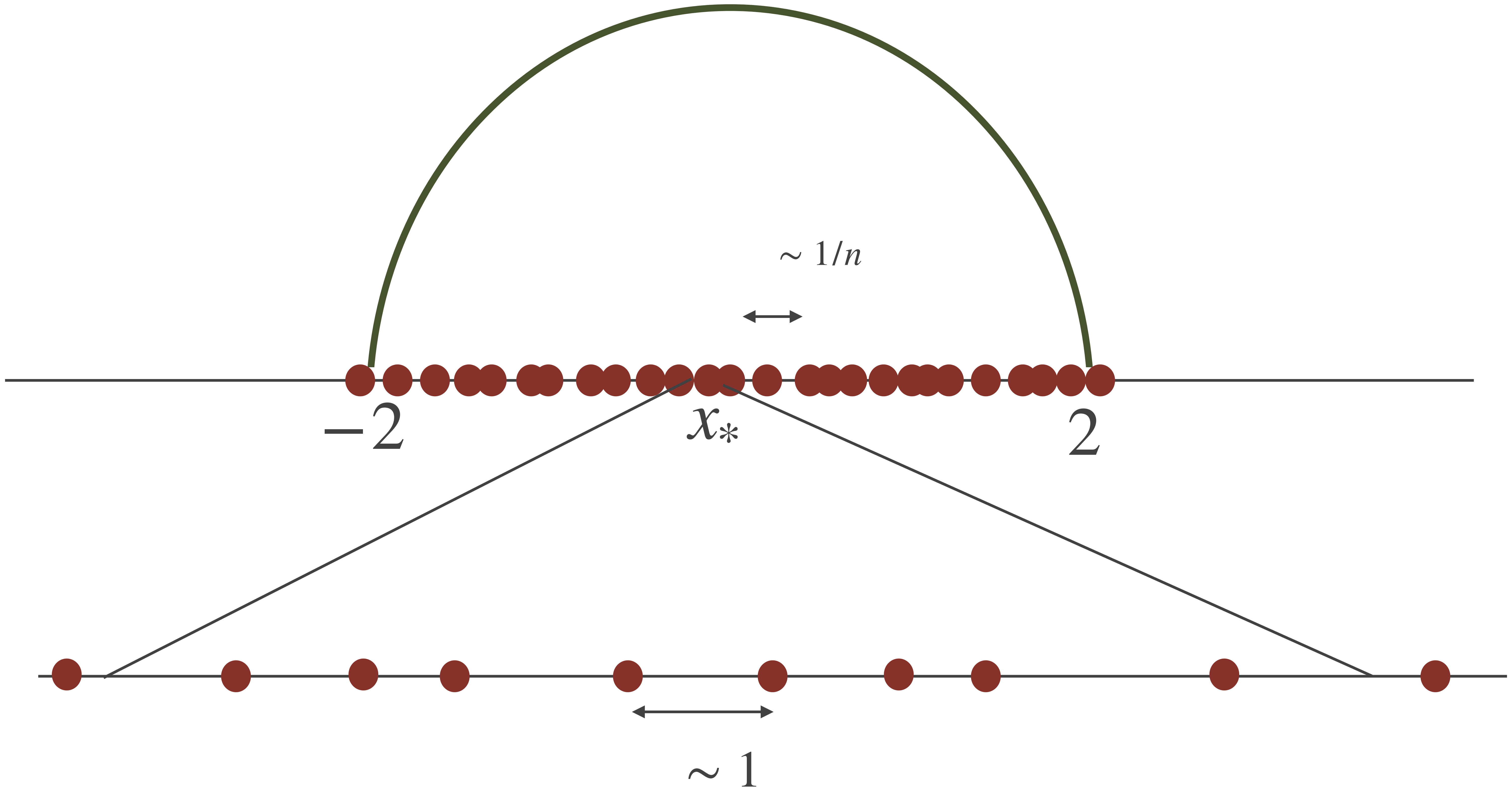




Energy levels of heavy nuclei

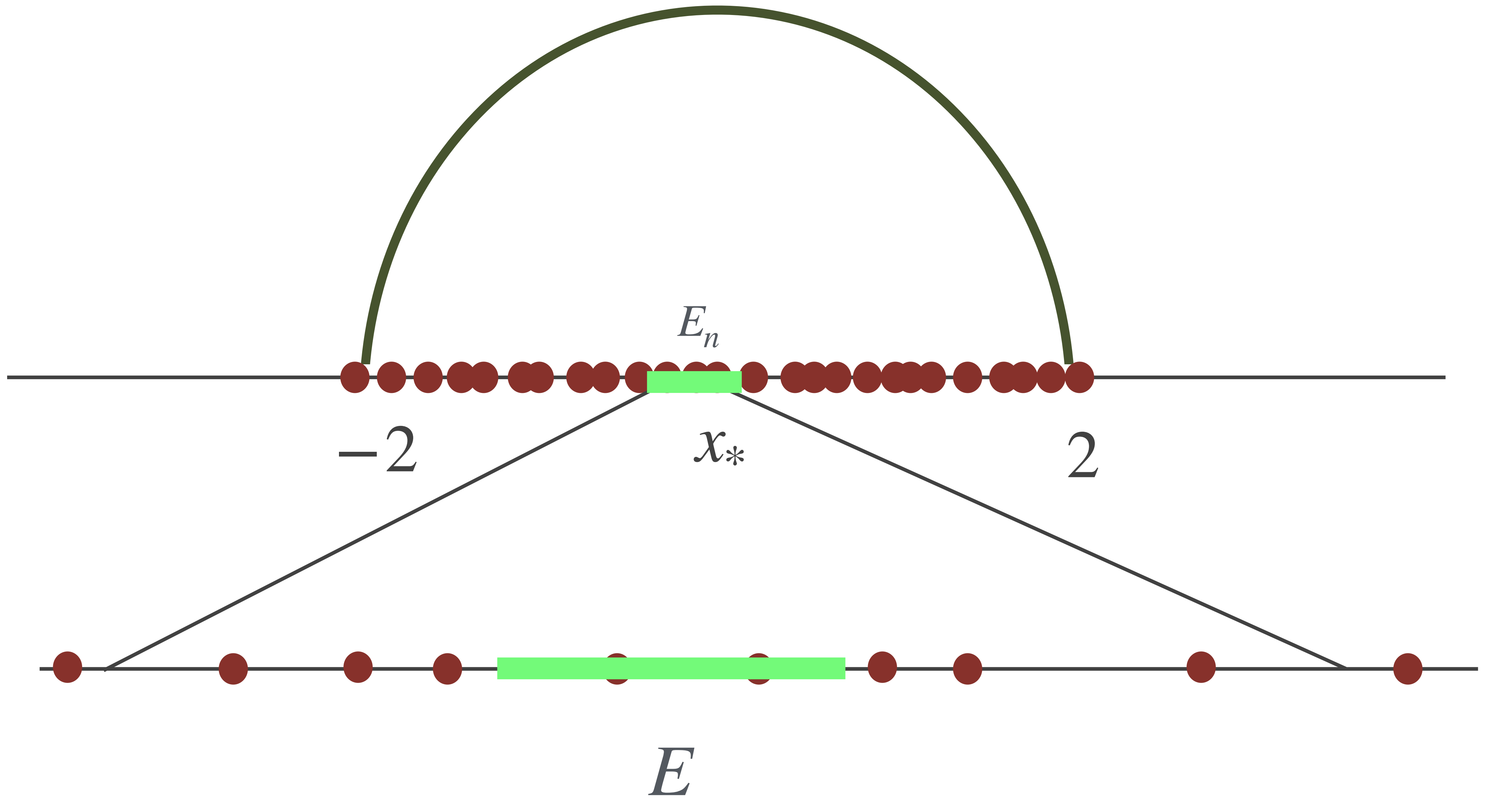
- Scaling at a bulk point $x_* \in (-2,2)$



Zeros Riemann-Zeta-function

- For $E \subset \mathbb{R}$ define the rescaled interval:

$$E_n = x^* + \frac{2\pi n}{\sqrt{4 - x_*^2}} E$$



- Then we have the asymptotic behavior of gap probabilities:

$$\mathbb{P}\left(\text{no eigenvalues in } E_n\right) \to 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!} \int_E \cdots \int_E \det\left(K_{sine}(x_i, x_j)\right)_{i,j=1}^{k} dx_1 \cdots dx_n$$

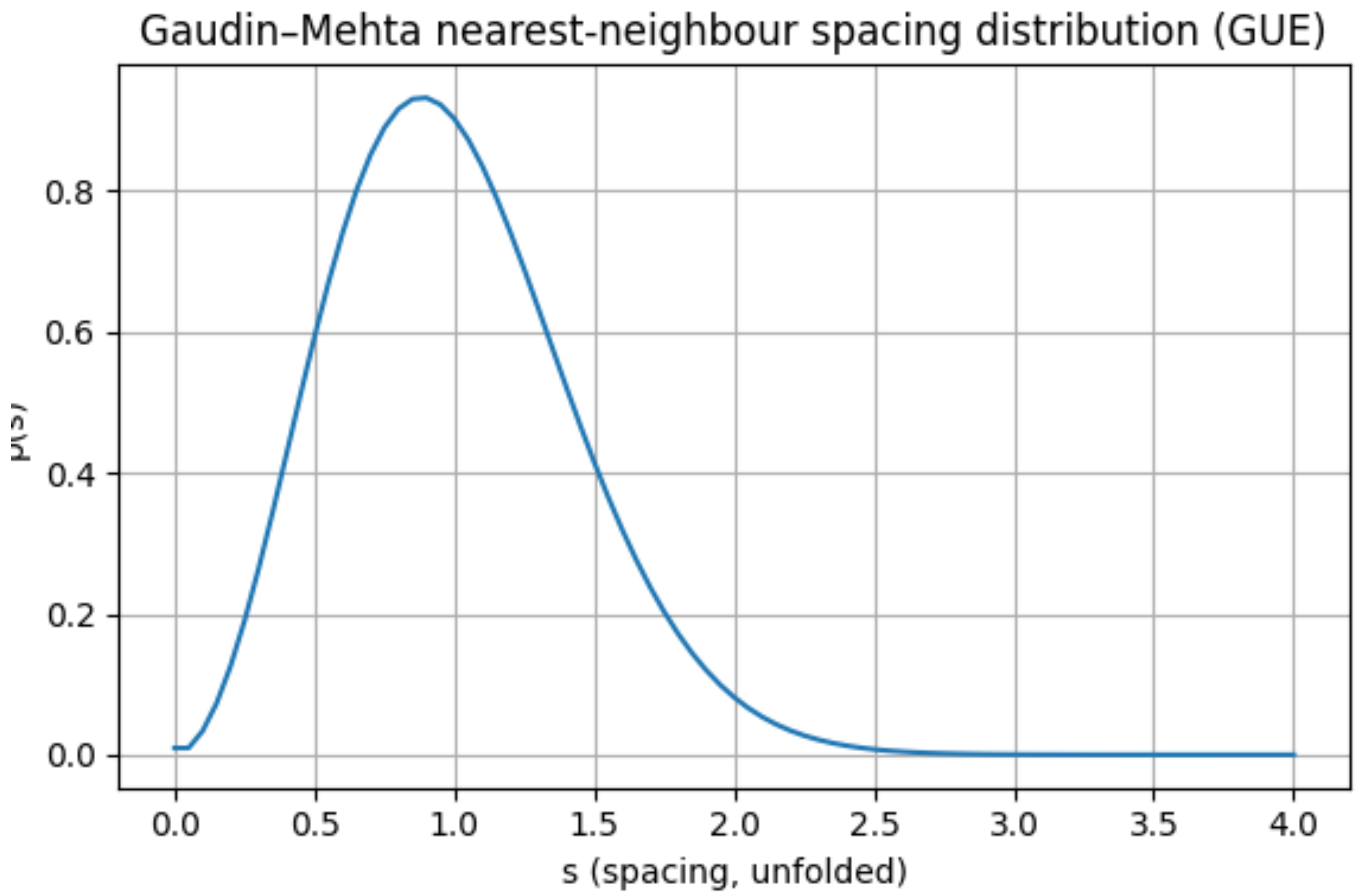
with $K_{sine}(x, y) = \dfrac{\sin \pi(x - y)}{\pi(x - y)}$

Gaudin-Mehta distribution

- The space give rise to a new law:

$$F(s) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!} \int_0^s \cdots \int_0^s \det\left(K_{sine}(x_i, x_j)\right)_{i,j=1}^{k} dx_1 \cdots dx_n$$

$$K_{sine}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}$$



- This law is also observed in
  - Energy levels of heavy nuclei
  - Zeros of the Riemann-Zeta function
  - Miscalleneous: Parking of cars……

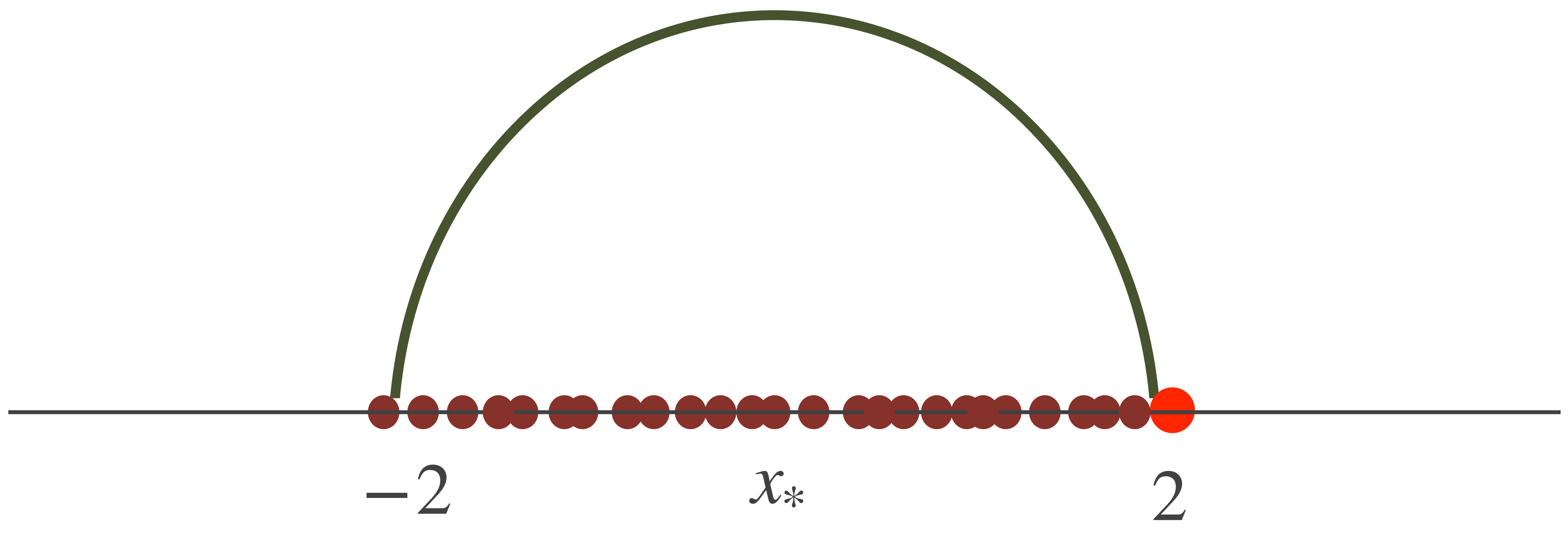
Tracy-Widom distribution

- Another very important distribution is the Tracy-Widom distribution for the largest eigenvalues.

- Let λ_{max} be the maximal eigenvalue. Then it it stick to the interval and fluctuates around the endpoint at order $n^{-2/3}$:

$$\mathbb{P}\left(\lambda_{max} < 2 + \frac{s}{n^{2/3}}\right) \to F_2(s)$$

where $F_2(s) = \ldots.$



Tracy-Widom distribution

- The Tracy-Widom distribution is another universal law that

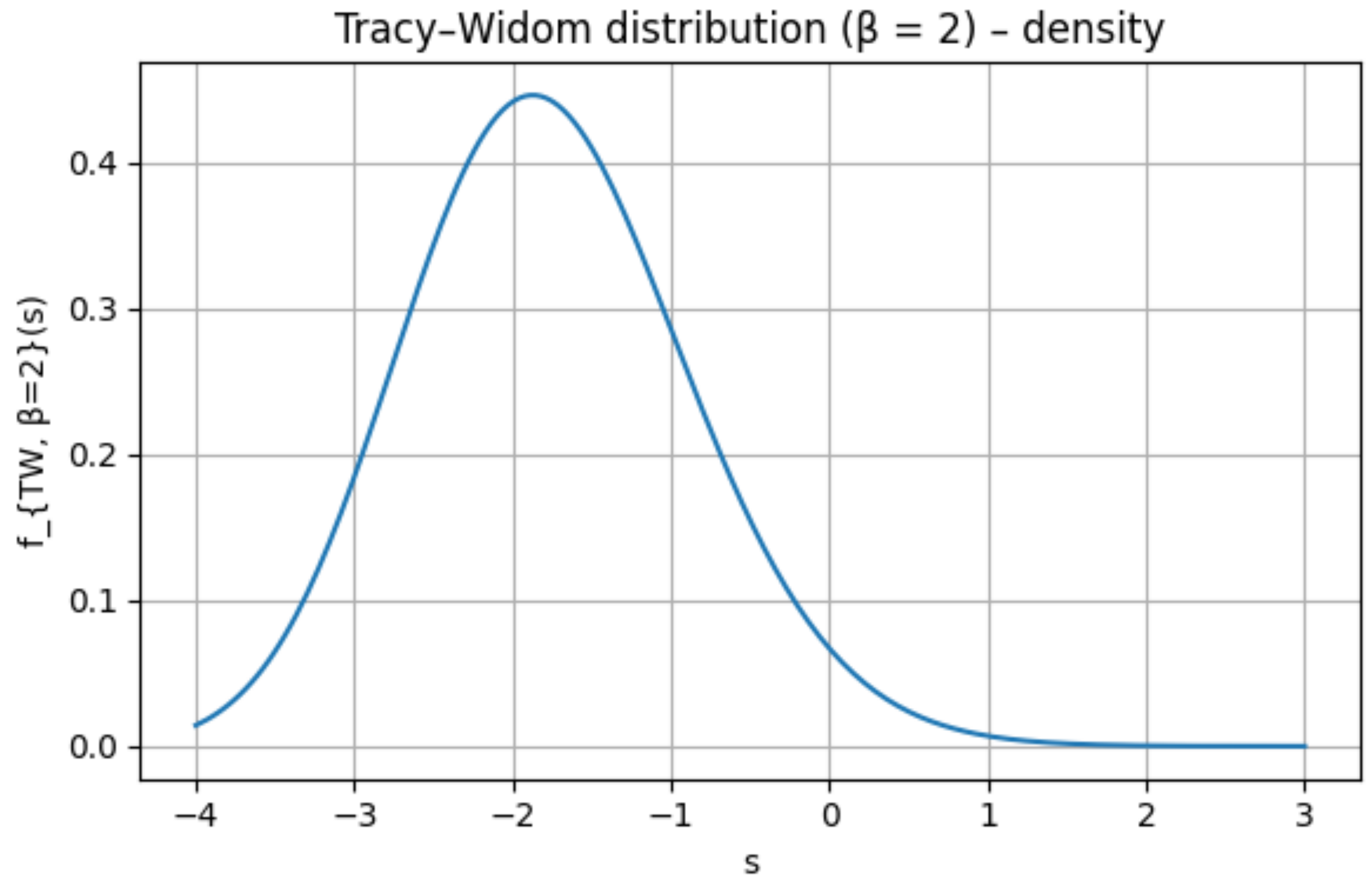
$$F_2(s) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!} \int_s^{\infty} \cdots \int_s^{\infty} \det\left(K_{Airy}(x_i, x_j)\right)_{i,j=1}^{k} dx_1 \cdots dx_n$$

where

$$K_{Airy}(x, y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y}$$

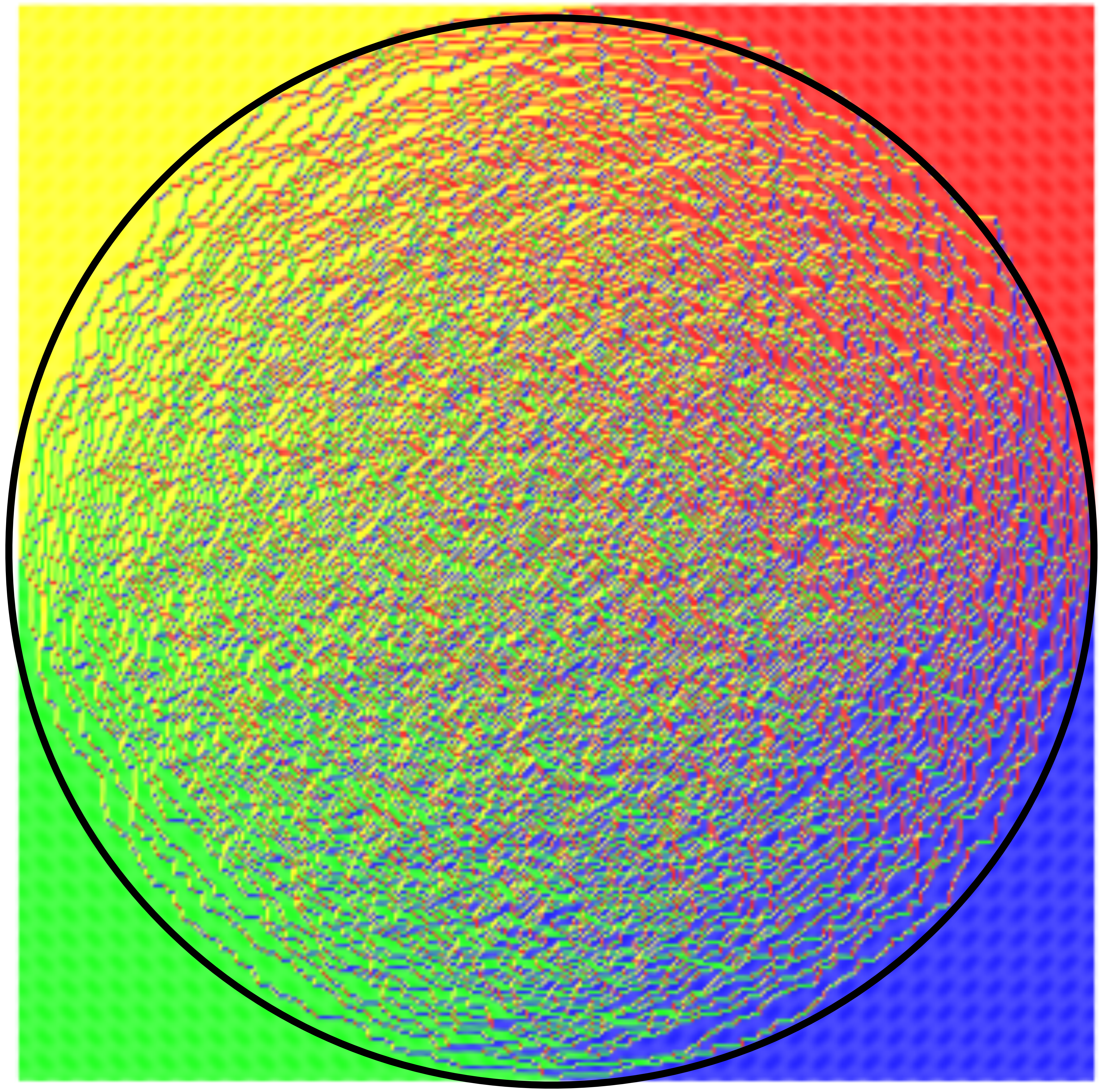
and $Ai(x)$ is the Airy function, ie the unique solution to

$$\begin{cases} Ai''(x) = xAi(x) \\ Ai(x) \to 0 \qquad x \to \infty \end{cases}$$

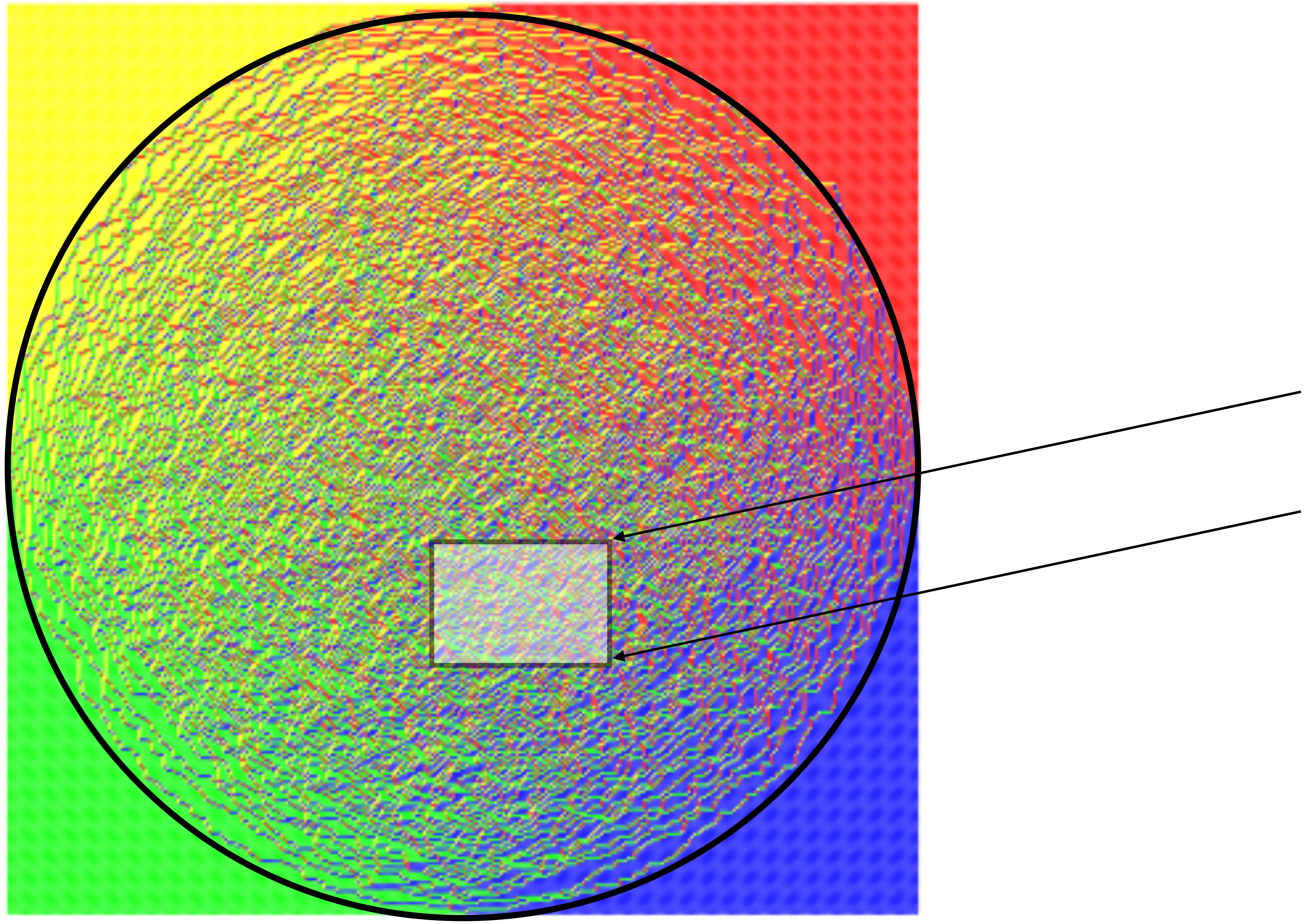


Back to tilings….

Large Aztec diamond

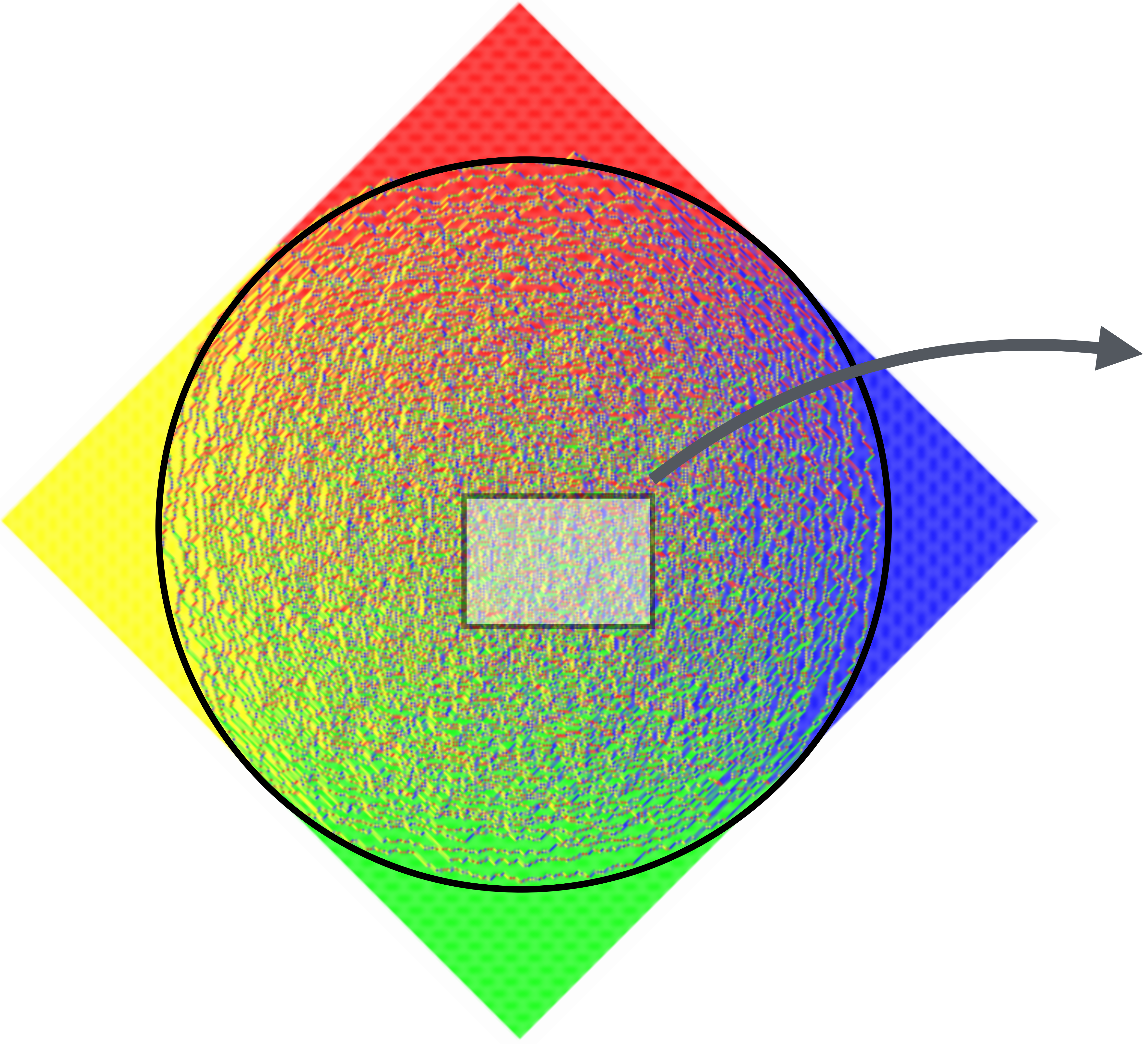


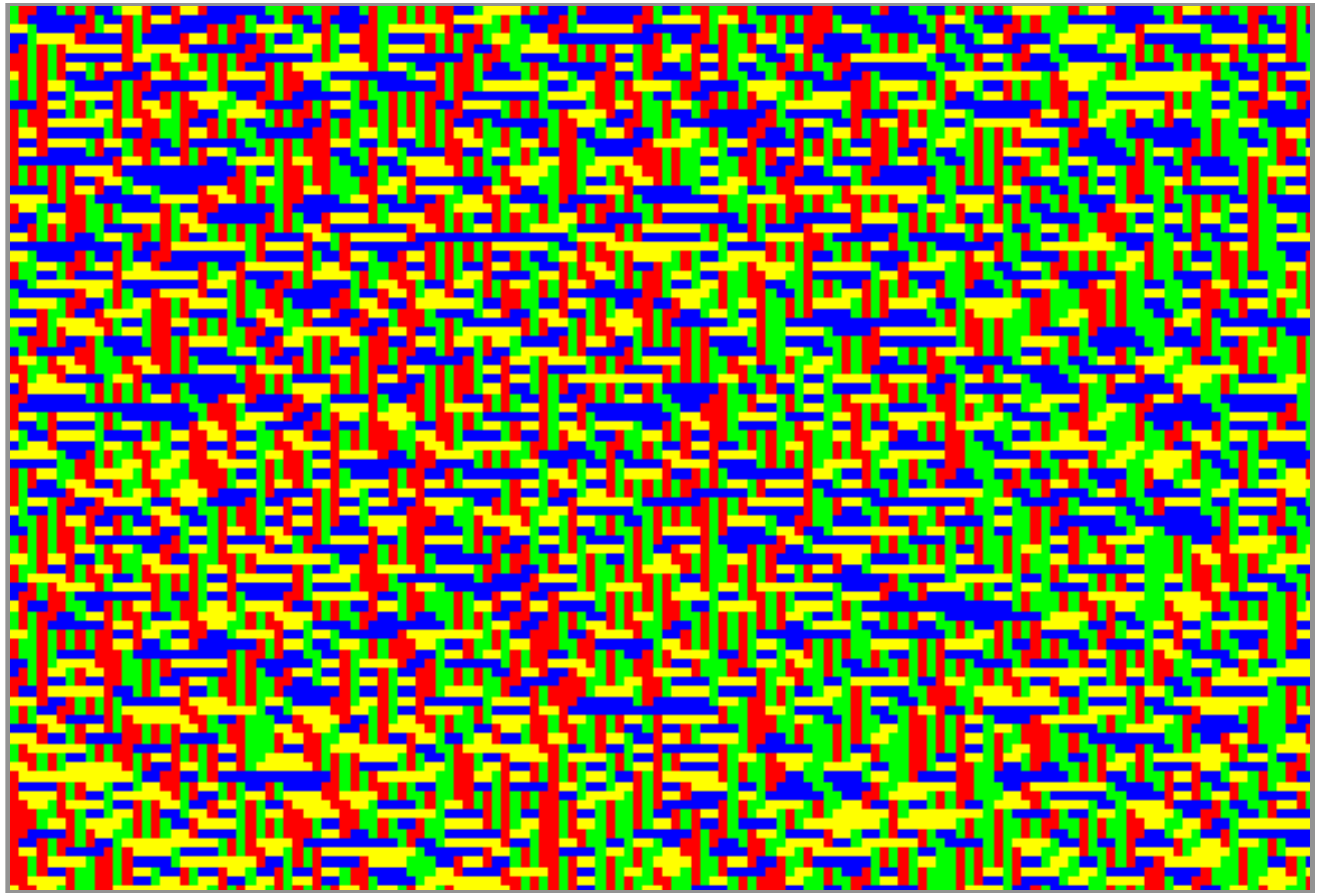
Bulk limits



Lets zoom in so that tiles are of order ~ 1

Bulk limits

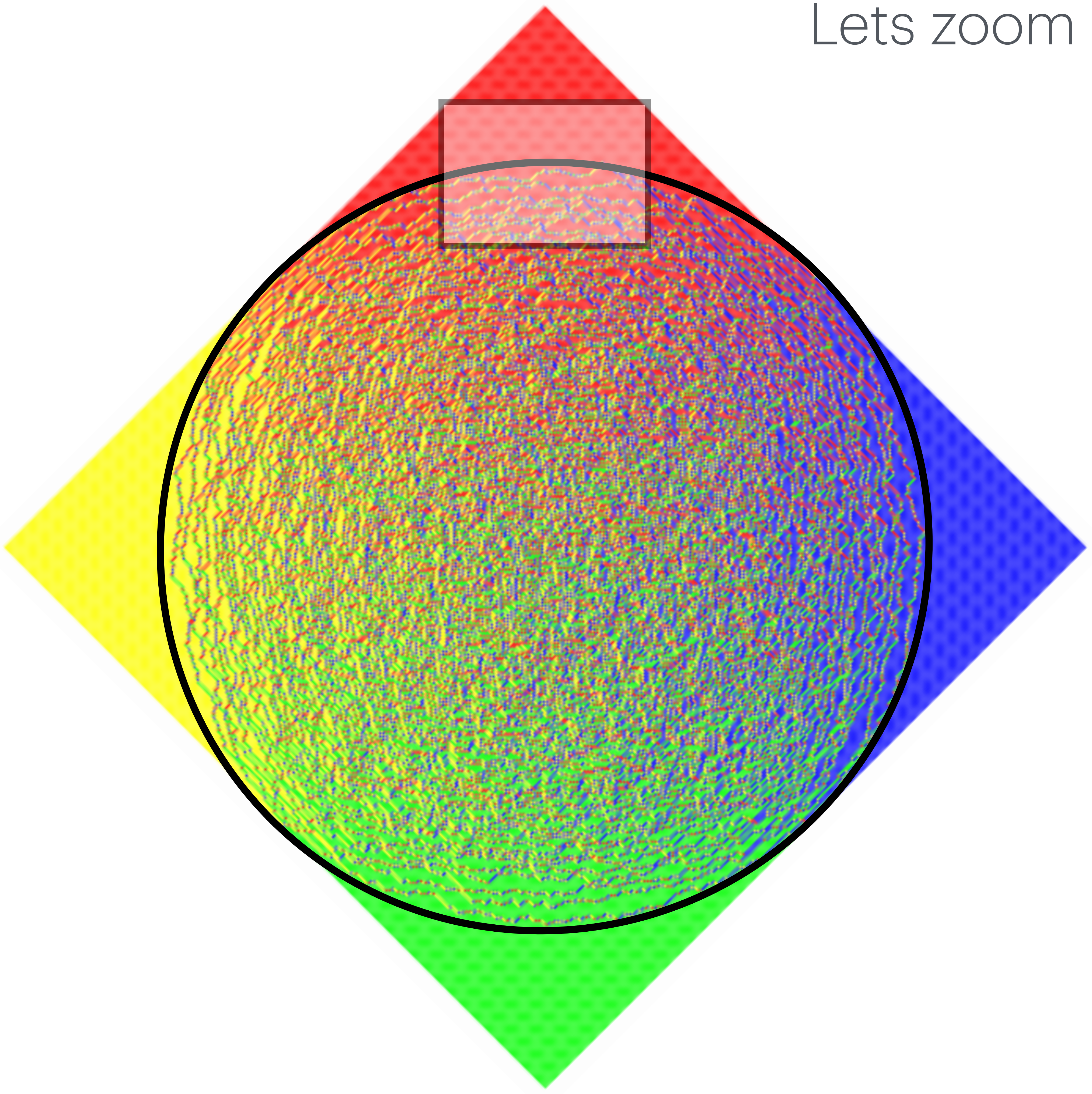


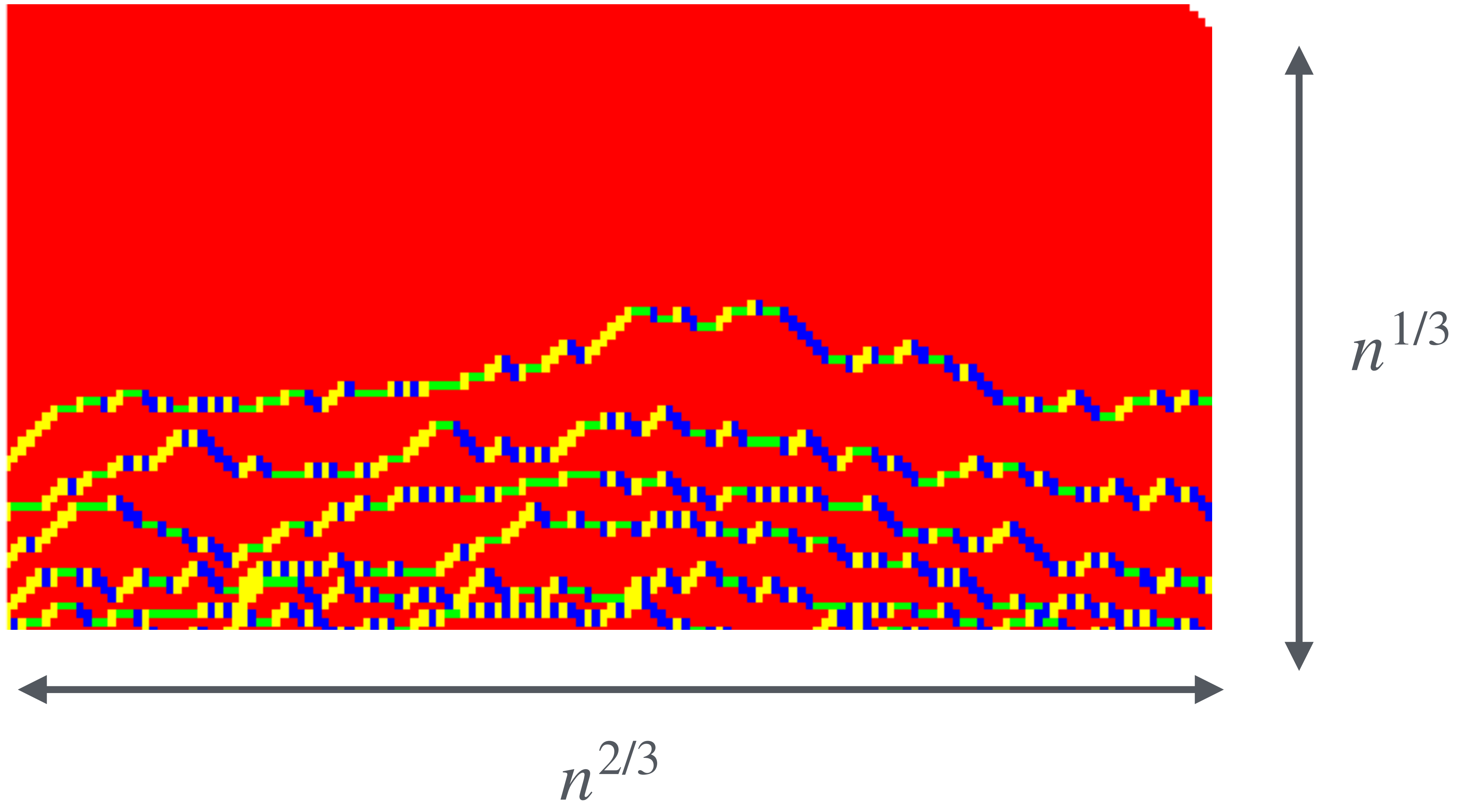


This will give a random domino tilings of the full plane. Can we describe it?

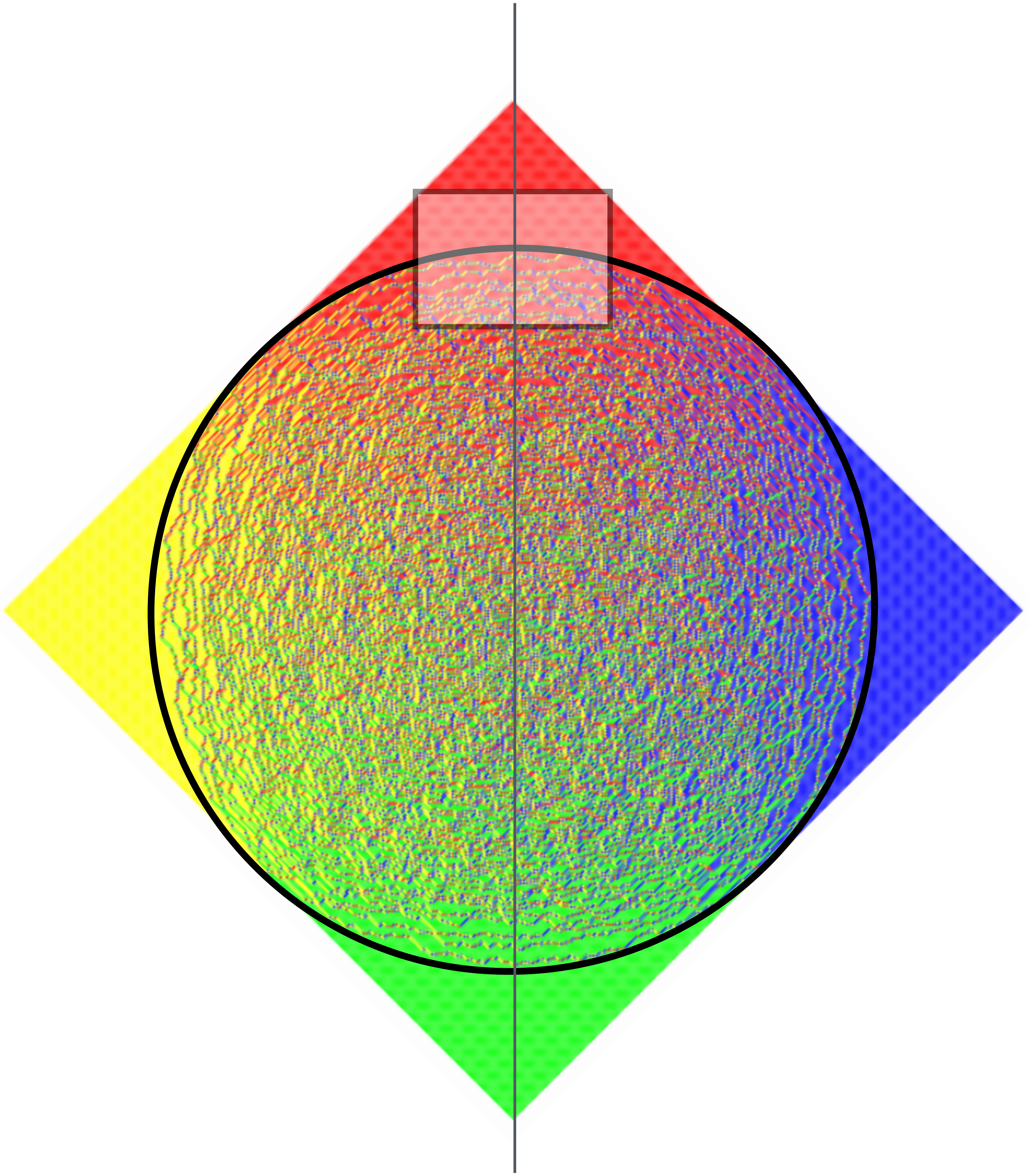
Limit near arctic circle

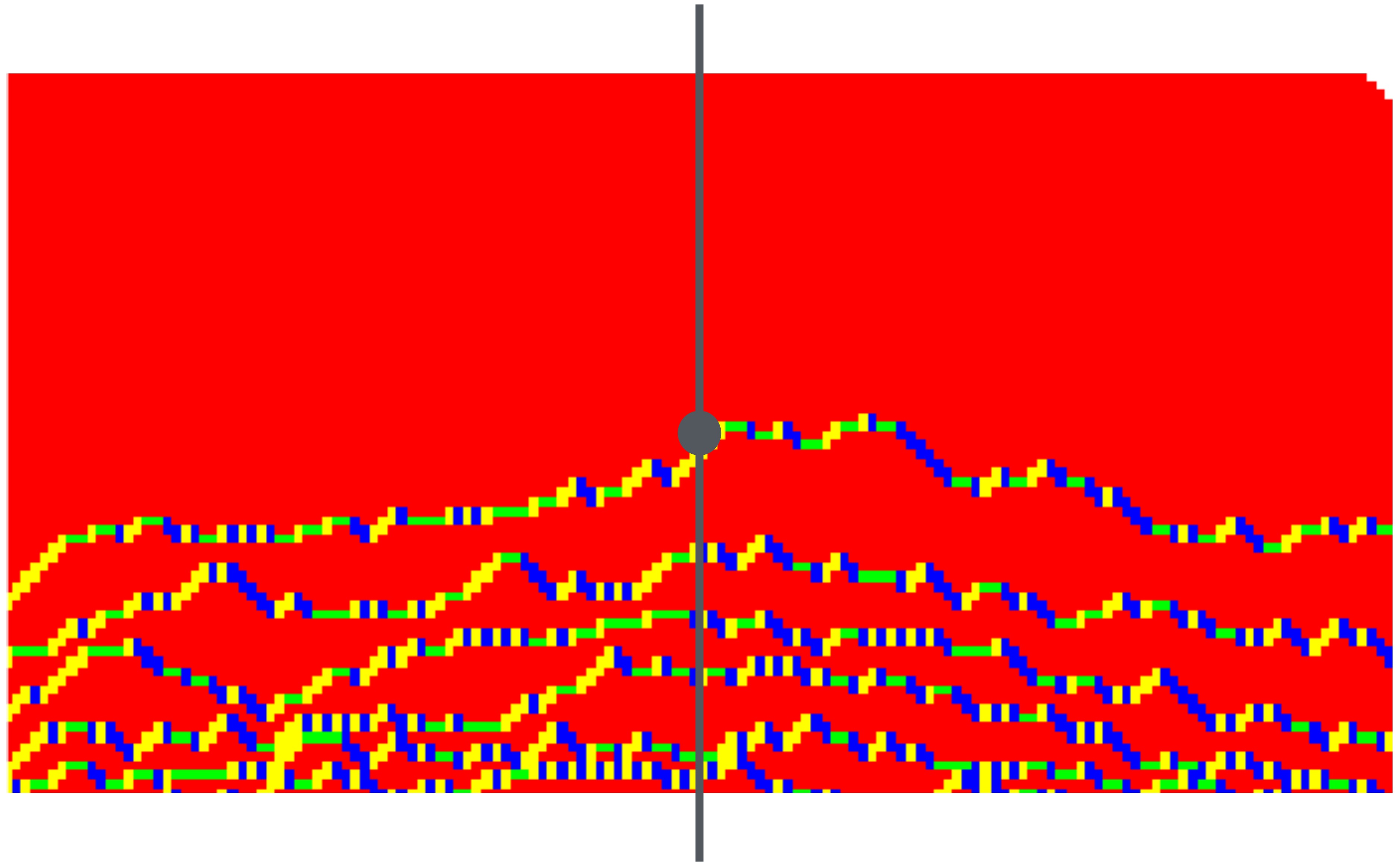
Lets zoom in at the top





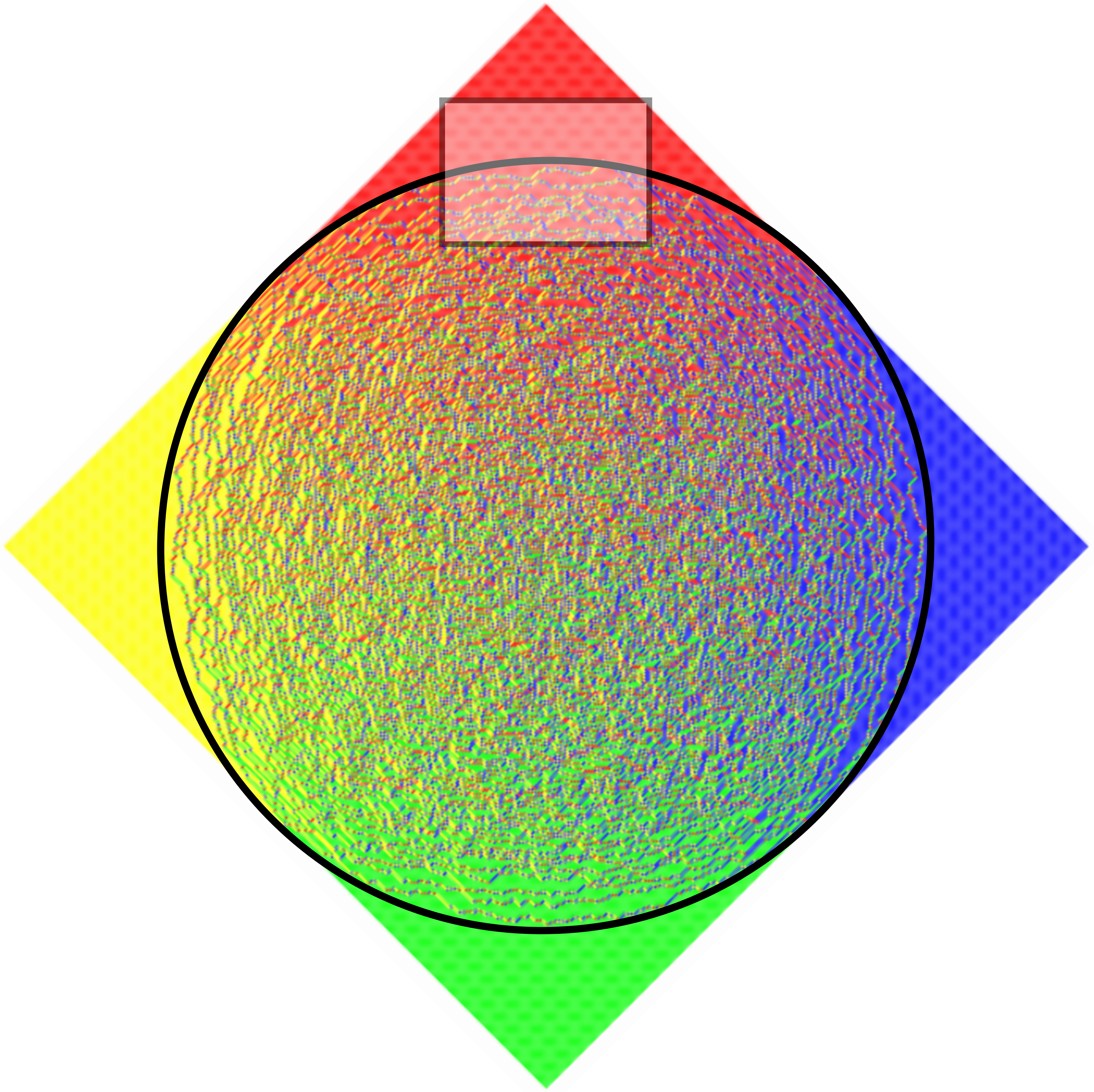
Limit near arctic circle

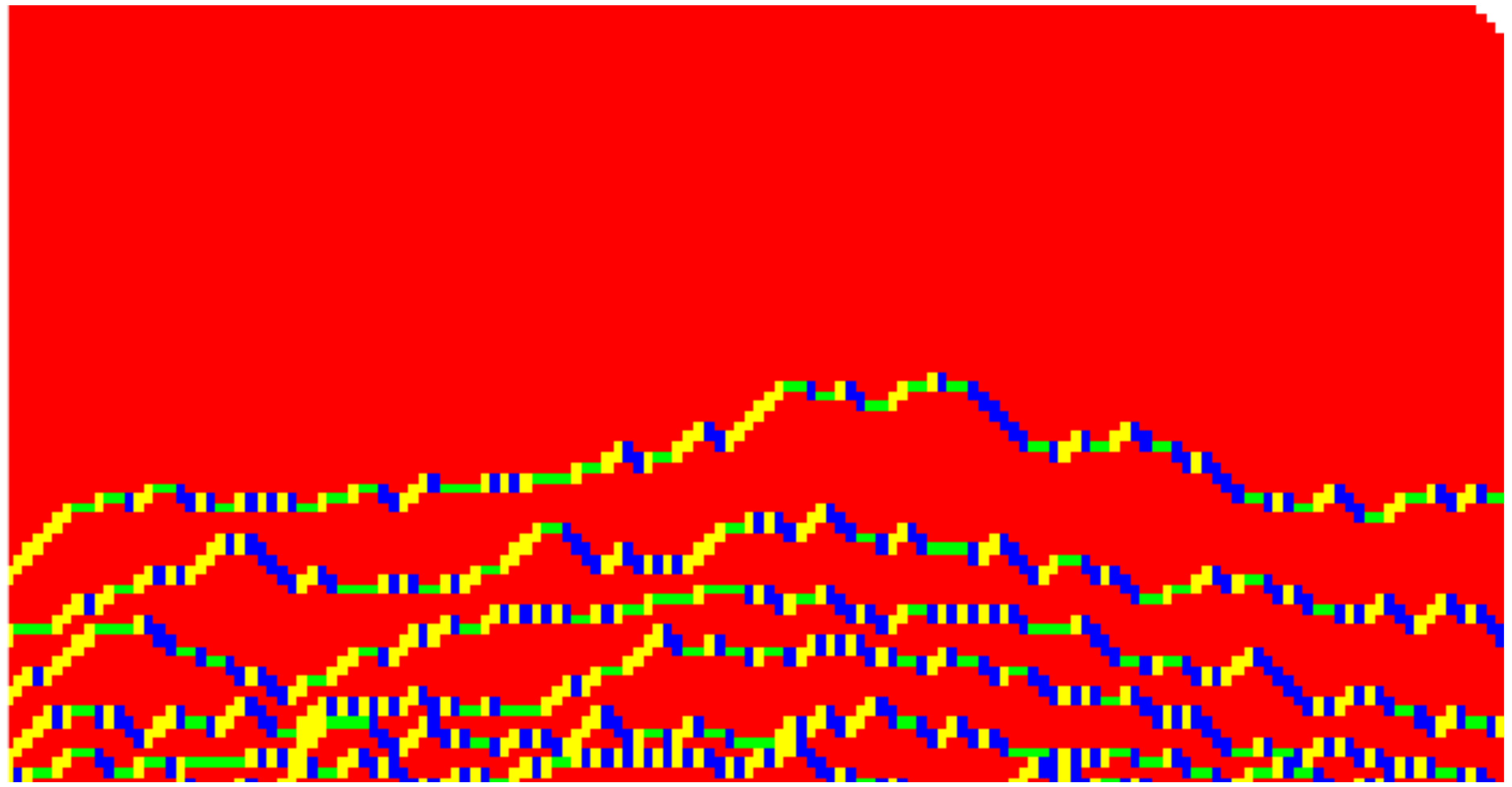




The fluctuations of the top part intersected at vertical line, will be the Tracy-Widom distribution, same distribution as the largest eigenvalue of a large GUE matrix

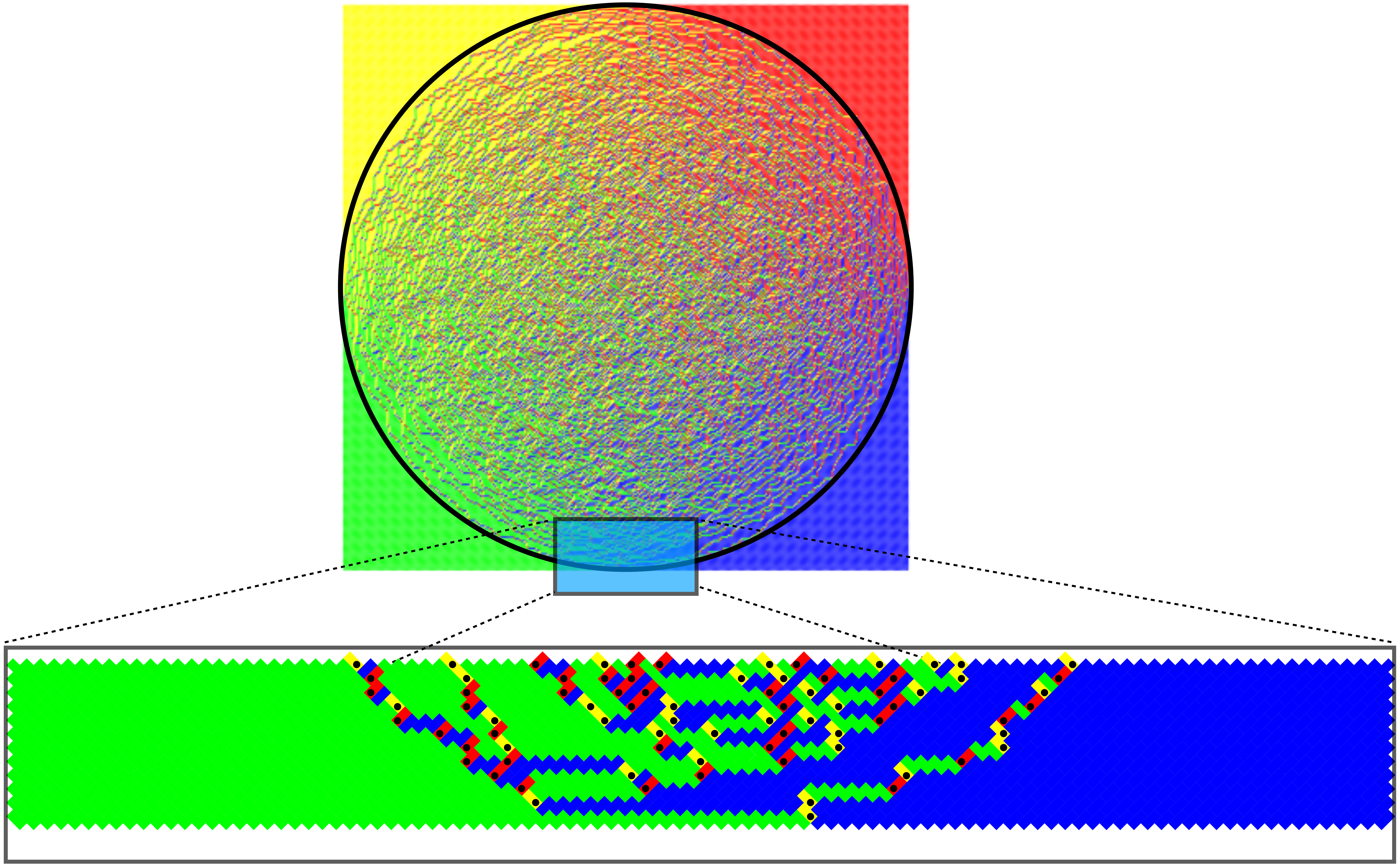
Limit near arctic circle



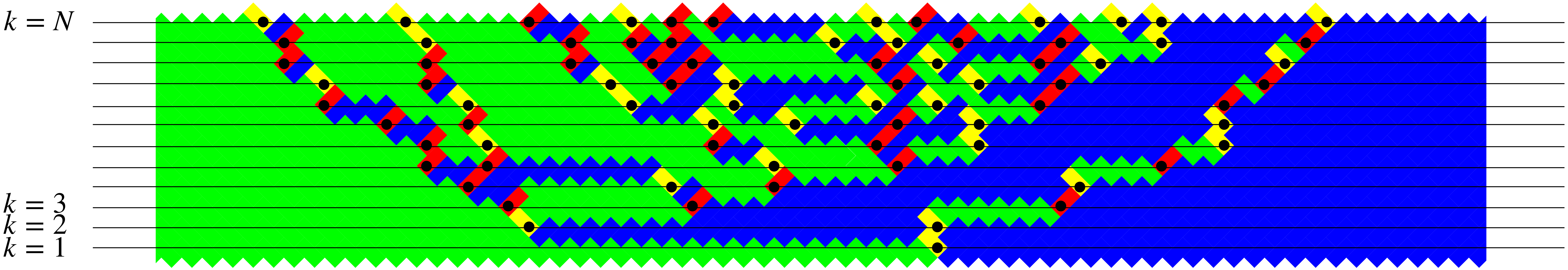


What is the law of the top path,
or top k paths?
—> Airy line-ensemble

Limit near turning point.



Limit near turning point



- Draw dots on the yellow and red dominoes
- For each horizontal k-section there are precisely k black dots.
- Let x_j^k be the position of the j-th dot counted from left to right in the k-th row, counted from bottom to top.

then $$x_j^{k+1} < x_j^k \leq x_{j+1}^{k+1} \quad \text{for } j = 1, \ldots, k$$

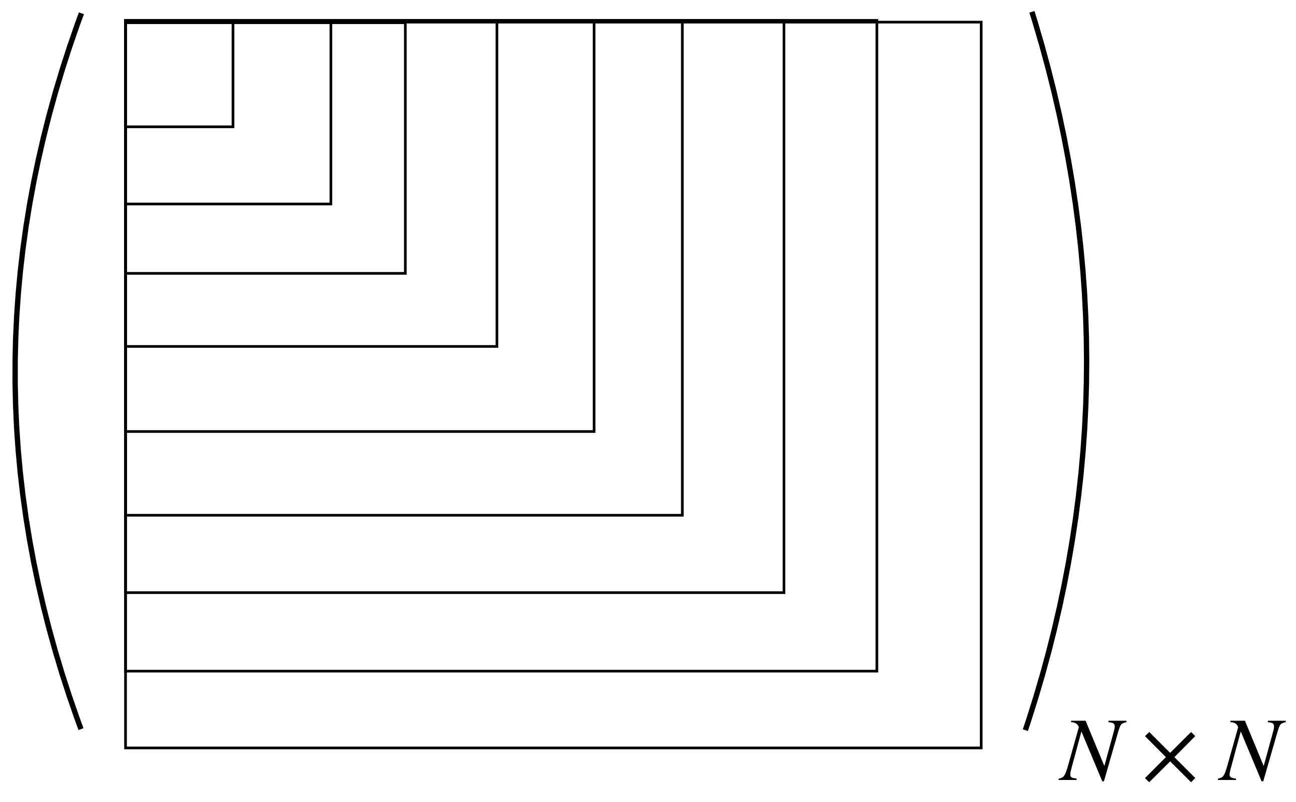
- Interlacing particle system.

GUE corner process

Let M_{ij} be an $N \times N$ Hermitian matrix with random entries such that

$$M_{ij} \sim N_{\mathbb{C}}(0,1) \qquad \text{for} \quad 1 \leq i < j \leq N$$
$$M_{ii} \sim N_{\mathbb{R}}(0,1) \qquad \text{for} \ 1 \leq i \leq N$$

$M =$



Then for $j = 1, \ldots, k$ let let λ_j^k be the eigenvalues of the $k \times k$ upper left sub matrix of M (delete the final $n - k$ rows and columns).

Linear algebra tells us that $\lambda_j^{k+1} < \lambda_j^k < \lambda_{j+1}^{k+1}$

Limit to GUE corner process

Theorem

Let x_j^k be the positions of the dots in the bottom N horizontal section of the Aztec diamond of size n, chosen uniformly over all matchings. And λ_j^k the eigenvalues of the submatrices of a GUE matrix. Then, as $n \to \infty$,

$$\left(\frac{x_j^k}{\sqrt{n}}\right)_{j,k=1}^{k,N} \xrightarrow{d} \left(\lambda_j^k\right)_{j,k=1}^{k,N}$$

Random matrices are limits of random tilings!

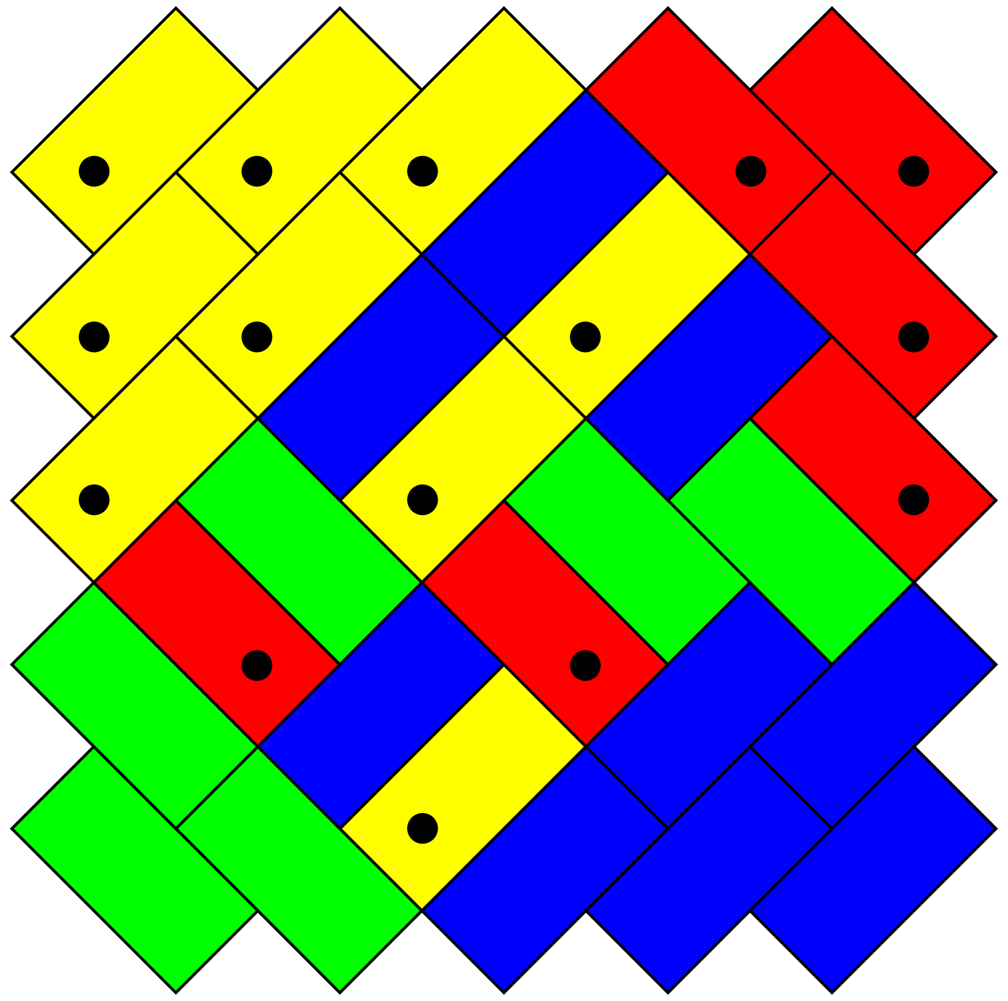
Limit Shape and global fluctuations

- First draw dots on red and yellow dominos, and then define the height function

$$h(x, k) = \#\{j \mid x_j^k \geq x\}$$

- The height function jumps at the position of the dots. The graph of the height function is a stepped surface.

- NB: This is not the standard way of defining the height function.

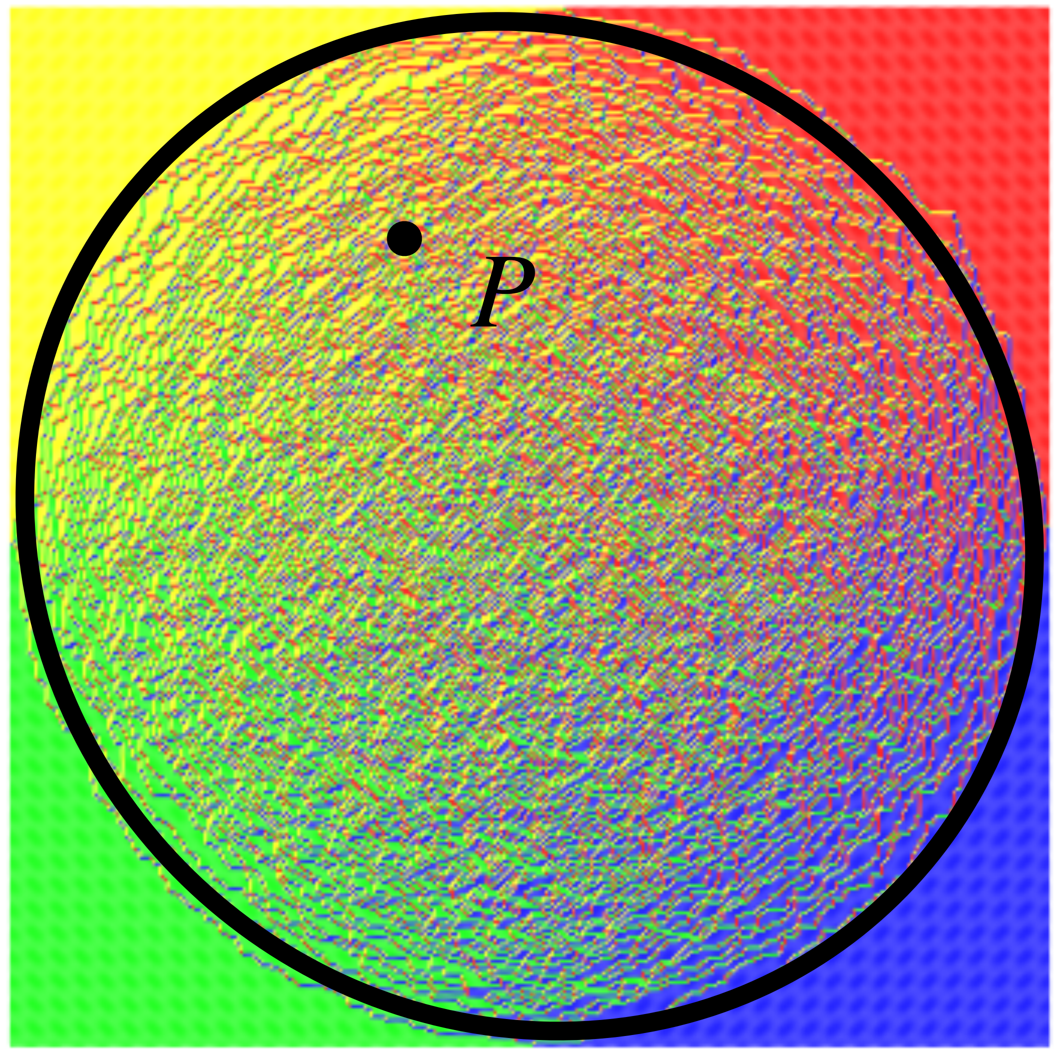


5	4	3	2	2	1
4	3	2	2	1	1
3	2	2	1	1	1
2	2	1	1	0	0
1	1	1	0	0	0

Limit Shape and global fluctuations

- The height function has a limit

$$\frac{1}{n}\mathbb{E}\left[h(P)\right] \to H(P)$$

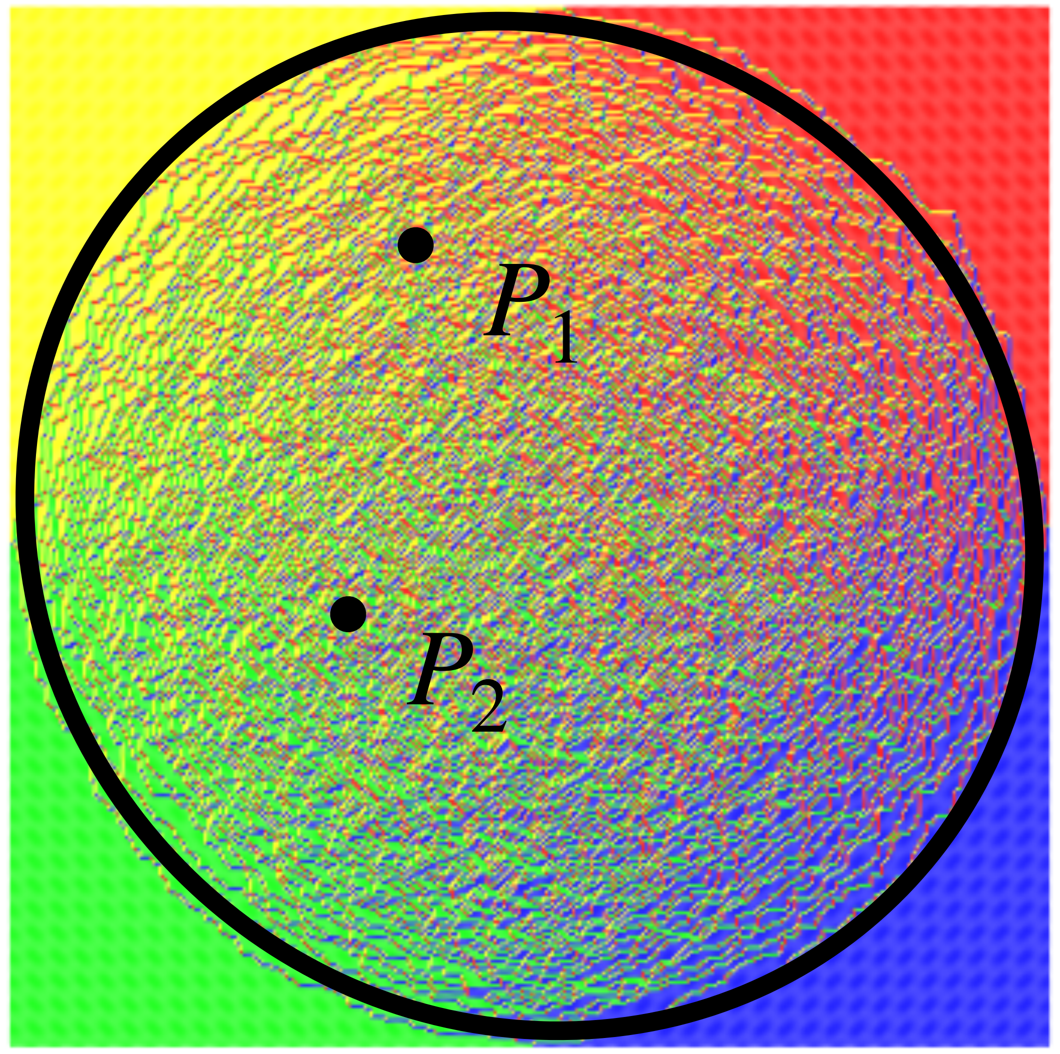


- Pointwise in the disordered region, the fluctuations of the height function diverge logarithmically.

$$\text{Var}\, h(P) \sim \log n$$

- But two point correlations remain finite!

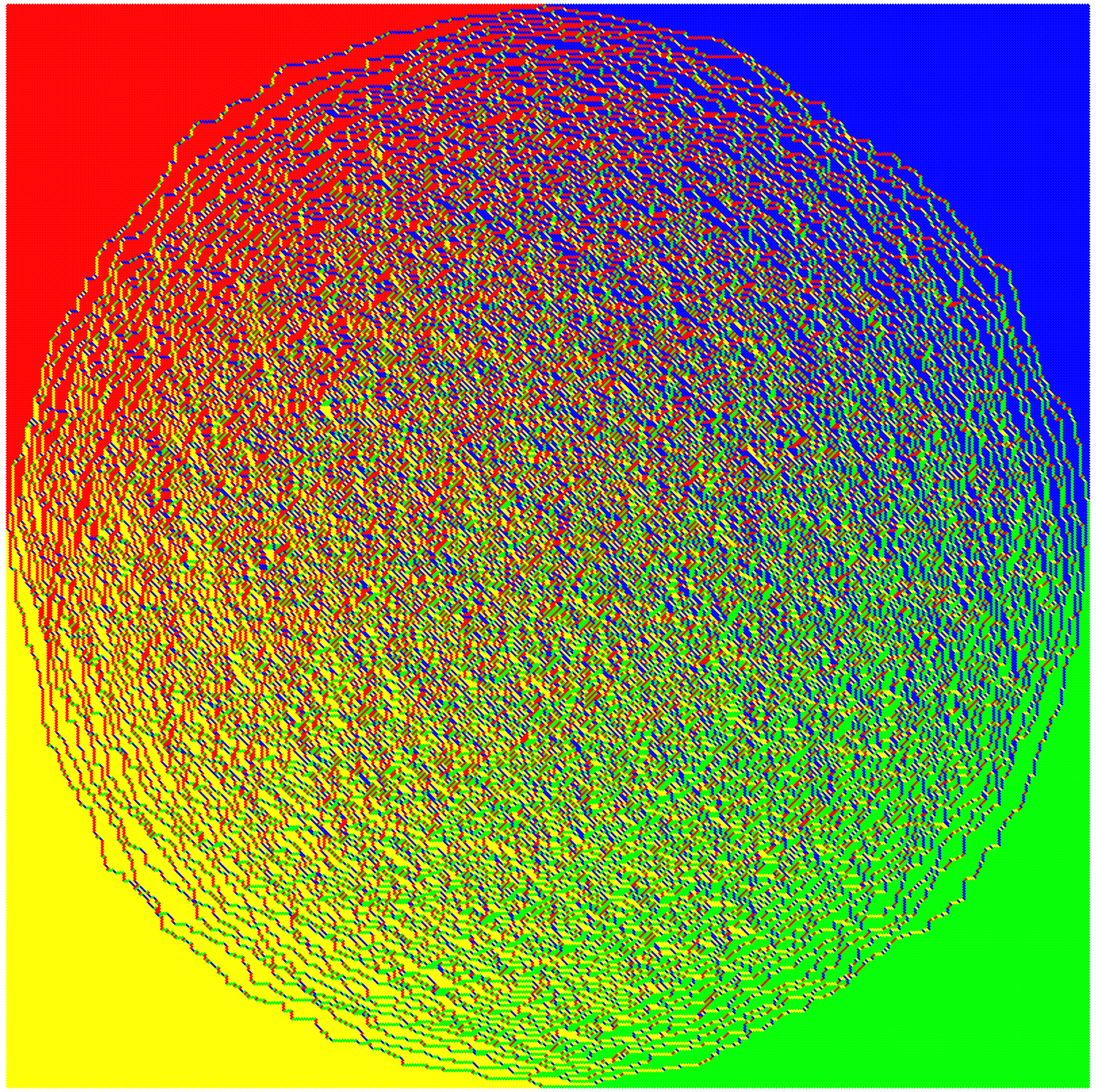
$$\text{C}ov[h(P_1), h(P_2)] \sim 1$$



- As $n \to \infty$ the height field, or random surface, is rough and converges to the ***Gaussian Free Field with Dirichlet boundary conditions.***

Content

- What will we cover in these lectures
    - Determinantal point processes, Schur process and extensions, biorthogonal ensembles, non-intersecting paths.
    - Asymptotic analysis: Arctic Circle Theorem, Limit shapes and fluctuations, Tracy-Widom fluctuations, Airy process, Bulk limits and measures on the plane. Random matrix limits.
    - Shuffling algorithm and dynamics.
    - Height fluctuations and Gaussian Free Field. Operator approach.
    - Orthogonal polynomial approaches.
- The perspective in this course leans heavily on ***analytic methods.***



Content

- What will we **not** cover in these lectures
  - Kasteleyn's theory
  - Pfaffian processes, or other process outside the determinant class
  - Representation theory
  - .....

Plan for the lectures (preliminary)

- Lecture 1: Determinantal point process
- Lecture 2: Schur processes, LGV, Eynard-Mehta, Toeplitz matrices
- Lecture 3: Asymptotic study of the Aztec diamond with uniform distribution
- Lecture 4: Shuffling algorithm and dynamics in 2 dimensions

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- Lecture 5: Doubly periodic weights
- Lecture 6: Height fluctuations and the Gaussian free field
- Lecture 7: Orthogonal polynomials, Painlevé transcendents in lozenge tilings of the hexagon.

I am to have each lecture self-contained