Lecture 9

Deift/Zhou Steepest descent, Part I

We now focus on the case of orthogonal polynomials for the weight

$$w(x) = \mathrm{e}^{-NV(x)}, \qquad V(x) = t\frac{x^2}{2} + \frac{x^4}{4}.$$

Since the weight depends on the parameter $N \in \mathbb{N}$ we will write

$$\pi_{n,N}, a_{n,N}, etc...$$

The dependence on $t \in \mathbb{R}$ will be omitted in the notation.

9.1 Weak asymptotics

Write $\pi_{n.n} = \prod_{j=1}^n (z - \eta_{j,n})$, so that $\eta_{j,n}$ are the zeros of $\pi_{n,n}$. We will show that for $t \geq -2$ these zeros accumulate on a single interval $[-a, a]$ and have density

$$\mathrm{d}\mu_V(x) = \frac{1}{\pi}(c + \frac{x^2}{2})\sqrt{a^2 - x^2}, \quad \text{on } [-a, a],$$

where

$$\begin{cases} a = \sqrt{\frac{-2t+2\sqrt{t^2+12}}{3}} \\ c = \frac{t+\sqrt{t^2/4+3}}{3} \end{cases}$$

One can show that with these constant, the total mass of the measure is 1, i.e.

$$\int_{-a}^{a} \mathrm{d}\mu_V(x) = 1.$$

For interpreting the following theory we note that

$$\frac{1}{n}\frac{\pi_{n,n}'(z)}{\pi_{n,n}} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{z-\eta_{j,n}} = \int \frac{1}{z-x}\mathrm{d}\mu_n(x).$$

where $\mu = \frac{1}{n}\sum_{j=1}^{n}\delta_{\eta_{j,n}}$ is the normalized sum of dirac masses at the zeros of $\pi_{n,n}$.

Theorem 9.1.1. *Let $t \geq -2$. Then*

$$\lim_{n\to\infty}\frac{1}{n}\frac{\pi_{n,n}'(z)}{\pi_{n,n}} = \int_{-a}^{a}\frac{1}{z-x}\mathrm{d}\mu_V(x),$$

uniformly for z in compact subsets of $\mathbb{C}\setminus[-a,a]$.

We will prove this Theorem, among stronger results, by using Riemann-Hilbert techniques. In this lecture we will always assume

$$t > -2.$$

The case $t = -2$ will be done later.

9.2 The g-function

Definition 9.2.1. We define $g : \mathbb{C}\setminus(\infty, a]$ as

$$g(z) = \int_{-a}^{a}\log(z-x)\mathrm{d}\mu_V(x),$$

where we choose the branch of the logarithm such that g is analytic and real for $z > a$.

Lemma 9.2.2.

$$g'(z) = \frac{V'(z)}{2} - (c + z^2/2)(z^2-a^2)^{1/2}$$

, where take the $z \mapsto (z^2-a^2)^{1/2}$ such that it is analytic in $\mathbb{C}\setminus[-a,a]$ and positive on (a,∞).

Proof. Define the function $h = \frac{1}{-2\mathrm{i}}g'$. Then h is, is up to constant, the Cauchy transform of $\mathrm{d}\mu_V(x)$, i.e.

$$h(z) = \frac{1}{2\mathrm{i}}\int_{-a}^{a}\frac{1}{x-z}\mathrm{d}\mu_V(x).$$

But that means that h is the unique solution to the following RHP (exercise!)

$$\begin{cases} h \text{ is analytic in } \mathbb{C}\setminus[-a,a] \\ h_+(x)-h_-(x)=(c+\frac{x^2}{2})\sqrt{a^2-x^2}, & x\in(-a,a), \\ h(z)=\mathcal{O}(1/z), & z\to\infty \\ h \text{ is bounded near } \pm a. \end{cases}$$

We claim that

$$-\frac{1}{2\mathrm{i}}\left(V'(z)/2-(c+z^2/2)(z^2-a^2)^{1/2}\right)$$

solves the same RHP problem and hence,by uniqueness of the solution, we proved the statement. To see that this indeed solve the RHP we need to check the jump condition, which follows by the choice of the square root (exercise), and the asymptotic behavior at infinity. After a computation one checks that $V'(z)$ cancels the polynomials part of the second term and hence the claim follows. □

The following properties will be crucial in the upcoming analysis.

Lemma 9.2.3. *The function g satisfies the following properties*

- $g_+(x)-g_-(x)=2\pi\mathrm{i}$ *for* $x<-a$
- $g_+(x)-g_-(x)=2\pi\mathrm{i}\int_x^a$ *for* $x\in[-a,a]$.
- $g_+(x)+g_-(x)-V=l$ *for* $x\in[-a,a]$
- $g_+(x)+g_-(x)-V<l$ *for* $x\in\mathbb{R}\setminus[-a,a]$.

for some constant $l\in\mathbb{R}$.

Proof. 1. Note that

$$\log_\pm(z-x)=\log|z-x|+\mathrm{i}\arg_\pm(z-x)$$

For $z,x\in\mathbb{R}$ we have

$$\arg_+(z-x)-\arg_-(z-x)=\begin{cases}2\pi\mathrm{i}, & z<x \\ 0, & z\geq x.\end{cases}$$

Since $\int \mathrm{d}\mu_V=1$, we find the statement.

2. Follows by the same arguments as 1.

3. This follows form the previous lemma. By the definition of the square root we have

$$(z^2-a^2)_\pm^{1/2} = \pm \mathrm{i}\sqrt{a^2-z^2}$$

for $z \in (-a,a)$ and hence

$$g'_+(x) + g'_-(x) - V'(x) = 0$$

for $x \in [-a,a]$.

4. This follows form the same arguments as before but now we have

$$g'_+(x) + g'_-(x) - V'(x) < 0$$

on (a,∞)

$$g'_+(x) + g'_-(x) - V'(x) > 0$$

on $(-\infty,a)$. □

9.3 Overview of the Deift/Zhou steepest descent analysis

We start with the RHP for the orthogonal polynomials as discussed int he previous lecture (with $w(x) = \mathrm{e}^{-nV(x)}$).

RH problem 9.3.1. *We seek for a function $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ such that*

- *Y is analytic in $\mathbb{C}\setminus\mathbb{R}$.*
- *$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & \mathrm{e}^{-nV(x)} \\ 0 & 1\end{pmatrix}$, for $x \in \mathbb{R}$.*
- *$Y(z) = (I+o(1))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}$ as $z\to\infty$.*

The asymptotic analytis consists of a sequence of transformations

$$Y \mapsto T \mapsto S \mapsto R.$$

Each transformation is simple and invertible. After each transformation we obtain a new RiHP. In the end, the goal is to end up with a RHP of the form

$$\begin{cases} R_+ = R_- J_r \\ R(z) = I + \mathcal{O}(1/z), z\to\infty, \end{cases}$$

with $J_R \to 1$ as $n\to\infty$. We then find $R\to I$ and moreover, an asymptotic expansion for R. By tracing the transformations back, we obtain an asymptotic expansion for Y.

9.4 First transformation : normalizing the RHP near infinity

In the first transformation

$$T(z) = \begin{pmatrix} e^{-nl/2} & 0 \\ 0 & e^{nl/2} \end{pmatrix} Y(z) \begin{pmatrix} e^{-n(g-l/2)} & 0 \\ 0 & e^{n(g-l/2)} \end{pmatrix}.$$

Then T satisfies RHP that we will now pose. First of all, we note that since μ_V has total mass 1 we haev

$$g(z) = \log z + \mathcal{O}(1/z), \quad z \to \infty.$$

That means that $T(z) = I + \mathcal{O}(1/z)$, which means that we have normalized the RHP at infinity. Of course, the price we pay is in terms of a more complicated jump structure which we can compute from

$$J_T = T_-^{-1} T_+ = \begin{pmatrix} e^{n(g_- - l/2)} & 0 \\ 0 & e^{-n(g_- - l/2)} \end{pmatrix} Y_-^{-1} Y_+ \begin{pmatrix} e^{-n(g-l/2)} & 0 \\ 0 & e^{n(g-l/2)} \end{pmatrix}.$$

A straightforward computation then gives that T solves the following RHP.

RH problem 9.4.1. *We seek for a function $T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ such that*

- *T is analytic in $\mathbb{C} \setminus \mathbb{R}$.*
- $T_+(x) = T_-(x) \begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & e^{n(g_+(x)+g_-(x)-V(x)-l)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix}$, *for $x \in \mathbb{R}$.*
- *$T(z) = I + \mathcal{O}(1/z)$ as $z \to \infty$.*

There is a simplification for the jump matrix for T by the properties of the g-function in Lemma 12.2.3. Then

$$\begin{aligned} &\begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & e^{n(g_+(x)+g_-(x)-V(x)-l)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix} \\ &\quad = \begin{cases} \begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & 1 \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix}, & x \in [-a,a] \\ \begin{pmatrix} 1 & e^{n(g_+(x)+g_-(x)-V(x)-l)} \\ 0 & 1 \end{pmatrix}, & x \in \mathbb{R} \setminus [-a,a]. \end{cases} \end{aligned} \tag{9.4.1}$$

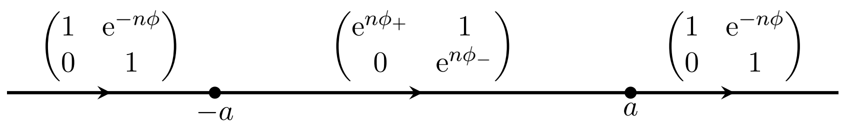

Figure 9.1: Jumps for T

We will introduce some notation for presentation purposes. We define, for $z \in \mathbb{C} \setminus (-\infty, a]$

$$\phi(z) = 2 \int_a^z (c + \frac{y^2}{2})(y^2 - a^2)^{1/2} \mathrm{d}y,$$

where the path of integration is $\mathbb{C} \setminus (-\infty, a]$. Then ϕ is analytic. Moreover, $\phi_+ - \phi_- = 0 \mod 2\pi \mathrm{i}$ on $(-\infty, a]$ and therefore we have that $\mathrm{e}^{n\phi}$ is analytic in $\mathbb{C} \setminus [-a, a]$. Moreover, by Lemmas 12.2.2 and 12.2.3 we see that we can rewrite the jump in terms of ϕ and obtain

$$T_+ = T_- \begin{pmatrix} \mathrm{e}^{n\phi_+} & 1 \\ 0 & \mathrm{e}^{n\phi_-} \end{pmatrix} \qquad \text{on } [-a, a]$$

and

$$T_+ = T_- \begin{pmatrix} 1 & \mathrm{e}^{-n\phi} \\ 0 & 1 \end{pmatrix} \qquad \text{on } \mathbb{R} \setminus [-a, a].$$

See also Figure **??**.

Since $\operatorname{Re} \phi > 0$ on $\mathbb{R} \setminus [-a, a]$ we see that the jumps on $\mathbb{R} \setminus [-a, a]$ are exponentially small in n. However, the jumps on $[-a, a]$ are not exponentially small. In fact, $\phi_\pm(x) = \pm 2\pi \mathrm{i} \int_x^a \mathrm{d}\mu_V(x)$ and hence the jumps on $[-a, a]$ are highly oscillating!

9.5 Second Transformation: opening of the lenses

To eliminate the highly oscillating jumps we perform a transformation that is known as the opening of the lens. It is based on the following factorization

$$\begin{pmatrix} \mathrm{e}^{n\phi_+} & 1 \\ 0 & \mathrm{e}^{n\phi_-} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{n\phi_-} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{n\phi_+} & 1 \end{pmatrix}.$$

Now we do the following. We open up a lens around the interval $[-a, a]$. That is, we take two simple contours connecting $-a$ to $+a$, one in the upper half plane and one in the lower half plane. We then define

$$S = \begin{cases} T \begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{n\phi} & 1 \end{pmatrix} & \text{in upper part} \\ T \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{n\phi} & 1 \end{pmatrix} & \text{in lower part} \\ T & \text{elsewhere.} \end{cases}.$$

See also Figure 12.2.

Now what did we gain from this transformation.

Proposition 9.5.1. *For $t > -2$ we can choose the lips of the lens such that $\operatorname{Re}\phi < 0$ on the lipses.*

Proof. Note that ϕ is analytic in $\mathbb{C} \setminus (-\infty, a]$ and, for $x \in [-a, a]$,

$$\phi_+(x) = -2\pi\mathrm{i} \int_x^a \mathrm{d}\mu_V(x).$$

Because of the Cuachy Riemann equations (note that ϕ can analytically continued to $[-a, a]$) we have with $z = x + \mathrm{i}y$

$$\frac{\mathrm{d}\operatorname{Re}\phi}{\mathrm{d}y}(x) = -\frac{\mathrm{d}\operatorname{Im}\phi}{\mathrm{d}x}(x) = -\frac{\mathrm{d}\mu_V}{\mathrm{d}x}(x) > 0$$

(since $t > -2$). But that means that we can indeed deform the interval $[-a, a]$ to a path from $-a$ to a in the upper half plane such that $\operatorname{Re}\phi < 0$ on that path.

The argument for the lower lips is analogous. $\square$

In Figure 12.4 we show the part of the complex plane where $\operatorname{Re}\phi < 0$ for $t = -1$. Clearly , the lense can be chosed such that it falls completely in the region.

Conclusion: all jumps for S other then the jump on $[-a, a]$ are exponentially small in n (pointwise). The jump on $[-a, a]$ does not depend on n. Hence we have made big progress in the understanding of the asymptotic behavior of Y.

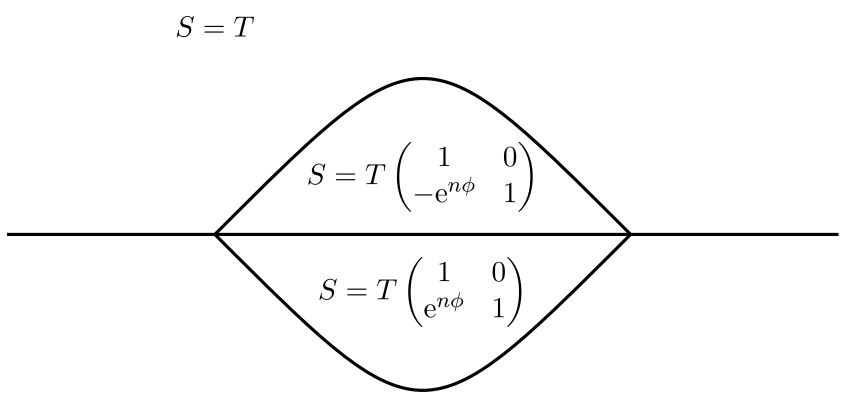

Figure 9.2: Opening of the lens

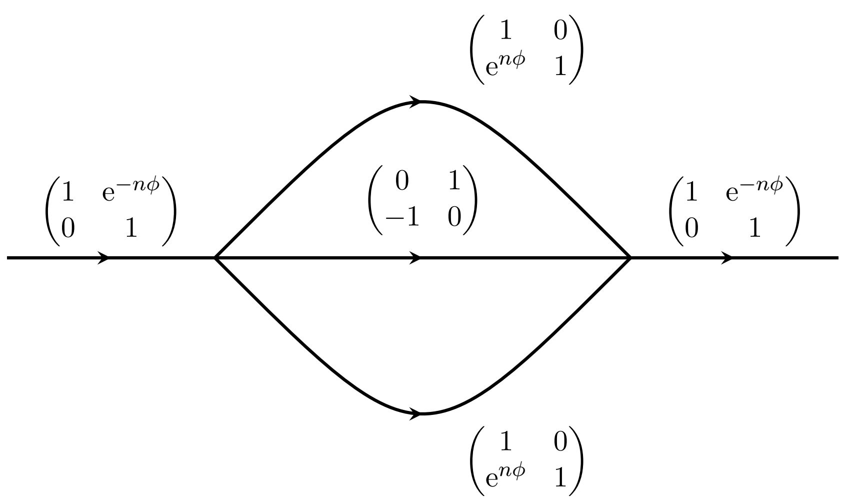

Figure 9.3: Jumps for S

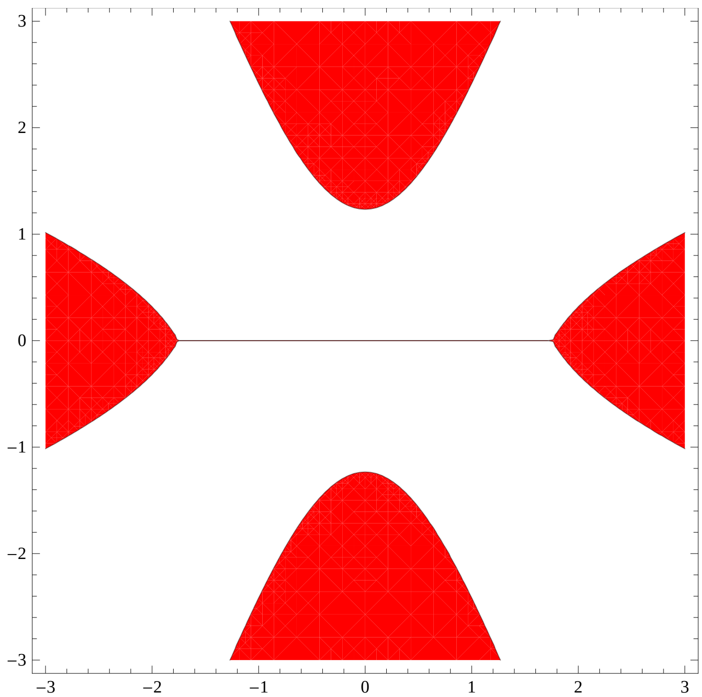

Figure 9.4: The white region in which is the region where $\mathrm{Re}\,\phi < 0$ for $t = -1$. The shaded region is the region $\mathrm{Re}\,\phi > 0$.

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