Lecture 8

Orthogonal polynomials

8.1 Orthogonal polynomials and some of their features

Let $w : \mathbb{R} \to [0, \infty)$ be a postive function with finite moments

$$\int_{-\infty}^{\infty} |x|^k w(x) \mathrm{d}x > \infty.$$

For simplicity we will assume that $w(x) \to 0$ rapidly as $x \to \pm\infty$.

The space of polynomials is a Hilbert supspace of $\mathbb{L}_2(R)$, spanned by the basis of monomials $\{1, x, x^2, \ldots\}$. By applying Gram-Schmidt to this basis we obtain the orthogonal polynomials. We denote the *monic* orthogonal polynomial of degree k by π_k. Hence $\pi_k(x) = x^k + \ldots$ is the unique monice polynomial such that

$$\int_{-\infty}^{\infty} \pi_k(x) x^j w(x) \mathrm{d}x = 0, \qquad j = 0, \ldots, k-1.$$

We will sometimes also use $p_k(x) = \kappa_k \pi_k(x)$ where $\kappa_k = \|\pi_k\|_2^{-1}$. Hence p_k are the orthonormal polynomials.

In the rest of the course we will be interested in the asymptotic behavior of the polynomials π_n as $n \to \infty$ and also some of their features. In particular, we are interested in the behavior of the recurrence coefficients and of the reproducing kernel. These objects are introduced in the next two lemma's.

Lemma 8.1.1. *There exists* $\{a_k^2 > 0, b_k \in \mathbb{R}\}_{k=0}^{\infty}$ *such that*

$$x\pi_k(x) = \pi_{k+1}(x) + b_k \pi_k(x) + a_k^2 \pi_{k-1}(x), \qquad k = 0, 1, 2, \ldots$$

(and $a_{-1} = 0$*).*

The a_k and b_k are called recurrence coefficients.

The *reproducing kernel* $K_n(x,y)$ is defined by

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)p_j(y).$$

The name of this kernel comes from the following result.

Lemma 8.1.2. *For any polynomial q of degree $\leq n-1$ we have*

$$\int_{-\infty}^{\infty} K_n(x,y)p(y)w(y)\mathrm{d}y = q(x).$$

Lemma 8.1.3. *We have that*

$$K_n(x,y) = \kappa_{n-1}^2 \frac{\pi_n(x)\pi_{n-1}(y) - \pi_n(y)\pi_{n-1}(x)}{x-y}$$

(Christoffel-Darboux formula).

8.2 Riemann-Hilbert problem for orthogonal polynomials

RH problem 8.2.1. *We seek for a function $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ such that*

- *Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.*
- $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$, *for $x \in \mathbb{R}$.*
- $Y(z) = (I + o(1)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ *as $z \to \infty$.*

Proposition 8.2.2. *The solution to the RHP **??** is unique and given by*

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\pi_n(x)w(x)}{x-z}\mathrm{d}x \\ -2\pi \mathrm{i}\kappa_{n-1}^2 \pi_{n-1}(z) & -\kappa_{n-1}^2 \int_{\mathbb{R}} \frac{\pi_{n-1}(x)w(x)}{x-z}\mathrm{d}x \end{pmatrix}. \tag{8.2.1}$$

Proof. The idea of the proof is to consider the individual entries for Y. We will prove the first row only and leave the second row (which is almost identical) to the reader.

The RHP give the following properties for the entry $Y_{11)}$

$$\begin{cases} Y_{11,+} = Y_{11,-} & \text{on } \mathbb{R} \\ Y_{11}(z) = z^n(1+\mathcal{O}(1/z)), & z \to \infty. \end{cases}$$

Hence, Y_{11} is an entire function and, by Liouville's Theorem, a monic polynomial q_n of degree n. We still need to show that $q_n = \pi_n$. This follows by looking at the entry Y_{12}. Here we find

$$\begin{cases} Y_{12,+} = Y_{12,-} + q_n w & \text{on } \mathbb{R} \\ Y_{12}(z) = \mathcal{O}(1/z^{n+1}), & z \to \infty. \end{cases}.$$

Given q_n, this problem has a unique solution given by the Cauchy transform

$$Y_{12}(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{q_n(y)w(y)}{y-z} \mathrm{d}y.$$

However, this only gives $Y_{12}(z) = \mathcal{O}(1/z)$ and we need a stronger condition. To this end, write

$$\frac{1}{y-z} = -\sum_{j=0}^{n-1} \frac{y^j}{z^{j+1}} + \frac{1}{z-y}\frac{y^n}{z^n},$$

and then

$$Y_{12}(z) = -\frac{1}{2\pi \mathrm{i}} \sum_{j=0}^{n-1} \frac{1}{z^{j+1}} \int_{-\infty}^{\infty} q_n(y) u^j w(y) \mathrm{d}y + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{q_n(y)w(y)}{z-y} \frac{y^n}{z^n} \mathrm{d}y.$$

Hence to have $Y_{12}(z) = \mathcal{O}(1/z^{n+1})$ as $z \to \infty$, we need

$$\int_{-\infty}^{\infty} q_n(y) u^j w(y) \mathrm{d}y$$

and hence $q_n = \pi_n$. This finishes the proof for the first row. □

Also the reproducing kernel K_n and the recurrence coefficients can be characterized in terms of the RH problem.

Lemma 8.2.3.

$$K_n(x,y) = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix}}{2\pi \mathrm{i}(x-y)} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.2.2}$$

Proof. Firs observe that the inverse for any 2×2 matrix can be found by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence, since $\det Y = 1$, we have

$$Y^{-1} = \begin{pmatrix} * & * \\ 2\pi \mathrm{i} \pi_{n-1} & \pi_n \end{pmatrix}$$

By working out the right-hand side of (8.2.2) we retrieve the Christoffel-Darboux formula for the kernel. □

Lemma 8.2.4. *Write*

$$Y(z) = \left(I + Y^{(1)}/z + Y^{(2)}/z^2 + \ldots\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Then

$$\begin{cases} a_n^2 = Y_{12}^{(1)} Y_{21}^{(1)} \\ b_n = \frac{Y_{12}^{(2)}}{Y_{12}^{(1)}} - Y_{22}^{(1)} \end{cases}$$

Proof. We start by noting that $Y_{21}^{(1)} = -2\pi \mathrm{i} \kappa_{n-1}^2$.

To indicate the dependence on n we write $Y = Y_n$. Since the jump for Y_n does not depend on n, we have that $Y_{n+1}Y_n^{-1}$ has no jump on the real line and hence it is entire. By inserting the asymptotic condition and using Liouville's Theorem we find

$$Y_{n+1}Y_n^{-1} = (I + \mathcal{O}(1/z))\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}(I + \mathcal{O}(1/z)) = \begin{pmatrix} z + * & * \\ * & 0 \end{pmatrix}$$

where the $*$ denote constants that we will now determine. Let us first write

$$Y_{n+1} = \begin{pmatrix} z + * & * \\ * & 0 \end{pmatrix} Y_n$$

The 21-entry at both sides gives $-2\pi \mathrm{i} \kappa_n^2 \pi_n$ and this determines one constant. Moreover the 11-entry at both sides is precisely the recurrence relation for the orthgonal polynomials and hence

$$Y_{n+1} = \begin{pmatrix} z - b_n & \frac{a_n^2}{2\pi \mathrm{i} \kappa_{n-1}^2} \\ -2\pi \mathrm{i} \kappa_n^2 & 0 \end{pmatrix} Y_n.$$

Now multiply both sides with $\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}$ and compute the asymptotic behavior for $z \to \infty$ to get

$$(I + \mathcal{O}(1/z)) \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} z - b_n & \frac{a_n^2}{2\pi \mathrm{i} \kappa_{n-1}^2} \\ -2\pi \mathrm{i} \kappa_n^2 & 0 \end{pmatrix} \left(I + \frac{Y^{(1)}}{z} + \frac{Y^{(2)}}{z^2} + \dots \right).$$

By computing the 12-entry of both sides we find

$$\mathcal{O}(z^{-2}) = Y_{12}^{(1)} + \frac{a_n^2}{2\pi \mathrm{i} \kappa_{n-1}^2} + \frac{Y_{12}^{(2)} - b_n Y_{12}^{(1)} + \frac{a_n^2 Y_{22}^{(1)}}{2\pi \mathrm{i} \kappa_{n-1}^2}}{z} + \mathcal{O}(z^{-2}),$$

as $z \to \infty$. Hence $Y_{12}^{(1)} = -\frac{a_n^2}{2\pi \mathrm{i} \kappa_{n-1}^2}$ and by using $Y_{21}^{(1)} = -2\pi \mathrm{i} \kappa_{n-1}^2$ we obtain the formula for a_n^2. Finally,

$$b_n = \frac{Y_{12}^{(2)}}{Y_{12}^{(1)}} + \frac{1}{Y_{12}^{(1)}} \frac{a_n^2 Y_{22}^{(1)}}{2\pi \mathrm{i} \kappa_{n-1}^2} = \frac{Y_{12}^{(2)}}{Y_{12}^{(1)}} - Y_{22}^{(1)},$$

and this is the formula for b_n.

□

8.3 Varying weights and discrete Lax Pairs

A specially interesting class of orthogonal polynomials come from the weights

$$w(x) = \mathrm{e}^{-NV(x)},$$

where $N \in \mathbb{N}$ and V is a polynomial of even degree and positive leadin coefficient (so that all monents exist). In that case, we have that the solution Y to the RHP satisfies a differential equation and, even more, a discrete Lax Pair.

Indeed, define the function Ψ by

$$\Psi_n(z) = Y_n(z) \begin{pmatrix} \mathrm{e}^{-NV(z)/2} & 0 \\ 0 & \mathrm{e}^{NV(z)/2} \end{pmatrix}.$$

Then Ψ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and has the properties

$$\begin{cases} \Psi_{n,+} = \Psi_{n,-} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \Psi_n(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n \mathrm{e}^{-NV(z)/2} & 0 \\ 0 & z^{-n} \mathrm{e}^{NV(z)/2} \end{pmatrix}. \end{cases}$$

Since the jump for Ψ does not depend on z we have that the function

$$\frac{\partial \Psi_n}{\partial z} \Psi_n^{-1} = A_n(z)$$

is entire. By computing the asymptotic behavior we find that A_n is a polynomial of degree $\deg V - 1$. Moreover, by the proof of Lemma 8.2.4 we know that

$$\Psi_{n+1}(z) = \begin{pmatrix} z - b_n & 2\pi \mathrm{i} a_n^2 \kappa_{n-1} \\ -2\pi \mathrm{i} \kappa_n^2 & 0 \end{pmatrix} \Psi_n(z) =: B_n(z) \Psi_n(z)$$

Hence we have the Lax Pair

$$\begin{cases} \frac{\partial \Psi_n}{\partial z} = A_n(z) \Psi_n(z) \\ \Psi_{n+1}(z) = B_n(z) \Psi_n(z) \end{cases} .$$

The compatibility equation reads

$$A_{n+1}(z) B_n(z) = B_n'(z) + B_n(z) A_n(z).$$

This equation gives a difference equation for the recurrence coefficients.

Example 1. *We end the lecture by claiming that for*

$$V(x) = x^4/4 + tx^2/2$$

this compatability equation (after quite a bit of work) leads to the following equation for a_n^2 (note that the fact that V is even implies $b_n = 0$ (exercise!))

$$a_n^2 (t + a_{n+1}^2 + a_n^2 + a_{n-1}^2) = \frac{n}{N}$$

This is known in the literature under various names: the Freud equation, string equation and discrete Painlevé equation. The validity of this equation can also be checked directly without the use of Lax pairs. Indeed,

$$\begin{aligned} n \|\pi_{n-1}\|_2^2 &= \int \pi_n'(x) \pi_{n-1}(x) \mathrm{e}^{-NV(x)} \mathrm{d}x \\ &= - \int \pi_n(x) \pi_{n-1}'(x) \mathrm{e}^{-NV(x)} \mathrm{d}x + N \int \pi_n(x) \pi_{n-1}(x) V'(x) \mathrm{e}^{-NV(x)} \mathrm{d}x \\ &= N \int V'(x) \pi_n(x) \pi_{n-1}(x) \mathrm{e}^{-NV(x)} \mathrm{d}x. \end{aligned}$$

and the last integral can be worked out using the recurrence relation leading to the Freud equation.

Bibliography

[1] M.J. Ablowitz and A.S. Fokas, Complex variables: introduction and applications. Second edition. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2003.

[2] K. F. Clancey and I. Gohberg, Factorization of matrix functions and singular integral operators. Operator Theory: Advances and Applications, 3. Birkhuser Verlag, Basel-Boston, Mass., 1981

[3] P.A. Deift,Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics, 3. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. viii+273 pp.

[4] A.S. Fokas, A. Its, A.A. Kapaev and V.Y. Novokshenov, Painlev Transcendents. The Riemann-Hilbert approach. Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006. xii+553 pp.

[5] I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. I. Introduction. Operator Theory: Advances and Applications, 53. Birkhuser Verlag, Basel, 1992.