Lecture 6

Painlevé II vs Airy

We start this lecture with the following RHP

RH problem 6.0.1. *Fix $s_1, \dots, s_6, \nu$ satisfying*

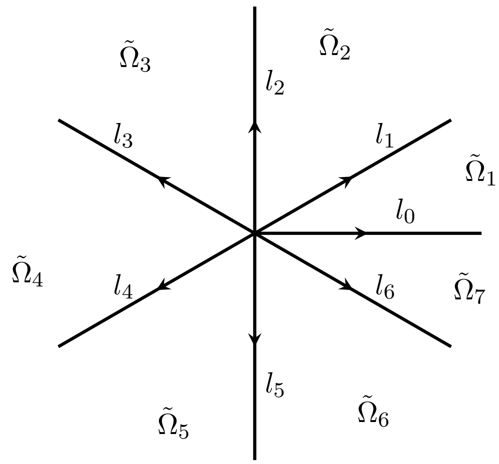
$$\begin{cases} 1 + s_1 s_2 = (1 + s_4 s_5)\mathrm{e}^{2\pi \mathrm{i}\nu}, \\ 1 + s_2 s_3 = (1 + s_5 s_6)\mathrm{e}^{-2\pi \mathrm{i}\nu}, \\ s_1 + s_3 + s_1 s_2 s_3 = -s_5 \mathrm{e}^{2\pi \mathrm{i}\nu}. \end{cases} \tag{6.0.1}$$

Search for 2×2 analytic function Ψ such that

- Ψ *is analytic in* $\mathbb{C} \setminus (\bigcup_{j=0}^{6} \tilde{l}_j)$
- $\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & 0 \\ s_{2j-1} & 1 \end{pmatrix}$ *for* $z \in l_{2j-1}$ *and* $j = 1, 2, 3$
- $\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & l_{2j} \\ 0 & 1 \end{pmatrix}$ *for* $z \in l_{2j}$ *and* $j = 1, 2, 3$
- $\Psi_+(z) = \Psi_-(z)\mathrm{e}^{-2\pi \mathrm{i}\nu\sigma_3}$
- $\Psi(z) = (I + \mathcal{O}(1/z)) \exp(-\frac{4}{3}z^3 - xz - \nu \log z)\mathrm{i}\sigma_3,$ *as* $z \to \infty$
- Ψ *is bounded near the origin.*

One may take $\nu = 0$ opun first readin. In that case the condition (6.0.1) is to ensure that the solution in each sector can be analytically extended near zero.

This RHP and its solution can be constructed out of canonical solutions the ODE with a non-Fuchsian singularity of rank 3 (which also give the



parameters, s_j, x, ν in terms of the parameters in the differential equation). However, in this lecture we take a different point of view. We start by looking at the RHP for some given s_j, x and ν and ask to re-derive the differential equation and express the parameters in the differential equation in terms of s_j, x_n and ν.

We start with the following theorem

Theorem 6.0.2. *Fix s_j and ν such that* (6.0.1). *Then there exists a countable set $X_s = \{x_j(s)\}$ so that the RHP has a solution* (6.0.1) *for every $x \in \mathbb{C} \setminus X_s$. Moreover,*

- $(z, x) \mapsto \Psi(z; x)$ *is holomorphic in* $\mathbb{C}(\bigcup_{j=0}^{6} \tilde{l}_j) \times \mathbb{C} \setminus X_s$.
- *If we denote the restriction of Ψ to $\tilde{O}mega_n$ by Ψ_n, then Ψ_n has an analytic continuation to an entire function. In particular, the jump conditions are satisfied pointwise.*
- *There exists Ψ_k such that for all $m \in \mathbb{N}$ we have*

$$\Psi(z) = \left(I + \sum_{k=1}^{m} \frac{\Psi_k}{z^k} + \mathcal{O}(z^{-m-1})\right) \exp(-\frac{4}{3}z^3 - xz - \nu \log z)\mathrm{i}\sigma_3,$$

  as $z \to \infty$. Moreover, each Ψ_k is a meromorphic function in x with poles in X_s.

Sketch of the proof. We sketch the Fredholm operator approach. A different approach can be found in [4].

Define

$$Y = \Psi \exp(\frac{4}{3}z^3 + \mathrm{i}xz + \nu \log z)\sigma_3.$$

Then for Y we find a RHP (with no jump on l_0)

RH problem 6.0.3. • *Y is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^{6} \tilde{l}_j)$*

- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & 0 \\ s_{2j-1}\mathrm{e}^{+\mathrm{i}\frac{4}{3}z^3 + \mathrm{i}xz + \mathrm{i}\nu \log z} & 1 \end{pmatrix}$ *for* $z \in l_{2j-1}$ *and* $j = 1, 2, 3$
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & l_{2j}\mathrm{e}^{-\mathrm{i}\frac{4}{3}z^3 - \mathrm{i}xz - \mathrm{i}\nu \log z} \\ 0 & 1 \end{pmatrix}$ *for* $z \in l_{2j}$ *and* $j = 1, 2, 3$
- $Y(z) = (I + \mathcal{O}(1/z))$, *as* $z \to \infty$
- *Y is bounded near the origin.*

More precisely, if we have a solution to the RHP for Y, then we have a solution for the RHP for Ψ. Write the jump condition as $Y_+ = Y_- J_Y$. Then it is not hard to check that the J_ψ satisfies the conditions in Lecture 3 on Fredholm operators. That is

- $\det J_\psi = 1$
- $J_\psi(z) - I$ decays rapidly for $z \to \infty$ along the rays.
- J_ψ is smooth away from 0
- $J_\psi - I$ is nilpotent.

Hence the fact that the solution exists and depends meromorphically on x follows from the analytic fredholm theorem.

Also the the solution to the RHP can be found in terms if a Cauchy transform as explained Since the jump is smooth, it can be shown from this representation of the solution that the pointwise limits exists and that Ψ_n is continuous in the closure of the sector $\tilde{\Omega}_n$. But then using the jump matrices it can also be analytically continued to the other sectors. Because of the cyclic condition, we end up at Ψ_n again after rotating over a full circle. There is also no pole at the origin and hence Ψ_n is entire. The expansion at infinity now also follows.

□

From the solution we can find the differential equation back (see also the Lecture notes for Lecture 4). Indeed, if Ψ is a solution then we can use the fact that $\frac{\mathrm{d}\Psi}{\mathrm{d}z}$ solves the same RHP (but with a different asymptotics). But that means that $\frac{\mathrm{d}\Psi}{\mathrm{d}z}\Psi^{-1}$ has no jumps extends to an entire function. The exact form can be found by inserting the asymptotic behavior of $\frac{\mathrm{d}\Psi}{\mathrm{d}z}$ and Ψ (see also the computations in Lecture 4) and this shows that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = (A_{-3}z^2 + A_{-2}z + A^{-1})\Psi$$

where A is a quadratic polynomial. The coefficients can be found in terms of the given data for the RHP. For future reference, we mention that we retrieve

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi = \left(-4\mathrm{i}z^2\sigma_3 + 4\mathrm{i}z\begin{pmatrix} 0 & u \\ -v & 0\end{pmatrix} + \begin{pmatrix} a & -2w \\ -2y & -a\end{pmatrix}\right)\Psi, \tag{6.0.2}$$

and

$$u = 2(\Psi_1)_{12}, \qquad v = 2(\Psi_1)_{21}.$$

6.0.1 Isomonodromy approach

Now it is clear that if we fix $s_1, \ldots, s_6$ and ν, that we have a different solution of the RHP for different values of $x \in \mathbb{C} \setminus X_s$. This means that if compute that the coefficients in the differential equation u, v, w, y, a all depend on x. Moreover, since $s_1, \ldots, s_6$ and ν form the essential monodromy data, by deforming the RHP with the parameter x we obtain different differential equations with the same essential monodromy. Hence we speak of a isomonodromic deformation.

The question rises how the parameter u, v, a, y, w depend on x. As it turns out. it is possible to give a set of equation for these parameters. This can be done as follows.

Since the jump matrices and jump contours for Ψ do not depend on x we can also derive a differential equation with respect to x and get

$$\frac{\partial}{\partial x}\Psi = B\Psi.$$

A calculation shows that B is the polynomial in z of degree 1

$$B = z\mathrm{i}\sigma_3 + \mathrm{i}\begin{pmatrix} 0 & u \\ -v & 0\end{pmatrix}$$

This means we have the following system of equations

$$\begin{cases} \frac{\partial}{\partial z}\Psi = A\Psi \\ \frac{\partial}{\partial x}\Psi = B\Psi. \end{cases}$$

We refer to this systems as the *Lax Pair* in view of the correspondence of the Lax Pairs that arise in PDE's and integrable systems. The relevance of the Lax Pair is the following. Since Ψ depends holomorphically on z and x one can show that

$$\frac{\partial^2}{\partial z \partial x}\Psi = \frac{\partial^2}{\partial x \partial z}\Psi.$$

By inserting the differential equation for $\partial/\partial z$ and $\partial/\partial x$, we obtain the *compatibility equation*

$$\frac{\partial A}{\partial x} + [A, B] = \frac{\partial B}{\partial z},$$

where $[A, B] = AB - BA$ is the commutator of A with B.

In the above setting it can be worked out (exercise!) that we have the following system of equations

$$\begin{cases} w = u_x \\ y = v_x \\ \mathrm{i}a = x + 2uv \end{cases} \tag{6.0.3}$$

and

$$\begin{cases} u_{xx} = xu + 2u^2 v \\ v_{xx} = xv + 2v^2 u. \end{cases} \tag{6.0.4}$$

Note that the last two equations determine everything. The precise solution to this equations are singled out by the precise values of $s_1, \ldots, s_6$ and ν. Concluding, if we want to deform the parameters in A such that we do no change the essential monodromy, we need the u, v, w, y, a to satisfy (6.0.3) and (6.0.4).

Now suppose we have a u, v solution to (6.0.4) and we define the other parameters according to (6.0.3). Then any solution of

$$\frac{\partial}{\partial z}\Psi = A\Psi,$$

also satisfies the

$$\frac{\partial}{\partial x}\Psi = B\Psi, \tag{6.0.5}$$

Indeed, from (6.0.3) and (6.0.4) we deduce that the compatibility equation holds, and therefore

$$\frac{\partial}{\partial z}(\frac{\partial}{\partial x}\Psi - B\Psi) = \frac{\partial A}{\partial x}\Psi + A\frac{\partial \Psi}{\partial x} - \frac{\partial B}{\partial z}\Psi - BA\Psi = A\left(\frac{\partial}{\partial x}\Psi - B\Psi\right)$$

Hence, there exists a C such that

$$\frac{\partial}{\partial x}\Psi - B\Psi = \Phi C$$

where Φ is a fundamental solution of the system. From the asymptotic behavior at infinity for solution of the equation near the singularity it follows that $C = 0$ and hence Ψ satisfies (6.0.5). But then for canonical solution Ψ_n and Ψ_{n+1} is overlapping Stokes sectors, we have for the Stokes matrix

$$\frac{\partial}{\partial x}S_n = \frac{\partial}{\partial x}\Psi_n^{-1}\Psi_{n+1} = O,$$

and hence $s_1, \dots, s_n, \nu$ do not depend on x.

We summarize the above in the following theorem.

Theorem 6.0.4. *Fix* $s_1, \dots, s_6, \nu$ *such that we have* (6.0.1). *Let* Ψ *be the solution to the RHP 6.0.1 (see also Theorem 6.0.2). Then*

$$\begin{cases} u = 2(\Psi_1)_{12}, \\ v = 2(\Psi_1)_{21}, \end{cases}$$

solve the system

$$\begin{cases} u_{xx} = xu + 2u^2 v \\ v_{xx} = xv + 2v^2 u. \end{cases} \tag{6.0.6}$$

Moreover, every solution to (6.0.6) *can be obtained in this way by choosing* $s_1, \dots, s_6$ *and* ν *appropriately. Finally,* Ψ, u *and* v *depend meromorphically on* x *with poles in* X_s.

6.0.2 Reductions

We discuss some reductions of the general parameter and show that (6.0.6) can be turned in both the Airy equation and the Painlevé II equation. In all reduction we will take

$$\nu = 0,$$

which we will assume from now on.

Reduction to Airy

If we take $v = 0$, then(6.0.6) turns into

$$u_{xx} = xu,$$

which is the Airy equation.

We claim that this corresponds to taking

$$s_1 = s_3 = s_5 = 0$$

and

$$s_2 + s_4 + s_6 = 0,$$

Indeed, in this case the RHP for Ψ has jumps that are strictly upper triangular. We leave it as an exercise (see also the lecture notes from lecture 1) that it can be check that the solution is given by $\Psi = \Phi \exp(-\sigma_3 \theta)$ with

$$\Phi(z) = \begin{pmatrix} 1 & y(z,x) \\ 0 & 1 \end{pmatrix},$$

and

$$y(z,x) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{3} \int_{l_{2j}} s_{2j} \mathrm{e}^{-\frac{8\mathrm{i}}{3}t^3 - 2\mathrm{i}xt} \frac{\mathrm{d}t}{t-z}.$$

In particular, since the solution is upper triangular, we have that $v = 0$.

Reduction to Painlevé II

Now let us take $u = v$, which means that (6.0.6) turns into

$$u_{xx} = xu + 2u^3.$$

Then there is a symmetry in A, namely

$$\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} A(-z) \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = -A(z).$$

This also implies that we have

$$\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \Psi(-z) \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \Psi(z).$$

For the Riemann-Hilbert problem this is equivalent (exercise) to

$$s_{j+3} = -s_j$$

and

$$s_2 + s_4 + s_6 = s_2 s_4 s_6.$$

An interpolating equation

Now let us take $u = \varepsilon v$ for some $\varepsilon > 0$, wich leads to

$$u_{xx} = xu + 2\varepsilon u^3.$$

Hence we interpolate between Airy $\varepsilon = 0$ and $\varepsilon = 1$. Then there is a symmetry in A namely

$$\begin{pmatrix} 0 & \frac{\mathrm{i}}{\sqrt{\varepsilon}} \\ -\mathrm{i}\sqrt{\varepsilon} & 0 \end{pmatrix} A(-z) \begin{pmatrix} 0 & \frac{\mathrm{i}}{\sqrt{\varepsilon}} \\ -\mathrm{i}\sqrt{\varepsilon} & 0 \end{pmatrix} = -A(z).$$

This also implies that we have

$$\begin{pmatrix} 0 & \frac{\mathrm{i}}{\sqrt{\varepsilon}} \\ -\mathrm{i}\sqrt{\varepsilon} & 0 \end{pmatrix} \Psi(-z) \begin{pmatrix} 0 & \frac{\mathrm{i}}{\sqrt{\varepsilon}} \\ -\mathrm{i}\sqrt{\varepsilon} & 0 \end{pmatrix} = \Psi(Z)$$

For the Riemann-Hilbert problem this is equivalent (exercise) to

$$s_1 = -\varepsilon s_4, s_3 = -\varepsilon s_6, \qquad s_5 = -\varepsilon s_2$$

and

$$s_2 + s_4 + s_6 = \varepsilon s_2 s_4 s_6.$$

Bibliography

[1] M.J. Ablowitz and A.S. Fokas, Complex variables: introduction and applications. Second edition. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2003.

[2] K. F. Clancey and I. Gohberg, Factorization of matrix functions and singular integral operators. Operator Theory: Advances and Applications, 3. Birkhuser Verlag, Basel-Boston, Mass., 1981

[3] P.A. Deift,Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics, 3. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. viii+273 pp.

[4] A.S. Fokas, A. Its, A.A. Kapaev and V.Y. Novokshenov, Painlev Transcendents. The Riemann-Hilbert approach. Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006. xii+553 pp.

[5] I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. I. Introduction. Operator Theory: Advances and Applications, 53. Birkhuser Verlag, Basel, 1992.