Lecture 3

Fredholm Operators

In this Lecture we continue the discussion form Lecture and work in the same setting (in particular, Assumption **??** apply).

We start by recalling some results about Fredholm operators.

Definition 3.0.1. A bounded operator A on a (separable) Hilbert space $\mathcal{H}$ is called *Fredholm* if there exists a bounded operator B such that $AB - I$ and $BA - I$ are compact.

Proposition 3.0.2. *An operator A is Fredholm if and only if* $\dim \operatorname{Ker} A < \infty$ *and* $\dim \operatorname{Ran} A^{\perp} < \infty$.

Definition 3.0.3. For a Fredholm operator A the *index* $\operatorname{Ind}(A) \in \mathbb{Z}$ is defined as

$$\operatorname{Ind}(A) = \dim \operatorname{Ker} A - \dim \operatorname{Ran} A^{\perp}.$$

Proposition 3.0.4. *Let A be a Fredholm operator. There exists a $\rho > 0$ such that for any X with $\|X - A\| < \rho$ we have that X is also a Fredholm and* $\operatorname{Ind}(A) = \mathrm{X}$.

Lemma 3.0.5. *Let A be a Fredholm operator with index zero. Then A is invertible if and only if A is injective.*

Back to the Riemann-Hilbert problem. In addition to $w_{\pm}$ we also define

$$\tilde{w}_{\pm} = \pm(J_{\pm}^{-1} - I).$$

Then, by assumption on J the functions $\tilde{w}_{\pm}$ are well defined, bounded, smooth and converge rapidly to zero as $z \to \infty$. Moreover, we have the identities

$$w_{-}\tilde{w}_{-} = w_{-} + \tilde{w}_{-} \text{ and } w_{+}\tilde{w}_{+} = -w_{+} - \tilde{w}_{+}.$$

Lemma 3.0.6. *Then the operator on* $\mathbb{L}_2(\Gamma)$ *defined by*

$$\mathcal{T}_w\phi = \mathcal{C}_+\left((\mathcal{C}_-\left(\phi w\right))\tilde{w}_-\right) + \mathcal{C}_-\left((\mathcal{C}_+\left(\phi w\right))\tilde{w}_+\right)$$

is compact as an operator on $\mathbb{L}_2$*.*

Proof. First, suppose that $\tilde{w}_\pm$ are rational functions with poles outside Γ. Then we claim that $\mathcal{T}_w$ is of finite rank. To this end, note that for any $z' \in \Omega_+$, we have by analyticity

$$\begin{aligned}\mathcal{C}\left((\mathcal{C}_-\left(\phi w\right))\tilde{w}_-\right)(z') &= \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{(\mathcal{C}_-\left(\phi w\right))(s)\tilde{w}_-(s)}{s-z'}\mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_-} \frac{(\mathcal{C}\left(\phi w\right))(s)\tilde{w}_-(s)}{s-z'}\mathrm{d}s,\end{aligned}$$

where Γ_- is a slight deformation of Γ that is now a finite collection of simple and closed contours in Ω_-. Since $\mathcal{C}\phi w$ is analytic in Ω_- and $\tilde{w}_-$ is rational, we can compute the integral by a residue calculus of each of the poles $\{a_t\}$ of $\tilde{w}_-$ in Ω_-. Hence

$$\mathcal{C}\left((\mathcal{C}_-\left(\phi w\right))\tilde{w}_-\right)(z') = \sum_{a_t}\frac{A_t}{a_t - z'},$$

where A_t are the residues at $s = a_t$. Therefore, by taking the boundary value at the $+$-side,

$$\mathcal{C}_+\left((\mathcal{C}_-\left(\phi w\right))\tilde{w}_-\right)(z') = \sum_{a_t}\frac{A_t}{a_t - z},$$

for $z \in \Gamma$. Since the number of poles is finite, we indeed see that $\psi \mapsto \mathcal{C}_+(\mathcal{C}_-(\phi w)\tilde{w}_-)$ is of finite rank. Of course, a similar statements holds for $\psi \mapsto \mathcal{C}_-(\mathcal{C}_+(\phi w)\tilde{w}_+)$ and hence $\mathcal{T}_w$ is of finite rank.

For general $\tilde{w}$ we remark that by smoothness and decay at infinity, we can always approximate $\tilde{w}_\pm$ by rational functions in a uniform way (apply a suitable version of Mergelyan's Theorem). Hence $\mathcal{T}_w$ is a limit of finite rank operators and hence compact. □

Theorem 3.0.7. *The operator* $I - \mathcal{C}_w$ *is Fredholm.*

Proof. We show that we have

$$I + \mathcal{T}_w = (I - \mathcal{C}_{\tilde{w}})(I - \mathcal{C}_w). \tag{3.0.1}$$

To this end, observe that

$$\begin{aligned}\mathcal{C}_{\tilde{w}}\mathcal{C}_w\phi &= \mathcal{C}_+((\mathcal{C}_w\phi)\tilde{w}_-) + \mathcal{C}_-((\mathcal{C}_w\phi)\tilde{w}_+)\\ &= \mathcal{C}_+((\mathcal{C}_+\phi w_-)\tilde{w}_-) + \mathcal{C}_+((\mathcal{C}_-\phi w_+)\tilde{w}_-) + \mathcal{C}_-((\mathcal{C}_+\phi w_-)\tilde{w}_+) + \mathcal{C}_-((\mathcal{C}_-\phi w_+)\tilde{w}_+)\end{aligned}$$

and now use $\mathcal{C}_+ = \mathcal{C}_- + I$, $\mathcal{C}_- = \mathcal{C}_+ - I$, $w_-\tilde{w}_- = w_- + \tilde{w}_-$ and $w_+\tilde{w}_+ = -w_+ - \tilde{w}_-$ to arrive at

$$\begin{aligned}\mathcal{C}_{\tilde{w}}\mathcal{C}_w\phi &= \mathcal{C}_+((\mathcal{C}_-\phi w_-)\tilde{w}_-) + \mathcal{C}_+((\mathcal{C}_-\phi w_+)\tilde{w}_-) + \mathcal{C}_-((\mathcal{C}_+\phi w_-)\tilde{w}_+)\\ &\qquad + \mathcal{C}_-((\mathcal{C}_+\phi w_+)\tilde{w}_+) + \mathcal{C}_+(\phi w_-\tilde{w}_-) - \mathcal{C}_-(\phi w_+\tilde{w}_+)\\ &= \mathcal{T}_w\phi + \mathcal{C}_+(\phi w_- + \tilde{w}_-) + \mathcal{C}_-(\phi w_+ + \tilde{w}_+)\\ &= \mathcal{T}_w\phi + \mathcal{C}_w\phi + \mathcal{C}_{\tilde{w}\phi}\end{aligned}$$

and from here (3.0.1) follows. This means that $(I - \mathcal{C}_{\tilde{w}})(I - \mathcal{C}_w) - I$ is compact. By reversing the roles of w and $\tilde{w}$ we can argue similarly and find that $(I - \mathcal{C}_w)(I - \mathcal{C}_{\tilde{w}}) - I$ is compact and hence $I - \mathcal{C}_w$ is Fredholm. □

Theorem 3.0.8. *Assume that $w_\pm$ are nilpotent. Then $I - \mathcal{C}_w$ is Fredholm with index* 0*.*

Proof. If $w_\pm$ are nilpotent, i.e. $w_\pm^{k+1} = 0$ for some k, then for any $t \in [0, 1]$ we have $(I \pm tw_\pm)^{-1} = \sum_{j=0}^{k}(-t)^j(\pm 1)^j w_\pm^j$ and $\det(1 \pm tw_\pm) = 1$. Hence, for any $t \in [0, 1]$ the above discussion applies and $I - \mathcal{C}_{tw} = I - t\mathcal{C}_w$ is a Fredholm operator. Moreover, $t \mapsto I - t\mathcal{C}_w$ is a continuous map that connect $I - \mathcal{C}_w$ to I. By Proposition 3.0.4 the index is also continuous and does not change. Clearly, the index of I is zero and therefore the same is true for $I - \mathcal{C}_w$. □

Assume that $w_\pm$ are nilpotent. Then $I - \mathcal{C}_W$ is invertible iff $I - \mathcal{C}_W$ is injective. Hence let m^h be a solution to $(I - \mathcal{C}_w)m = 0$. Then, as before, one can check that

$$Y^h(z) = \mathcal{C}(m^h w)(z),$$

is a solution to the *homogeneous* RHP

RH problem 3.0.9. *We seek a $k \times k$-matrix valued function such that*

1. *Y^h is analytic in $\mathbb{C} \setminus \Gamma$.*
2. *$Y_+^h = Y_-^h J$ on $\Gamma \setminus \Gamma_p$*
3. *$Y^h(z) = o(1)$ as $z \to \infty$.*

This means the following principle:

Corollary 3.0.10. *Assume that $w_\pm$ are nilpotent and assume that the the only solution to the homogeneuos RHP is the trivial solution $Y^h = 0$. Then there exists a solution to the inhomogoneous RHP for Y.*

Finally, we mention (without proof) the following theorem.

Theorem 3.0.11 (Analytic Fredholm Theorem)**.** *Assume that $w_\pm$ are nilpotent and that $w_\pm = w_{\pm,\zeta}$ are entire as functions of ζ. Then $I - \mathcal{C}_{w_\zeta}$ is invertible for no $\zeta \in \mathbb{C}$ or $(I - \mathcal{C}_{W_\zeta})^{-1}$ is a meromorphic function of ζ.*

Exercise 3.0.12. Show that the discussion in this section applies to the RHP for the Painlevé II equation in Lecture 1.

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