Lecture 2

Singular integral operators

In this lecture we will discuss the relation between Riemann-Hilbert problems and integral equations. In particular, this will gives a a useful criterion when a Riemann-Hilbert problem has a solution. Since the emphasize of the present course is on the use of Riemann-Hilbert problems for asymptotic analysis, we will be brief in our discussion and often only provide sketches of some of the proofs.

2.1 Integral equation: heuristics

The Riemann-Hilbert problems discussed in this course are equivalent to integral equations on Γ. Let us first proceed in a formal way, to see the point of the upcoming analysis. Write $J = I + W$ and $Y = I + X$. Then the jump condition in RHP **??** can be rewritten to

$$X_+(z) = X_-(z) + X_-(z)W(z) + W(z).$$

Moreover, $X(z) = o(1)$ as $z \to \infty$. But then, using the Cauchy Transform (i.e. Sokhotski-Plemelj formula), we can write

$$X(z) = \mathcal{C}X_-W(z) + \mathcal{C}W(z). \tag{2.1.1}$$

By taking the limiting value at the $-$-side we have

$$X_-(z) = \left(\mathcal{C}X_-W\right)_-(z) + \left(\mathcal{C}W\right)_-(z).$$

And this is a (singular) integral equation for X_-. If we can solve this equation then we can retrieve X from (2.1.1). Clearly, the above heuristic derivation needs clarifications/justifications at several points.

2.2 The Riemann-Hilbert problem revisited

We will now exclude endpoints from our discussion on RHP's. Hence we consider $\Gamma = \cup\gamma_j$ a finite union of simple closed (in $\bar{\mathbb{C}}$) curves. We allow the curves to intersect, but we assume that there are only finite number of intersection points and that each intersection is transversal. In that case we can obtain partition $\mathbb{C} \setminus \Gamma$ as $\Omega_- \cup \Omega_+$ such that $\Omega_\pm$ are disjoint and $\Gamma = \partial\Omega_\pm$. This also provides Γ with an orientation by insisting that Ω_+ is always at the left-hand side. Finally, we assume that the unbounded parts of Γ will converge to straight lines at ∞.

We recall the Riemann-Hilbert problem in standard form.

RH problem 2.2.1. *Let $k \in \mathbb{N}$, $\Gamma \subset \mathbb{C}$ be a contour and $J : \Gamma \to \mathbb{C}^{k\times k}$. Find a function $Y : \mathbb{C} \to \mathbb{C}^{k\times k}$ with the following properties*

1. *Y is analytic in $\mathbb{C} \setminus \Gamma$*
2. *$Y_+(z) = Y_-(z)J(z)$ for $z \in \Gamma$*
3. *$Y(z) = 1 + o(1)$ as $z \to \infty$.*

We will always work here with the following assumptions on J

Assumptions 2.2.2. *We assume the following on J*

- *J is smooth on each arc of $\Gamma \setminus \Gamma_p$.*
- *$J(z) - I \to 0$ rapidly as $z \to \infty$ along Γ.*
- *J is bounded.*
- $\det J = 1$.

These assumptions are stronger than necessary in the following discussion and can easily be relaxed by following the proofs. On the other hand, they are not too restrictive for our purposes and they hold in most RHP's of interest to us.

We are going to redefine what we mean with the limiting values at Γ. Let f be an analytic function on $\mathbb{C} \setminus \Gamma$ and $f_\pm \in \mathbb{L}_2(\Gamma, |ds|)$. Let $z \in \Gamma \setminus \Gamma_p$ and $\{\zeta(t) | -t_0 < t < t_0\}$ be a parametrization of a small part of the arc around $\zeta(0) = z$. Set $\hat{\zeta}_0 = \zeta'(0)/|\zeta'(0)$ be the tangent to the arc at z. The for sufficiently small ε we have that $\{\zeta(t) \pm \mathrm{i}\varepsilon\hat{\zeta}_0 | -t_0 < t < t_0\}$ are at the $\pm$-side of the arc. Then we say that

$$\lim_{\substack{z' \to z \\ z \text{ at } \pm\text{-side}}} f(z') = f_\pm(z), \tag{2.2.1}$$

iff, for some $t_0 > 0$, we have

$$\lim_{\varepsilon \downarrow 0} \int_{-t_0}^{t_0} |f(\zeta(t) \pm \mathrm{i}\varepsilon\hat{\zeta}_0) - f_\pm(\zeta(t))|^2 |\mathrm{d}\zeta(t)| = 0.$$

The jump condition in the RHP 2.2.1 has to be understood in this sense.

To ensure uniqueness we need to put certain conditions on the behavior of the solution of the RHP near the points of intersection. There are various way of doing this. One way is the following, taken from [3].

Let $z \in \Gamma_p$ and consider first the limiting value in one of the components say $\tilde{\Omega}$ that has z as a (irregular) boundary point. Then fix v such that $z + \varepsilon \mathrm{i} v \in \tilde{\Omega}$ and as $\varepsilon \downarrow 0$ approaches z non-tangential to $\partial\tilde{\Omega}$. Let $\{\zeta(t) = z + tv + \mathrm{i}vh(t) \mid -t_0 < t < t_0\}$ be a parametrization of part of $\partial\tilde{\Omega}$ containing $z = \zeta(0)$. Here h is piecewise real analytic function with $h(0) = 0$. Then we say that $f(z')$ converges as $z' \to z$ in $\tilde{\Omega}$ iff $f(z + tv + \mathrm{i}vh(t)(1+\varepsilon))$ converges as $\varepsilon \to 0$ in $\mathbb{L}_2((-t_0, t_0))$. Then we say that $f(z')$ converges as $z' \to z$ if this holds in any component $\tilde{\Omega}$ that has z as a boundary point.

In the RHP we pose the extra condition that $Y(z')$ converges as $z' \to z$ at every point of intersection.

These technical definitions are chosen such that we have the following.

Theorem 2.2.3. *If $n = 2$, then the solution to the RHP is unique, if it exists.*

Proof. See [3, Th. 7.18] □

2.3 Boundary values of the Cauchy transform

Proposition 2.3.1. *Let $f \in \mathbb{L}_2(\Gamma)$. Then there exists $\mathcal{C}_\pm f \in \mathbb{L}_2(\Gamma)$ such that*

$$\lim_{\substack{z' \to z \\ z \text{ at } \pm\text{-side}}} \mathcal{C}f(z') = \mathcal{C}_\pm f_\pm(z), \tag{2.3.1}$$

Moreover, the maps $\mathcal{C}_\pm : f \mapsto \mathcal{C}_\pm f$ are bounded linear operators on $\mathbb{L}_2(\Gamma)$. Finally, $\mathcal{C}_+ - \mathcal{C}_- = I$ and $\mathcal{C}_+\mathcal{C}_- = \mathcal{C}_-\mathcal{C}_+ = 0$, i.e. $\mathcal{C}_+$ and $\mathcal{C}_-$ are complementary projections.

Sketch of the proof. The fact that $\mathcal{C}_\pm$ are well-defined and bounded operators on $\mathbb{L}_2((\Gamma)$ is a fundamental result that, for arbitrary Γ require significant technical effort. The main message (i.e. the fact that the singularity in the integral when approaching the contour Γ still gives a bounded operator) can be more easily seen in the special case $\Gamma = \mathbb{R}$. Since Γ is smooth away from intersection points, this special case is representative for the general situation (although the extension to the general situation still requires a non-trivial effort).

So let us assume that $\Gamma = \mathbb{R}$. In that case, we have

$$\begin{aligned}\mathcal{C}f(x \pm \mathrm{i}\varepsilon) &= \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(y)}{y - x \mp \mathrm{i}\varepsilon} \mathrm{d}y \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} f(y) \frac{y - x \pm \mathrm{i}\varepsilon}{(y-x)^2 + \varepsilon^2} \mathrm{d}y \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} f(y) \frac{y - x}{(y-x)^2 + \varepsilon^2} \mathrm{d}y \pm \frac{1}{2\pi} \int_{\mathbb{R}} f(y) \frac{\varepsilon}{(y-x)^2 + \varepsilon^2} \mathrm{d}y\end{aligned}$$

Now $y \mapsto \frac{1}{\pi} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}$ converges to the δ-function at x, so the second integral converges back to $\frac{1}{2} f$ (in $\mathbb{L}_2$) sense. The first integral also converges. Indeed, by taking the Fourier transform, we see that the limit converges in $\mathbb{L}_2$ to the inverse Fourier transform (up to a multiple of $\frac{-1}{2i}$) of

$$\omega \mapsto -\frac{\mathrm{i}}{2} \mathrm{sign}(\omega) \hat{f}(\omega),$$

where $\hat{f}$ is the Fourier transform of f. This is known in the literature as the Hilbert transform

$$Hf(x) = \frac{1}{\pi} P.V. \int \frac{f(y)}{x - y} \mathrm{d}y := \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{|y-x| \geq \delta} \frac{f(y)}{x - y} \mathrm{d}y. \tag{2.3.2}$$

Since in the Fourier domain it is a multiplication operator with a unimodular function, it is a bounded operator on $\mathbb{L}_2(\mathbb{R})$. Concluding

$$\mathcal{C}_{\pm} f = \pm \frac{1}{2} f - \frac{1}{2\mathrm{i}} Hf,$$

and we see that both $\mathcal{C}_{\pm}$ are bounded. Also the identities $\mathcal{C}_+ - \mathcal{C}_- = I$ and $\mathcal{C}_+ \mathcal{C}_- = \mathcal{C}_- \mathcal{C}_+ = 0$ are easily verified (using $H^2 = -I$).

We will not discuss the case of general contours Γ here. But only mention that the Hilbert transform can also be defined on $\mathbb{L}_2(\Gamma)$ using the Cauchy Principal value integral and this defines a bounded operator (see [5, Th. 3.1]). Also, the Cauchy transform can be shown to converge at every point of intersection in the sense that is given in the beginning of the section.

Finally, in an exercise in the previous set of lecture notes we proved that, for general contours Γ, we have $\mathcal{C}_+ f(z) - \mathcal{C}_- f(z) = f(z)$ if f is analytic at $z \in \Gamma \setminus \Gamma_p$. By using a continuity argument one can now show that $\mathcal{C}_+ f - \mathcal{C}_- f = f$ for general $f \in \mathbb{L}_2$. The fact that $\mathcal{C}_+\mathcal{C}_- = \mathcal{C}_-\mathcal{C}_+ = O$, can be seen as follows: let $f \in \mathbb{L}_2$. Then $\mathcal{C}_- f$ is the boundary value (in $\mathbb{L}_2$-sense) of a function that is analytic in Ω_-. If $z \in \Omega_+$, the we can deform the integral in

$$\mathcal{C}(\mathcal{C}_- f)(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathcal{C}_- f(y)}{y - z} \mathrm{d}y,$$

into a union of closed simple contours Γ_-, with each contour in a connected component of Ω_- and write

$$\mathcal{C}(\mathcal{C}_- f)(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_-} \frac{\mathcal{C} f(y)}{y - z} \mathrm{d}y.$$

By analyticity of $\mathcal{C} f$ in Ω_-, the integral vanishes. [1] $\square$

One can prove that for any $f \in \mathbb{L}_2(\Gamma)$ the non-tangential point wise limit $\mathcal{C}_\pm f(z)$ exist almost everywhere. If we assume some regularity we have a stronger statement.

Proposition 2.3.2. *Suppose that $f \in C^1$ and $f, f' \in \mathbb{L}_2(\Gamma)$. Then for all $z \in \Gamma \setminus \Gamma_p$ the pointwise limits $\mathcal{C}_\pm f(z)$ exist and $f(z) = \mathcal{C}_+ f(z) - \mathcal{C}_- f(z)$. Moreover, $\mathcal{C} f(z) \to 0$ as $z \to \infty$ uniformly in $\bar{\Omega}$ for any unbounded component Ω of $\mathbb{C} \setminus \Gamma$.*

Exercise 2.3.3. Consider the unit circle $\Gamma = \{|z| = 1\}$ and equip it with anti-clockwise orientation, so that $\Omega_+ = \{|z| < 1\}$ and $\Omega_- = \{|z| > 1\}$.

Let $f \in \mathbb{L}_2(\Gamma)$ and write f in a Fourier series $f(z) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k$. Show that $\mathcal{C}_+ f = \sum_{k=0}^{\infty} \hat{f}_k z^k$ and $\mathcal{C}_- f = -\sum_{-\infty}^{k=-1} \hat{f}_k z^k$, to conclude that $C_\pm$ are bounded, $\mathcal{C}_+ - \mathcal{C}_- = I$ and $\mathcal{C}_+\mathcal{C}_- = \mathcal{C}_-\mathcal{C}_+ = O$.

2.4 Solving the RHP

Assume that J can be factorized

$$J = J_-^{-1} J_+,$$

where $J_\pm$ are invertible, bounded and satisfy the Assumption 2.2.2. Then set

$$w_\pm = \pm(J_\pm - I).$$

[1] Exercise: use a similar procedure to compute $\mathcal{C}_-\mathcal{C}_-$. Also note that our assumptions on Γ in the beginning of this second lecture are crucial for this argument.

and

$$w = w_+ + w_-.$$

Now we define the operator

$$\mathcal{C}_w f = \mathcal{C}_+ \left(f w_- \right) + \mathcal{C}_- \left(f w_+ \right).$$

Since $w_\pm$ are bounded, the operator $\mathcal{C}_w$ is a bounded operator on $\mathbb{L}_2(\Gamma)$.

Theorem 2.4.1. *Suppose that $I - \mathcal{C}_w$ is invertible on $\mathbb{L}_2(\Gamma)$. Let $m \in \mathbb{L}_2(\Gamma)$ be the solution to*

$$(I - \mathcal{C}_w) m = C_+ w_- + C_- w_+.$$

Then

$$Y(z) = I + \mathcal{C} \left((I + m) w \right)(z)$$

solve the RHP 2.2.1.

Proof. (See also [3, Th. 7.103]. Here we only check the jump conditions. By using $w = w_+ + w_-$, $\mathcal{C}_+ - \mathcal{C}_- = I$ and the definition of m we find

$$\begin{aligned} Y_+ &= I + \mathcal{C}_+(mw) + \mathcal{C}_+ w \\ &= I + \mathcal{C}_+(mw_+) + \mathcal{C}_+(mw_-) + \mathcal{C}_+ w_+ + \mathcal{C}_+ w_- \\ &= I + \mathcal{C}_-(mw_+) + \mathcal{C}_+(mw_-) + \mathcal{C}_- w_+ + \mathcal{C}_+ w_- + m w_+ + w_+ \\ &= I + \mathcal{C}_w m + \mathcal{C}_- w_+ + \mathcal{C}_+ w + m w_+ \\ &= I + m + m w_+ + w_+ = (I + m)(I + w_+). \end{aligned}$$

In the same way one can prove

$$Y_- = (I + m)(I - w_-).$$

Therefore

$$Y_+ = Y_- (I - w_-)^{-1} (1 + w_+) = Y_- J,$$

and hence the jump follows. $\square$

Remark 2.4.2. An important situation for which we know that $1 - \mathcal{C}_w$ is invertible, is when $\|\mathcal{C}_w\|_\infty < 1$. In that case the inverse can be given by a Neumann series. For this to happen, we need $\|w_\pm\|_\infty$ to be small. In particular, we see that when the jump matrices are a small perturbation of the identity then the Riemann-Hilbert problem can be solved by a Neumann series. This is an important observation when we deal with asymptotics.

Remark 2.4.3. The operator C_w depends on the factorization $J = J_-^{-1}J_+$. For each different factorization we obtain a different operator for the RHP. But we arrive a the same solution. (As it is unique). The freedom in the choice of factorization turns out to be useful in certain cases.

Proposition 2.4.4. *Consider $J^{(\infty)}$ and a sequence $J^{(n)}$ for $n \in \mathbb{N}$ and assume that $w_\pm^{(n)} \to w_\pm^{(n)}$ as $n \to \infty$ both in $\mathbb{L}_2$ and $\mathbb{L}_\infty$. Assume that $I - \mathcal{C}_{w^{(\infty)}}$ is invertible. Then $I - \mathcal{C}_{w^{(n)}}$ for sufficiently large n and $Y^{(n)}(z) \to Y^{(\infty)}(z)$ uniformly on compact subsets of $\mathbb{C} \setminus \Gamma$.*

Proof. The map $(w_+, w_-) \mapsto \mathcal{C}_w$ is linear and $\|\mathcal{C}_w\| \leq c(\|w_+\|_\infty + + \|w_-\|_\infty)$ for some $c > 0$. But then by assumption that $w_\to^{(n)} \to w^{(\infty)}$ in $\mathbb{L}_\infty$ we have

$$(I - \mathcal{C}_{w^{(n)}})^{-1} = \sum_{j=0}^{\infty} (I - \mathcal{C}_{w^{(\infty)}})^{-j} (\mathcal{C}_{w^{(n)} - w^{(\infty)}})^j (I - \mathcal{C}_{w^{(\infty)}})^{-1},$$

and the series converges for large enough n.

Moreover, by the assumption that $w_\to^{(n)} \to w^{(\infty)}$ both in $\mathbb{L}_\infty$ and $\mathbb{L}_2$

$$m^{(n)} = (I - \mathcal{C}_{w^{(n)}})^{-1} \left(C_+ w_-^{(n)} + C_- w_+^{(n)} \right) \to (I - \mathcal{C}_{w^{(\infty)}})^{-1} \left(C_+ w_-^{(\infty)} + C_- w_+^{(\infty)} \right) = m^{(\infty)},$$

as $n \to \infty$. Finally, for the same reasons,

$$\begin{aligned} Y^{(n)}(z) &= I + \mathcal{C}\left((I + m^{(n)}) w^{(n)} \right)(z) \\ &\to I + \mathcal{C}\left((I + m^{(\infty)}) w^{(\infty)} \right)(z) = Y^{(\infty)}(z), \end{aligned} \tag{2.4.1}$$

as $n \to \infty$. Moreover, the converge is uniform for z in compact subset of $\mathbb{C} \setminus \Gamma$. $\square$

Bibliography

[1] M.J. Ablowitz and A.S. Fokas, Complex variables: introduction and applications. Second edition. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2003.

[2] K. F. Clancey and I. Gohberg, Factorization of matrix functions and singular integral operators. Operator Theory: Advances and Applications, 3. Birkhuser Verlag, Basel-Boston, Mass., 1981

[3] P.A. Deift,Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics, 3. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. viii+273 pp.

[4] A.S. Fokas, A. Its, A.A. Kapaev and V.Y. Novokshenov, Painlev Transcendents. The Riemann-Hilbert approach. Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006. xii+553 pp.

[5] I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. I. Introduction. Operator Theory: Advances and Applications, 53. Birkhuser Verlag, Basel, 1992.