Lecture 10

Deift/Zhou steepest descent, Part II

We continue the analysis from the last lecture.

10.1 Construction of the global parametrix

By taking the pointwise limit of the jump matrices for S as $n \to \infty$, we obtain the following Riemann-Hilbert problem

RH problem 10.1.1. *We seek for a function $P_\infty : \mathbb{C} \setminus [-a, a] \to \mathbb{C}^{2\times 2}$ such that*

- *P_∞ is analytic in $\mathbb{C} \setminus [-a, a]$.*
- *$P_{\infty,+}(x) = P_{\infty,-}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $x \in [-a, a]$.*
- *$P_\infty(z) = I + o(1)$ as $z \to \infty$.*

Lemma 10.1.2. *The RHP 10.1.1 admits the following solution*

$$P_\infty(z) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \left(\frac{z+a}{z-a}\right)^{1/4} & 0 \\ 0 & \left(\frac{z-a}{z+a}\right)^{1/4} \end{pmatrix} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix} \tag{10.1.1}$$

for $z \in \mathbb{C} \setminus [-a, a]$. Here the quartic roots are taking such that $z \mapsto \left(\frac{z+a}{z-a}\right)^{1/4}$ and $z \mapsto \left(\frac{z-a}{z+a}\right)^{1/4}$ are analytic in $\mathbb{C} \setminus [-a, a]$ and positive on (a, ∞).

Proof. The proof follows by direct verification, but we will give a more constructive argument. We note that the jump matrix in the RHP can be diagonalized as follows

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}.$$

Hence, by defining

$$D(z) = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix} P_\infty(z) \begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix},$$

then D has the properties

$$\begin{cases} D_+ = D_- \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, & \text{on } (-a,a) \\ D(z) = I + o(1), & z \to \infty. \end{cases}$$

This problem is easily seen to have a diagonal solution $D = \operatorname{diag}(d_1, d_2)$, where

$$\begin{cases} d_{1,+} = -\mathrm{i} d_{1,-}, & \text{on } (-a,a) \\ d_1(z) = 1 + o(1), & z \to \infty. \end{cases}$$

and

$$\begin{cases} d_{2,+} = \mathrm{i} d_{2,-}, & \text{on } (-a,a) \\ d_2(z) = 1 + o(1), & z \to \infty. \end{cases}$$

Now note that if we define $t : z \mapsto z^{1/4}$ such that t is analytic in $\mathbb{C} \setminus (\infty, 0]$ and positive on $(0, \infty)$ then $t_+ = \mathrm{i} t_-$. Hence we can solve the scalar RHP's for d_1 and d_2 by

$$d_1 = \left(\frac{z+a}{z-a}\right)^{1/4}, \qquad d_2 = \left(\frac{z-a}{z+a}\right)^{1/4}.$$

By expressing P_∞ in terms of D we obtain the solution in the lemma. $\square$

10.2 A first try

As $n \to \infty$ we expect that S is close to P_∞. The reason for this is the following. If we consider $\tilde{R} = S P_\infty^{-1}$ then R satisfies the RHP

RH problem 10.2.1. *We seek for a function $\tilde{R} : \mathbb{C} \setminus \Sigma_{\tilde{R}} \to \mathbb{C}^{2\times 2}$ such that*

- R *is analytic in* $\mathbb{C} \setminus [-a,a]$.
- $\tilde{R}_+(x) = \tilde{R}_- J_{\tilde{R}}(x)$, *for* $x \in J_{\tilde{R}} = J_S \setminus [-a,a]$
- $\tilde{R}(z) = I + o(1)$ *as* $z \to \infty$.

and $J_{\tilde{R}} = P_\infty J_S P_\infty^{-1}$ for $J_{\tilde{R}}$. Since J_S converges I exponentially fast pointwise to at $J_{\tilde{R}}$ we have

$$J_{\tilde{R}} \to I,$$

as $n \to \infty$ pointwise at every point of $J_{\tilde{R}}$. Hence we would naively expect that $\tilde{R} \to I$ as $n \to \infty$.

However, pointwise convergence is not sufficient for this conclusion! We need uniform convergence. But we do no have uniform convergence near the points $z = \pm a$. Indeed, $\phi(a) = 0$ and therefore $\mathrm{e}^{\pm n\phi(a)} = 1$. As we will see, this is not just a technical issue. The function P_∞ in (10.1.1) is not a good approximation near $\pm a$, but only a good approximation away from these points. Hence we will find an alternative approximation near $\pm$, called local parametrices, and this is what we will do next.

10.3 Construction of the local parametrices

Let $U_\pm$ be a small neighborhoods around $z = \pm a$. By symmetry, we will take $U_{-a} = -U_a$. Then we want the to construct solutions $P_\pm$ such that

1. $P_{\pm a}$ has the exactly the same jump conditions as S in $U_{\pm a}$
2. $P_{\pm a}$ satisfies the *matching condition* on the boundary

$$P_{\pm a}(z) = P_\infty(z)(I + \mathcal{O}(1/n)), \text{ as } n \to \infty$$

uniformly for $z \in \partial U_\pm$.

We will construct such solutions.

Note that locally we have

$$\phi(z) = c(z-a)^{3/2}(1 + \mathcal{O}(z-a)), \qquad z \to a,$$

for some positive constant $c > 0$. We start by posing a *model RHP*, containing all the essential local information of $P_{\pm a}$.

RH problem 10.3.1. *We seek for a function* $\Phi : \mathbb{C} \setminus \Sigma_\Phi \to \mathbb{C}^{2\times 2}$ *such that*

- Φ *is analytic in* $\mathbb{C} \setminus \Sigma_\Phi$.

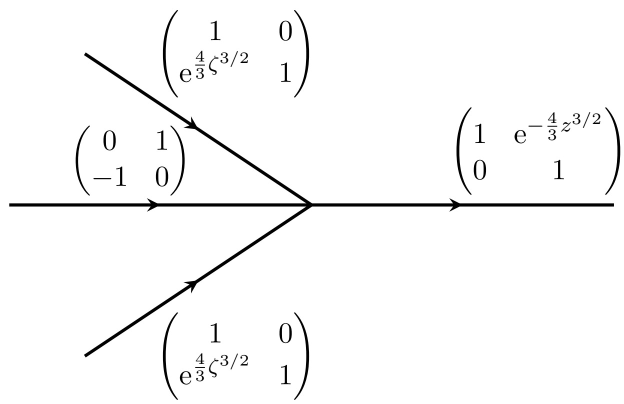

Figure 10.1: Jump contours and matrix for Φ

- $\Phi_+(\zeta) = \Phi_-(\zeta)J_\Phi$ *for* $\zeta \in \Sigma_\zeta$.
- $\Phi(\zeta) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}\left(I + \mathcal{O}(\zeta^{-3/2})\right)$ *as* $\zeta \to \infty$.
- Φ *bounded near the origin.*

We will construct an explicit solution to this RHP. To this end, we will derive an ODE for Φ as we have done before.

First, we are going to transfer this into a constant jump RHP by define

$$\Psi(\zeta) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}\Phi(\zeta)\begin{pmatrix} e^{-\frac{2}{3}\zeta^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}\zeta^{3/2}} \end{pmatrix}.$$

RH problem 10.3.2. *We seek for a function* $\Psi : \mathbb{C} \setminus \Sigma_\Psi \to \mathbb{C}^{2\times 2}$ *such that*

- Ψ *is analytic in* $\mathbb{C} \setminus \Sigma_\Psi(\zeta)$.
- $\Psi_+(\zeta) = \Psi_-(\zeta)J_\Psi$ *for* $\zeta \in \Sigma_\Psi$.
- $\Psi(\zeta) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}\left(I + \mathcal{O}(\zeta^{-3/2})\right)\begin{pmatrix} e^{-\frac{2}{3}\zeta^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}\zeta^{3/2}} \end{pmatrix}$ *as* $\zeta \to \infty$.
- Ψ *bounded near the origin.*

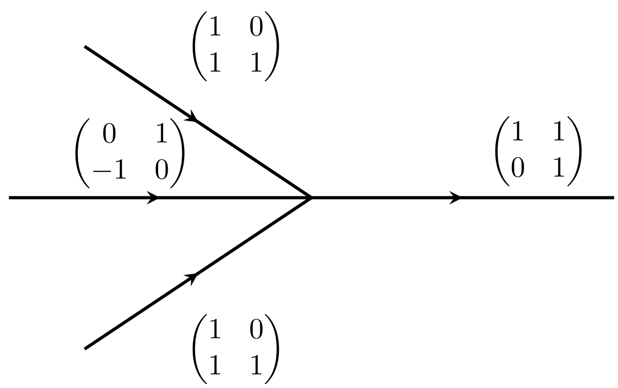

Figure 10.2: Jump contours and matrix for Ψ

Then, we employ our usual strategy for finding an ODE for Ψ and, after some computation (exercise!), this gives

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\zeta} = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} \Psi(\zeta).$$

But that means that entries of Ψ satisfy

$$\begin{cases} \Psi'_{1j}(\zeta) = \Psi_{2j}(\zeta), \\ \Psi'_{2j}(\zeta) = \zeta\Psi_{1j}(\zeta). \end{cases}$$

Hence, in particular we have,

$$\Psi''_{1j}(\zeta) = \zeta\Psi_{1j}\zeta).$$

which means that the entries in the first row of Ψ are solution to the Airy equation! Hence, we are searching for a solution of the model RHP problem in terms of Airy functions.

Let us now define $\omega = \mathrm{e}^{2\pi \mathrm{i}3}$ and

$$y_j(x) = \omega^j \operatorname{Ai}(\omega_j x), \qquad , j = 0, 1, 2.$$

Then each y_0, y_1 and y_2 are solutions to the Airy equation. They satisfy the relation

$$y_0 + y_1 + y_2 = 0.$$

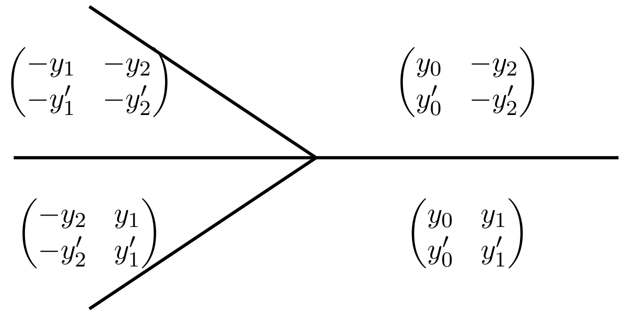

Figure 10.3: Solution for Ψ

Moreover, classical analysis on the Airy function gives

$$y_0(\zeta) = \frac{1}{2\sqrt{\pi}}\zeta^{-1/4}\left(1+\mathcal{O}(\zeta^{-3/2})\right)\mathrm{e}^{-\frac{2}{3}\zeta^{3/2}},$$

as $\zeta \to \infty$. Moreover, for any $\varepsilon > 0$ the order is uniform or $-\pi+\varepsilon < \arg\zeta < \pi-\varepsilon$.

Then it follows after some computation (exercise) that if we choose Ψ according to Figure 10.3 then Ψ solve the RHP 10.3.2. Hence,

$$\Phi(\zeta) = \frac{\sqrt{\pi}}{2}\begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix}\Psi(\zeta)\begin{pmatrix} \mathrm{e}^{\frac{2}{3}\zeta^{3/2}} & 0 \\ 0 & \mathrm{e}^{-\frac{2}{3}\zeta^{3/2}} \end{pmatrix},$$

solves the model RHP 10.3.1.

In the next step we are going map this model RHP onto neighborhoods $U_{\pm a}$ and construct our local solution $P_{\pm a}$. We will start with $z = +a$.

We use the conformal map

$$\beta(z) = (\frac{3}{4}\phi(z))^{3/2}.$$

Here the fractional power is taken such that $\beta(z) = c(z-a)(1+\mathcal{O}(z-a)))$ as $z \to \pm a$. Strictly speaking, β is only defined with a cut at the left of a but it can be continued to be an analytic function in a sufficiently small neighborhood of a. In fact, this neighborhood can be take sufficiently small so that β is conformal. We then choose

$$P_a(z) = E_n(z)\Phi(n^{2/3}\beta(z)),$$

Here E_n is an analytic function that we specify in a moment. By choosing the lips of the lens so that they match with the jump contour of Φ (check that this can done!) and the fact that E_n (and hence has no effect on the jump structure), we find that P_a o indeed solves the RHP inside U_a. We use the freedom in E_n to ensure that we have the matching condition.

We define

$$E_n(z) = \frac{1}{2} P_\infty(z) \begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} n^{1/6}\beta(z)^{1/4} & 0 \\ 0 & n^{-1/6}\beta(z)^{-1/4} \end{pmatrix} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}$$

Then we claim that E_n is (or better, extends) to an analytic functions in U_a. It is important to note that the fractional powers cancel against each other and there is no branching around a (check!). By invoking the asymptotic behavior of Φ from the RHP 10.3.1 we see that

$$P_a(z) = E_n(z)\Phi(n^{2/3}\beta(z)) = P_\infty(z)(I + \mathcal{O}(1/n),$$

uniformly for $\in \partial U_a$ which is the matching condition.

This finishes the contruction of P_a. The construction of P_{-a} goes similar. BUt we can also exploit the symmetry and define (check!)

$$P_{-a}(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_a(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

10.4 Final transformation $S \mapsto R$

We now define

$$R(z) = \begin{cases} S(z)P_\infty(z)^{-1}, & z \in \mathbb{C} \setminus (U_a \cup U_{-a} \cup \Sigma_S) \\ S(z)P_a(z)^{-1}, & z \in U_a \setminus \Sigma_S \\ S(z)P_{-a}(z)^{-1}, & z \in U_{-a} \setminus \Sigma_S. \end{cases}$$

Then R satisfies the following RHP

RH problem 10.4.1. *We seek for a function $R : \mathbb{C} \setminus [-a, a] \to \mathbb{C}^{2\times 2}$ such that*

- *R is analytic in $\mathbb{C} \setminus [-a, a]$.*
- *$R_+(x) = R_- J_R(x)$, for $x \in J_R$*
- *$(z) = I + o(1)$ as $z \to \infty$.*

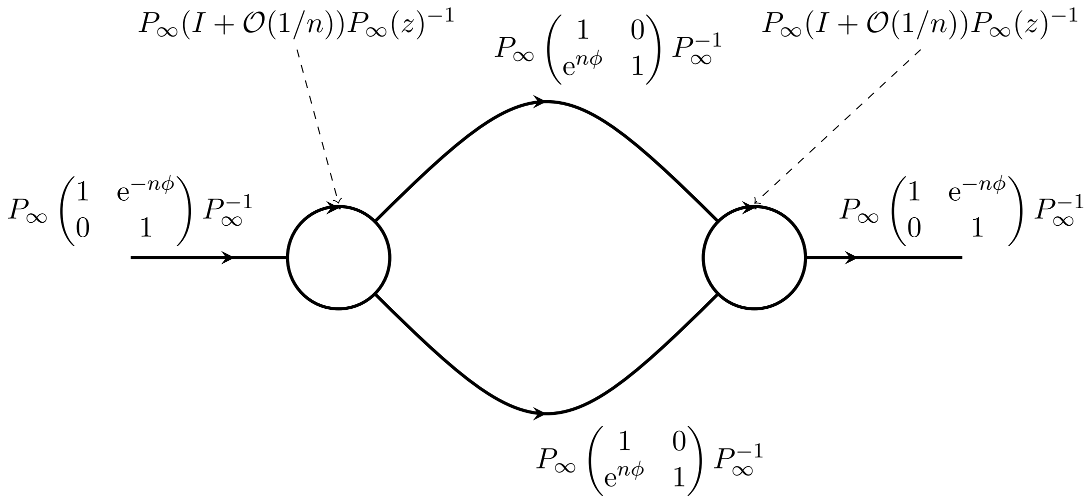

Figure 10.4: Jumps for R

The jump contour Σ_R and the jumps J_R are indicated in Figure 10.4. Now we have

$$\|J_R - I\|_{\infty,2} = \mathcal{O}(1/n),$$

as $n \to \infty$. That means that we have

$$R(z) = I + \mathcal{O}(1/n),$$

as $n \to \infty$, uniformly for z in compact subsets of $\mathbb{C} \setminus \Sigma_R$.

Note that this is only one conclusion for R, In fact, we have a series expansiion for R as discussed in lectures 2 and 3.

10.5 Conclusions

We now follow the transformation $Y \mapsto T \mapsto S \mapsto R$ and obtain the asymptotic behavior for the orthogonal polynomials and their features. At this point, these just boil down to simple computations. We will present a full presentation of the various asymptotic results we can obtain, but leave this as a very usefull exercise to the reader and encourage to do all the exercises below.

For example, for the polynomial $\phi_{n,n}(z)$, with $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\pi_{n,n}(z) = Y_{11}(z) = T_{11}(z)e^{ng(z)}$$

The unfolding of the next transformation $T \mapsto S$ depends on the location in the plane we are looking. Suppose that z is away from the lens. In fact, we still have some freedom inour choice of the lips of the lenses, hence the followin argument works for z in compact subset of $\mathbb{C} \setminus \mathbb{R}$ (check!).

$$\begin{aligned}
\pi_{n,n}(z) &= S_{11}(z)\mathrm{e}^{ng(z)} \\
&= (RP_\infty)_{11}\mathrm{e}^{ng(z)} \\
&= (P_\infty)_{11}\mathrm{e}^{ng(z)}(1 + \mathcal{O}(1/n)) \\
&= \frac{1}{2}\left(\left(\frac{z-a}{z+a}\right)^{1/4} + \left(\frac{z+a}{z-a}\right)^{1/4}\right)\mathrm{e}^{ng(z)}(1 + \mathcal{O}(1/n))
\end{aligned}$$

as $n \to \infty$, uniformly for z in compact subsets $\mathbb{C} \setminus \mathbb{R}$. Note that this proves Theorem 1.1.1 from last lecture (check!).

A similar analysis proves the following

Lemma 10.5.1. $\lim_{n\to\infty} a_{n,n} = a^2/4$

Proof. Exercise. □

Exercise 10.5.2. Compute the asymptotic behavior of

- $\pi_n(x)$ for x in compact subset of $[-a, a]$.
- $\pi_n(x)$ for x near $\pm a$
- $K_n(x, x)$ for $x \in \mathbb{R}$
- $K_n(x_0 + x/n, x_0 + y/n)$ for $x_0 \in (-a, a)$ and x, y in compact subsets of $\mathbb{R}$.

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