Lecture 1

Introduction

We start these lecture notes with an *informal* discussion on Riemann-Hilbert problems.

1.1 What is a Riemann-Hilbert problem?

Roughly speaking a Riemann-Hilbert problem is the following. Suppose we have a (matrix-valued) function in the complex plane that is analytic except at a some given contour. When passing this contour the function makes a jump. Suppose that we know the contour and the jump explicitly. To what extend can we reconstruct the function from these data?

Let us consider a finite union of analytic arcs $\Gamma \subset \mathbb{C}$. The arcs may have endpoints and they might intersect The collection of endpoints and points of intersection will be denoted by Γ_p. For simplicity, we will assume that such intersection is always transversal. We then equip each arc with an orientation. When traveling along the contour in positive directions we call the right-hand side the $-$-side and the left-hand side the $+$-side. Note that this assignment is local and that the $+$-side for one arc, may be the minus-side of another arc. Examples of contours are the real line $\Gamma = \mathbb{R}$, the unit circle $\Gamma = \mathbb{T}$ or the combination $\Gamma = \mathbb{R} \cup \mathbb{T}$. See also Figure 1.1.

With these assumptions we define for a function Y on $\mathbb{C} \setminus \Gamma$ its limiting values by

$$Y_\pm(z) = \lim_{\substack{z' \to z \\ z' \text{ at } \pm\text{-side}}} Y(z'),$$

for $z \in \Gamma$, provided that the limit exists. The following is a standard form of a Riemann-Hilbert problem that we will discuss in this course.

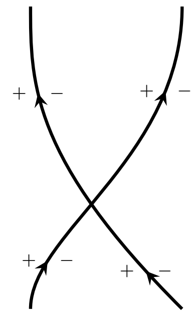

Figure 1.1: An example of a configuration of jump contours.

RH problem 1.1.1. *Let $k \in \mathbb{N}$, $\Gamma \subset \mathbb{C}$ be a contour and $J : \Gamma \to \mathbb{C}^{k\times k}$. Find a function $Y : \mathbb{C} \to \mathbb{C}^{k\times k}$ with the following properties*

1. *Y is analytic in $\mathbb{C} \setminus \Gamma$*
2. *$Y_+(z) = Y_-(z)J(z)$ for $z \in \Gamma \setminus \Gamma_p$*
3. *$Y(z) = 1 + o(1)$ as $z \to \infty$.*

Note that part of the Riemann-Hilbert problem is that the limiting values $Y_\pm$ exist. For now we will always assume that J is sufficiently smooth. It is perfectly fine that in the first reading the reader assumes J is an entire function. Later we will also assume that J is in $\mathbb{L}_2(\Gamma)$ and the jump condition has to be understood to hold in $\mathbb{L}_2$ sense. We will discuss a precise definition later.

1.1.1 Existence and uniqueness

At this point the reader may wonder about the existence and uniqueness for such a problem. In this course, we will mostly start with a Riemann-Hilbert problem for which the uniques solution can be characterized explicitly and this is not an important issue at the start. But, afterwards we will manipulate the Riemann-Hilbert problem so that it is easy to deduce the (asymptotic) behavior of the solution when changing the parameters involved. It is important that we have a thorough understanding of what a correctly stated Riemann-Hilbert problem should look like. We will briefly comment on this here.

The existence of a solution is typically non-trivial. It is usually proved by an explicit construction of the solution or by general principles, some of which we will discuss later.

The uniqueness of the solution is often much easier to show. If $\det J = 1$, this is guaranteed by the condition at infinity and choosing appropriate conditions near points of self-intersection of Γ.

Let us assume that $\Gamma_p = \emptyset$ for the moment. In that case we see that for any solution Y we have

$$\det Y_+(z) = \det\left(Y_-(z)J(z)\right) = \det Y_-(z),$$

for $z \in \Gamma$ and hence $\det Y$ is an *entire* function satisfying $\det Y(z) = 1 + o(1)$ as $z \to \infty$. By Liouville's Theorem we then deduce that $\det Y(z) = 1$. In particular, Y^{-1} exists.

Now let Y_1 and Y_2 be two solutions to the RH problem and define $Z = Y_1 Y_2^{-1}$. Then Z is analytic in $\mathbb{C} \setminus \Gamma$ and

$$\begin{aligned} Z_+(z) = Y_{1+}(z)(Y_2^{-1})_+(z) = Y_{1-}(z)J(z)(Y_{2+}(z))^{-1} \\ = Y_{1-}(z)(Y_2^{-1})_-(z) = Z_-(z), \quad (1.1.1) \end{aligned}$$

for $z \in \Gamma$. Hence, if $\Gamma_p = \emptyset$, we see that Z is an entire function and $Z(z) = I + o(1)$ as $z \to \infty$. Again by Liouville's Theorem we have $Z(z) = I$ and hence $Y_1 = Y_2$, establishing the uniqueness of the solution of the Riemann-Hilbert problem.

In case $\Gamma_p \neq \emptyset$, we do no immediately know that $\det Y$ is entire, but only that $\det Y$ is meromorphic with possible singularities at Γ_p. However, by choosing appropriate conditions on the precise behavior of Y near these points, the singularities for $\det Y$ are removable and we do have that $\det Y$ is entire again. Similarly for Z.

1.1.2 The Cauchy integral

The RH problem is *multliplicative*. We will also frequently encounter *additive* RH problems in this course, which can be solved in terms of the Cauchy operator that maps a function f on the contour Γ to a function on $\mathbb{C} \setminus \Gamma$ by

$$\mathcal{C}f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(y)}{z - y} \mathrm{d}y.$$

Then, under some smoothness conditions on f we have that $g = \mathcal{C}f$ is a solution to the following Riemann-Hilbert problem.

RH problem 1.1.2. *Find g with the following properties*

- *g is analytic in $\mathbb{C} \setminus \Gamma$.*

- *$g_+(z) = g_-(z) + f(z)$ for $z \in \Gamma \setminus \Gamma_p$.*
- *$g(z) = o(1)$ as $z \in \infty$.*

Again to ensure uniqueness we need to pose conditions near the points Γ_p.

Exercise 1.1.3. Prove that if f is analytic at $z \in \Gamma \setminus \Gamma_p$ we indeed have $(\mathcal{C}f)_+(z) - (\mathcal{C}f)_-(z) = f(z)$. What can you say about $(\mathcal{C}f)_+(z) + (\mathcal{C}f)_-(z)$?

Remark 1.1.4. One may hope to reduce a multiplicative RHP into an additive one by the logarithmic map. Indeed, if Y solves the RHP 1.1.1, then

$$\begin{cases} \log Y_+(z) = \log Y_-(z) + \log J(z), & z \in \Gamma \\ \log Y(z) = o(1/z), & z \to \infty \end{cases}$$

and hence

$$Y(z) = \exp\left(\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\log J(w)}{z-w} \mathrm{d}w\right), \tag{1.1.2}$$

is a candidate for a solution to RHP 1.1.1. However, we need to explain what we mean $\log J(w)$. And even if we do, the formula may not be correct.

For example, take $J = \mathbb{R}$, $k = 1$ and assume that J is continuous on $\mathbb{R}$ with $J(x) \to 1$ as $x \to +\infty$ and $J(x) \neq 0$ for $x \in \mathbb{R}$. Then it is clear what we mean with $\log J(x)$ by using the branch of the logarithm such that $\log J(x) \to 0$ as $x \to +\infty$. However, when we continue to $-\infty$, we may end up at a different branch if $\log J(x) \to 2\pi \mathrm{i} N$ as $x \to -\infty$ for an integer N (the winding number). If $N \neq 0$, then the Cauchy integral diverges and the right-hand side of (1.1.2) is ill-defined.

Exercise 1.1.5. Consider the following Riemann-Hilbert problem

RH problem 1.1.6. *Find a function $Y : \mathbb{C} \setminus [-1,1] \to \mathbb{C}^{2\times 2}$ with the following properties*

1. *Y is analytic in $\mathbb{C} \setminus [-1,1]$*
2. *$Y_+(x) = Y_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $x \in (-1,1)$*
3. *$Y(z) = I + o(1)$ as $z \to \infty$.*
4. *$Y(z) = \mathcal{O}((z \pm 1)^{-1/4})$ as $z \to \pm 1$.*

Solve the following exercises:

a) Show that the solution, if exists, is unique.

b) Find a RHP for the function

$$Z(z) = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} Y(z) \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

(The jump-matrix for Z should be diagonal).

c) Solve the RHP for Z (Hint: first take the logarithm formally and use the Cauchy transform to find $\log Z$. Justify your answer afterwards.)

d) Find the unique solution for Y

1.2 Why do we care about Riemann-Hilbert problems?

Many objects of interest in analysis or mathematical physics can be formulated as a Riemann-Hilbert problem. In this course we will show that having such a formulation can be good starting point for an asymptotic analysis of the system. We will do this by considering three concrete examples:

- Toeplitz determinants.
- Painlevé II equation.
- Orthogonal polynomials.

Historically, these three examples have been an important motivation for the development of Riemann-Hilbert methods in asymptotic analysis. It should also be noted that all three examples have important applications to Random Matrix Theory.

There is another important source of problems that can be studied using Riemann-Hilbert problems that we will not study in these lectures: the study of solutions to non-linear PDEs, such as the long time behavior of the NLS or KdV equations.

1.2.1 From Airy to Painlevé II

We start by discussing the Painlevé II equation. The Painlevé equations are non-linear second order differential equations, whose only moveable singularities (moveable means that their location varies with the initial conditions) are poles. That means that the location of other type of singularities do not depend on the precise solution. This definition was given by Painlevé in an attempt to extend the realm of classical functions that are solutions to linear second order differential equations, such as Hermite polynomials, Bessel Functions, Airy functions, etcetera. Up to transformations, there are only six such equations (apart from the linear equations). After an initial spark of activity in this area around 1900, interest decreased in the years afterwards. They were re-discovered in the mahtematical physics literature where they appear naturally in several places. For instance, the Painlevé II equation appears in the search of self-similarity solutions to the KdV equation. Their importance was quickly recognized and to this date, the study of Painlevé euqation remains an important challenge. They are sometimes referred to as the "special functions of the 21th century".

As mentioned, classical special functions are solutions to second order linear differential equation and typically can be represented in terms of integrals, which is very useful for asymptotic analysis. However, when we deal with solutions to non-linear equations, we can no longer expect to have such integral representations. In that case, one may try to formulate the solution as a Riemann-Hilbert problem and start the asymptotic analysis from there.

Let us start with the following second order differential equation

$$y''(x) = xy(x). \tag{1.2.1}$$

This equation is called the Airy equation. It is the simplest equation for which there is a turning point. With this we mean the following. If x large bot postive, we expect the solution to grow or decay exponentially since y''/y is positive. On the other hand, for x large but negative, we have that y''/y is negative and we expect its solution to oscillate. Hence, solutions behave very differently at $\pm\infty$.

First, note that it is easy to write the general solution to this equation in integral form. To this end, let γ_j be contours connecting $\mathrm{e}^{(2j-1)\pi\mathrm{i}/3}\infty$ to $\mathrm{e}^{(2j+1)\pi\mathrm{i}/3}\infty$, for $j = 0, 1, 2$, and define the function y_j by the complex integrals

$$y_j(x) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_j} \mathrm{e}^{\frac{1}{3}t^3 - xt}\mathrm{d}t, \qquad j = 0, 1, 2. \tag{1.2.2}$$

By integration by parts, it is easy to see that all y_j are solutions to the Airy equation. Of course, the three together should be linearly dependent and indeed we have

$$y_0 + y_1 + y_2 = 0.$$

Any two of the three are independent and span the solution space.

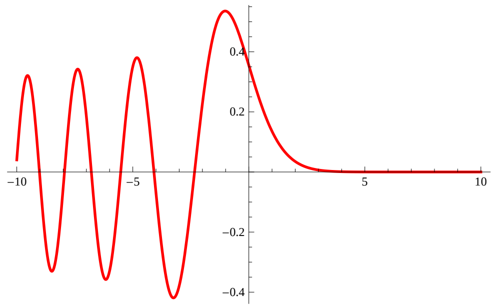

Figure 1.2: A plot of the Airy function $\mathrm{Ai}(x)$. For negative x we see oscillations and for x positive there is exponential decay.

The solution y_0 is a special solution and is called the Airy function, usually denoted by Ai. The Airy function has the following asymptotic expansions

$$\mathrm{Ai}(x) = y_0(x) \sim \begin{cases} \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, & x \to +\infty \\ \frac{\sin\left(\frac{2}{3}(-x)^{3/2}+\pi/4\right)}{\sqrt{\pi}(-x)^{1/4}}, & x \to -\infty \end{cases}$$

A natural way to prove these expansions is to use steepest descent techniques (also called stationary phase method) for the integral representation (1.2.2) (see Chapter 6 of [1] for a discussion on this method).

Now let us modify the Airy equation and include a cubic term

$$y''(x) = xy(x) + 2(y(x))^3. \tag{1.2.3}$$

This non-linear equation is known in the literature as the Painlevé II equation (with $\alpha = 0$). There is no integral representation for the general solution for his equation and an asymptotic analysis is therefore non-standard. Nevertheless, we would like to answer question such as the following

1. What can we say about the asymptotic behavior of a solution $y(x)$ as $x \to \infty$ and $x \to -\infty$?

2. From the differential equation, one may conjecture that for every $c \in \mathbb{C}$ there exists solution to the Painlevé II equation for which

$$y(x) \sim c\,\mathrm{Ai}(x)$$

as $x \to +\infty$. Is that indeed true?

3. Solutions to the Painlevé II equation are in general not entire functions and will have poles. What can we say about the locations of the poles?

4. If we view the Painlevé II equation as a a perturbed Airy equation and introduce a parameter ε and consider $y''(x) = xy(x) + \varepsilon(y(x))^3$. Can we find a family of solutions $\{y_\varepsilon\}_\varepsilon$ so that in the limit $\varepsilon \downarrow 0$, the converges to a solution to the Airy equation?

These are all natural questions that are very well suited for a Riemann-Hilbert approach, as we will see in this course. Moreover, such type of questions are not only interesting from a mathematical point of view, but often have a strong motivation from various applications from physics.

1.2.2 From integral representation to Riemann-Hilbert problem

We now show that both solutions to the Airy equation and the Painlevé II equation can be formulated in terms of a RHP.

We can write the general solution to the Airy equation as

$$y(x) = \frac{s_1}{2\pi i}\int_0^{\mathrm{e}^{\pi \mathrm{i}/3}\infty} \mathrm{e}^{\frac{1}{3}z^3 - xz}\mathrm{d}z + \frac{s_2}{2\pi i}\int_0^{-\infty} \mathrm{e}^{\frac{1}{3}z^3 - xz}\mathrm{d}z + \frac{s_3}{2\pi i}\int_0^{\mathrm{e}^{-\pi \mathrm{i}/3}\infty} \mathrm{e}^{\frac{1}{3}z^3 - xz}\mathrm{d}z \quad (1.2.4)$$

where we let $s_1, s_2, s_3 \in \mathbb{C}$ satisfying

$$s_1 + s_2 + s_3 = 0.$$

Because of this condition the function y solves the Airy equation, which can be proved using the integration by parts.

We now characterize the solutions to the Airy equation in terms of RHP. We use the notation

$$\gamma_1 = \mathrm{e}^{\pi i/3}[0,\infty), \gamma_2 = \mathrm{e}^{\pi i}[0,\infty), \text{ and } \gamma_3 = \mathrm{e}^{-\pi i/3}[0,\infty).$$

RH problem 1.2.1. *Let $x \in \mathbb{C}$. We seek for a function $Y(\cdot) = Y(\cdot; x)$ such that*

- *Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.*
- $Y_+(z) = Y_-(z)\begin{pmatrix} 1 & s_j \mathrm{e}^{-\frac{1}{3}z^3 + xz} \\ 0 & 1 \end{pmatrix}$ *for $z \in \gamma_j$.*

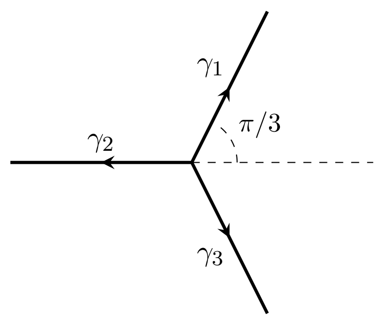

Figure 1.3: The jump contours for RHP 1.2.1

- $Y(z) = I + o(1)$ *as* $z \to \infty$.
- Y *is bounded near* 0.

with $s_1, s_2, s_3 \in \mathbb{C}$ *such that* $s_1 + s_2 + s_3 = 0$.

Proposition 1.2.2. *The solution to the RHP 1.2.1 is unique and given by*

$$Y(z) = \begin{pmatrix} 1 & \frac{1}{2\pi \mathrm{i}} \sum_{j=0}^{2} s_j \int_{\gamma_j} \frac{\mathrm{e}^{\frac{1}{3}t^3 - zt}}{t-z} \mathrm{d}t \\ 0 & 1 \end{pmatrix}, \tag{1.2.5}$$

for $z \in \mathbb{C} \setminus \Gamma$. *Moreover, if we write*

$$Y(x) = Y(z;x) = I + \frac{Y^{(1)}(x)}{z} + O(1/z^2), \qquad z \to \infty.$$

Then $-Y_{11}^{(1)}(x)$ *is the solution to the Airy equation with parameters* s_0, s_1 *and* s_2.

Proof. Write

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

Then it follows from the jump condition that

$$\begin{cases} Y_{11,+}(z) = Y_{11,-}(z), \\ Y_{21,+}(z) = Y_{21,-}(z), \end{cases}$$

for $z \in \Gamma$. Since Y is bounded near the origin we see that Y_{11} and Y_{21} are entire functions. By the asymptotic condition at infinity we then conclude $Y_{11} = 1$ and $Y_{21} = 0$ which proves the first column.

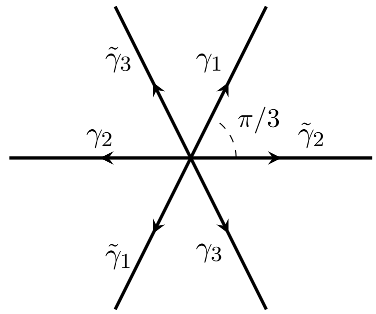

Figure 1.4: The jump contours for RHP 1.2.3

Let us come to the second column. From the jump condition we have

$$Y_{22,+}(z) = Y_{22,-}(z) + s_j Y_{21_-}(z)\mathrm{e}^{z^3-tz},$$

for $z \in \gamma_j$. Since $Y_{21} = 0$ we also have that Y_{22} is entire and hence $Y_{22} = 1$. For the remaining entry Y_{12} we find from the jump condition and using $Y_{11} = 1$ that

$$Y_{12,+}(z) = Y_{12,-}(z) + s_j \mathrm{e}^{z^3-tz}$$

and $Y_{12}(z) = o(1)$ as $z \to \infty$. Since at the intersection point Y is bounded, the unique solution to this additive RHP is given by the Cauchy transform

$$Y_{12}(z) = \sum_{j=1}^{3} \frac{s_j}{2\pi i} \int_{\gamma_j} \frac{\mathrm{e}^{t^3-xt)}}{t-z} \mathrm{d}t,$$

as given in the statement. This proves (1.2.5).

That $-Y_{11}^{(1)}$ solves the Airy equation with given parameters follows by expanding $1/(t-z)$ for z near infinity. □

It is clear from the proof that this result extends to other integral representations. In other words, all classical special functions that admit an integral representation can be characterized by a RHP of the above form. Moreover, such RHP characterizations also exist for special functions that fall outside this class, such as the Painlevé transcendents. As an exmaple we now formulate a RHP for the Painlevé II equation (1.2.3), by introducing extra jumps on the contours $\tilde{\gamma}_j = -\gamma_j$ in the RHP (1.2.1). See also Figure 1.4

RH problem 1.2.3. *Let $x \in \mathbb{C}$. We seek for a function $Y(\cdot) = Y(\cdot; x)$ such that*

- *Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.*
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & s_j \mathrm{e}^{-\frac{1}{3}z^3 + xz} \\ 0 & 1 \end{pmatrix}$ *for $z \in \gamma_j$.*
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \\ s_j \mathrm{e}^{\frac{1}{3}z^3 - xz} & 1 \end{pmatrix}$ *for $z \in \tilde{\gamma}_j$.*
- $Y(z) = I + o(1)$ *as $z \to \infty$.*
- *Y is bounded near 0,*

with $s_1, s_2, s_3 \in \mathbb{C}$ such that $s_1 + s_2 + s_3 + s_1 s_2 s_3 = 0$.

Proposition 1.2.4. *The solution to the RHP 1.2.3 is unique and depends meromorphically on x. Moreover, if we write*

$$Y(x) = Y(z;x) = I + \frac{Y^{(1)}(x)}{z} + O(1/z^2), \qquad z \to \infty.$$

Then $-2Y^{(1)}_{11}(x)$ is a solution to the Painlevé II equation.

A proof of this fact will be given in a later stage of the course. Here it comes out of the blue, but we will see that there is a general mechanism that given both RHP 1.2.1 and (1.2.3) as special cases of a more general RHP.

An important difference betwee RHP (1.2.1) and (1.2.3) is that in the latter, we include jump with lower triangular jump matrices. In particular, the jump matrices for the different contours do no always commute. For RHP's that have commuting jump matrices one may hope to find an explicit (integral) representations in terms of elementary function for the solution as in RHP 1.2.1 and Exercise 1.1.3. But for the RHP 1.2.3 this is a different story altogether. Nevertheless, the RHP is well-suited for asymptotic analysis.

1.2.3 Orthogonal polynomials

Another important example, where Riemann-Hilbert techniques have proved to be useful is the study of orthogonal polyomials.

Let w be a non-negative function on $\mathbb{R}$ such that $\int |x|^k w(x) \mathrm{d}x < \infty$ for all $k \in \mathbb{R}$. Then we can view the space of polynomials as a subspace of $\mathbb{L}_2(\mathbb{R}, w(x)\mathrm{d}x)$. Clearly, the monomials

$$\{1, x, x^2, x^3, x^4, \ldots\}$$

form a basis for the subspace of polynomials. By applying Gramm-Schmidt we can turn this into an orthogonal basis of monic polynomials

$$\{\pi_0(x), \pi_1(x), \pi_2(x), \ldots\}.$$

In other words $\pi_k = x^k + \cdots$ is the unique monic polynomial of degree k such that

$$\int_{\mathbb{R}} \pi_k(x) x^j w(x) \mathrm{d}x = 0, \qquad j = 0, 1, \ldots, k-1.$$

We will also use the notation

$$h_k^2 = \int (\pi_k(x))^2 w(x) \mathrm{d}x.$$

RH problem 1.2.5. *We seek for a function $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ such that*

- *Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.*
- $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$, *for $x \in \mathbb{R}$.*
- $Y(z) = (I + o(1)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ *as $z \to \infty$.*

Proposition 1.2.6. *The solution to the RHP 1.2.5 is unique and given by*

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\pi_n(x) w(x)}{x - z} \mathrm{d}x \\ -\frac{2\pi i}{h_{n-1}^2} \pi_{n-1}(z) & -\frac{1}{h_{n-1}^2} \int_{\mathbb{R}} \frac{\pi_{n-1}(x) w(x)}{x - z} \mathrm{d}x \end{pmatrix}. \tag{1.2.6}$$

Proof. The spirit of the proof is similar to the proof of Proposition 1.2.2 and we leave it as a very useful exercise. A full proof and extensive discussion will be given at a later stage of the course. □

1.2.4 Toeplitz determinants

Denote the unit circle with $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $a : \mathbb{T} \to \mathbb{C}$ be an integrable function and

$$a_k = \frac{1}{2\pi} \int a(z) \frac{dz}{z^{k+1}}, \qquad k \in \mathbb{Z}.$$

Then the Toeplitz matrix $T_n(a)$ of size $n \in \mathbb{N}$ is defined as the $n \times n$matrix

$$T_n(a) = (a_{j-k})_{j,k=1}^n = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & & & & \\ a_1 & a_0 & a_{-1} & a_{-2} & & & \\ a_2 & a_1 & a_0 & \ddots & \ddots & & \\ & a_2 & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & a_{-2} \\ & & & \ddots & \ddots & \ddots & a_{-1} \\ & & & & a_2 & a_1 & a_0 \end{pmatrix}_{n\times n}$$

Another words, a Toeplitz matrix is a square matrix wthat is constant along the diagonals.

The function $a(z)$ is called the symbol of the Toeplitz matrix.

Toeplitz matrices play an important role in various parts of mathematical physics. A particular question that one often wants to understand, is about the behavior of the Toeplit determinant as the size tends to infinity.

An example of a celebrated result for determinant of Toeplitz matrices is the Strong Szegő Limit Theorem.

Theorem 1.2.7. *Let $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n$ such that $\sum_{n=-\infty}^{\infty} |n||f_n|^2 < \infty$, then*

$$\det T_n(e^f) = e^{nf_0} e^{\sum_{n=1}^{\infty} nf_n f_{-n}} (1 + o(1)),$$

as $n \to \infty$

This theorem has important consequences. One such consequence is a Central Limit Theorem for linear statistics for the eigenvalues of a randomly chosen unitary matrix (wih respect to the Haar measure).

In this course we will see a proof of this statement for symbols that are analytic in an annulus using Riemann-Hilbert techniques

The Riemann-Hilbert problem that is associated to the Toeplitz determinant is the following

RH problem 1.2.8. *Find a function $Y : \mathbb{C} \setminus \mathbb{T} :\to \mathbb{C}^2$ such that*

1. *Y is analytic $C \setminus \mathbb{T}$*
2. *$Y_+(z) = Y_-(z) \begin{pmatrix} a(z) & -(a(z)-1)z^n \\ (a(z)-1)z^{-n} & 2-a(z) \end{pmatrix}$*
3. *$Y(z) = I + o(1)$ as $z \to \infty$*

Here the +-side is the interior and the −-side is in the exterior of the unit disk.

The Toeplitz determinant can be represented in terms of the solution to this Riemann-Hilbert problem. The exact expression and the proof of this statement are more involved and will be postponed for now.

1.2.5 Non-linear Schrödinger equation

Finally, Riemann-Hilbert problems also appear naturally from inverse scattering for PDE's. The following example is taken from [3]. For more examples, details and references we refer to [3].

Consider the non-linear Schrdinger equation

$$\begin{cases} \psi_t = \psi_{xx} - 2\psi|\psi|^2 \\ \psi(x,0) = \psi_0(x) \in \mathcal{S}(\mathbb{R}), \end{cases} \tag{1.2.7}$$

where $S(\mathbb{R})$ stands for the Schwarz class. By the princples of inverse scattering there is a bijection from $\mathcal{S}(\mathbb{R})$ to $\{r \in \mathcal{S}(\mathbb{R}) \mid |r(x)| < 1\}$ that maps ψ_0 to r.

Let $\mathbb{R}$ be oriented from left to right. Consider the followign RHP.

RH problem 1.2.9. *Find a function $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ such that*

1. *Y is analytic in $\mathbb{C} \setminus \mathbb{R}$.*
2. *$Y_+(z) = Y_-(z) \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r(z)} \mathrm{e}^{-2\mathrm{i}(2tz^2+xz)} \\ r(z)\mathrm{e}^{2\mathrm{i}(2tz^2+xz)} & 1 \end{pmatrix}$ for $z \in \mathbb{R}$.*
3. *$Y(z) = I + o(1)$ as $z \to \infty$.*

Proposition 1.2.10. *For (x,t) let $Y(\cdot; x,t)$ be the solution for the RHP 1.2.9. Expand*

$$Y(z; x,t) = I + \frac{Y^{(1)}(x,t)}{z} + \mathcal{O}(z^{-2}), \qquad z \to \infty.$$

Then $2\mathrm{i}Y^{(1)}(x,t)$ solves the non-linear Schrdinger equation with y_0 (determined by r) as initial data .

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