Comments

A Comment on "Graph-Based Algorithm for Boolean Function Manipulation"

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Abstract—In this paper, a slight error in the paper of Bryant [1] is corrected. It was stated in [1] that, under a certain ordering restriction, composition of two Reduced Ordered Binary Decision Diagrams (ROBDDs) results in a reduced OBDD. We show a counterexample and explore under which conditions this statement is incorrect.

Index Terms—Boolean function, Binary Decision Diagram, composition.

1 Introduction

It is quite common that a logic network describing a structured design contains repeating substructures. Bottom-up approaches exploit this regularity to produce a more economic logic description for such a network. The main facility of these approaches is functional composition. First, the subfunctions representing the individual substructures are derived and then the complete function is composed from these subfunctions.

Let f and g be Boolean functions of type $\{0,1\}^n \rightarrow \{0,1\}$, of the arguments $x_1, \ldots, x_n, n \geq 1$. *Composition* is the operation of replacing some argument x_i, $i \in \{1, \ldots, n\}$, of f by a function g, resulting in the following function:

$$f|_{x_i=g}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n).$$

Using the Shannon expansion of a function with respect to the variable x_i [2], namely:

$$f(x_1, \ldots, x_n) = x_i' \cdot f|_{x_i=0} + x_i \cdot f|_{x_i=1},$$

where $f|_{x_i=j} = f(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_n)$, we can derive the following expression for $f|_{x_i=g}$:

$$f|_{x_i=g}(x_1, \ldots, x_n) = g' \cdot f|_{x_i=0} + g \cdot f|_{x_i=1}. \quad (1)$$

Bryant [1] has presented an efficient algorithm $Compose$ for performing composition of two functions represented by ROBDDs. To compute (1), $Compose$ utilizes the ternary Boolean operation ITE (if-then-else):

$$ITE(a, b, c) = a \cdot b + a' \cdot c.$$

The algorithm has the worst-case complexity $O(|G_1|^2 \cdot |G_2|)$, where G_1 and G_2 are the ROBDDs representing f and g, respectively. Furthermore, if the following ordering restriction holds:

$$\text{There are no } j \in I_f \text{ and } k \in I_g \text{ such that } i < j \leq k \text{ or } i > j \geq k, \quad (2)$$

where I_f and I_g are the *dependence sets* (or *support sets*) of f and g, respectively, defined by $I_f = \{i \mid f|_{x_i=0} \neq f|_{x_i=1}\}$, then the composition can be performed in a simpler and more efficient way by substituting each vertex $v \in G_1$ having index i by a copy of G_2, replacing each branch to the terminal vertex 0 in G_2 by a branch to $low(v)$ and each branch to the terminal vertex 1 in G_2 by a branch to $high(v)$. Fig. 1 shows an example of this kind of substitution, where two vertices labeled by 3 in the ROBDD of $f(x_1, x_2, x_3, x_6)$ are substituted by the graph for the function $g(x_4, x_5)$. It was stated in [1] that, provided G_1 and G_2 are reduced, the graph resulting from the composition is also reduced [1, p. 686]. Many ROBDD-related works use this assumption, including [3] and [4]. However, we found that there are cases when the composition results in a nonreduced OBDD. For example, consider the functions $f = x_1 \oplus x_2$ and $g = x_3 \oplus x_4$, where "$\oplus$" denotes XOR. Let G_1 be ROBDD for f with the ordering $\langle x_1, x_2 \rangle$ (Fig. 2a) and G_2 be ROBDD for g with the ordering $\langle x_3, x_4 \rangle$ (Fig. 2b). If G_2 is substituted in the vertices labeled by 2 in G_1, then we get a graph with seven nonterminal vertices, which is not reduced (Fig. 2c). The reduced version is shown in Fig. 2d.

In the next section, we present necessary and sufficient conditions for the composition of two ROBDDs to result in a nonreduced graph. In a preliminary formulation, these conditions have appeared in [5] and [6]. A different technique for proving them can be found in [7].

2 Necessary and Sufficient Conditions

The following theorem presents necessary and sufficient conditions for the composition of two ROBDDS to result in a nonreduced OBDD. We use the notation $G_1(i \leftarrow G_2)$ for the graph representing $f|_{x_i=g}$, the indexed letters v, u, and w to denote the vertices of the graphs G_1, G_2, and $G_1(i \leftarrow G_2)$, respectively, and the terms f_v, g_u, and $f_w|_{x_i=g}$ for the functions represented by the subgraphs rooted by v, u and w, respectively.

Theorem 1. *Let G_1 and G_2 be ROBDDs representing the functions f and g, respectively, and let the ordering restriction (2) hold for some $i \in I_f$. Then, the OBDD $G_1(i \leftarrow G_2)$ for $f|_{x_i=g}$ is not reduced if and only if G_1 and G_2 satisfy the following two conditions:*

1. $\exists v_1, v_2 \in G_1$ *such that* $low(v_1) = high(v_2)$, $high(v_1) = low(v_2)$, *and* $index(v_1) = index(v_2) = i$.
2. $\exists u_1, u_2 \in G_2$ *such that* $g_{u_1} = g'_{u_2}$ *and* $index(u_1) = index(u_2) = j$, *for some* $j \in I_g$.

Proof 1) "if" part: Suppose the conditions of the theorem are satisfied. Then, by definition of ROBDD [1, p. 679], the subgraphs of G_1 having root vertices v_1 and v_2 represent the following functions:

$$\begin{aligned} f_{v_1} &= x_i' f_{low(v_1)} + x_i f_{high(v_1)}, \\ f_{v_2} &= x_i' f_{low(v_2)} + x_i f_{high(v_2)} \\ &= x_i' f_{high(v_1)} + x_i f_{low(v_1)} \ \{\text{by condition 1}\}. \end{aligned} \quad (3)$$

Consider the graph $G_1(i \leftarrow G_2)$, obtained after the replacement of the vertices v_1 and $v_2 \in G_1$ by G_2. Let $w_1 \in G_1(i \leftarrow G_2)$ be a copy of the vertex $u_1 \in G_2$ obtained after the replacement of G_2 in v_1, and $w_2 \in G_1(i \leftarrow G_2)$ be a copy of $u_2 \in G_2$ obtained after the replacement of G_2 in v_2. That is, w_1 and w_2 belong to the different copies of G_2.

Since the composition is performed by replacing each branch to the terminal vertices in G_2 by the branches to the correspondent children of the vertices being replaced, the

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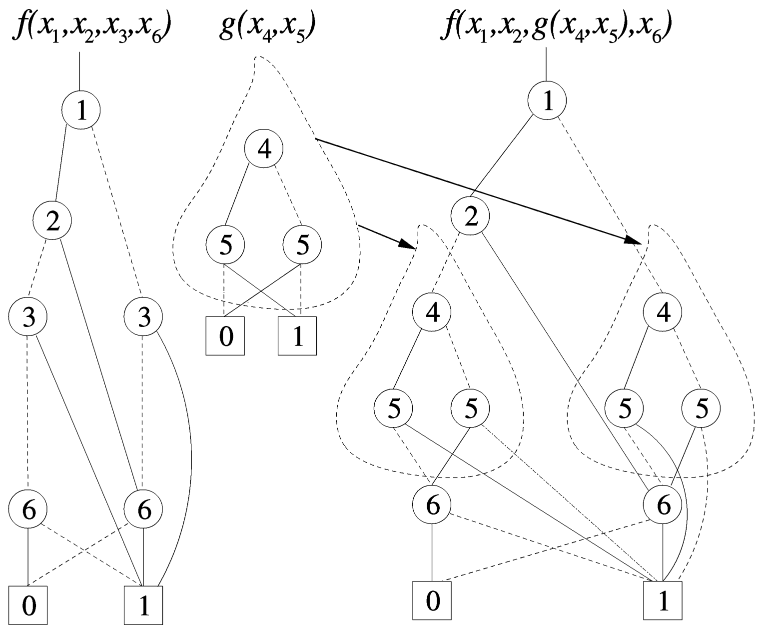

Fig. 1. Composition of ROBDDs satisfying ordering restriction (2).

subgraphs rooted by w_1 and w_2 represent the following functions:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_2} f_{high(v_1)} + g_{u_2} f_{low(v_1)} \\ &= g_{u_1} f_{high(v_1)} + g'_{u_1} f_{low(v_1)} \ \{\text{by condition 2}\}. \end{aligned} \quad (4)$$

Thus, $f_{w_1}|_{x_i=g} = f_{w_2}|_{x_i=g}$. From the way $G_1(i \leftarrow G_2)$ is constructed, it is easy to see that the subgraphs rooted by w_1 and w_2 match in their structure and their attributes. Thus, by definition [1, p. 679], they are isomorphic.

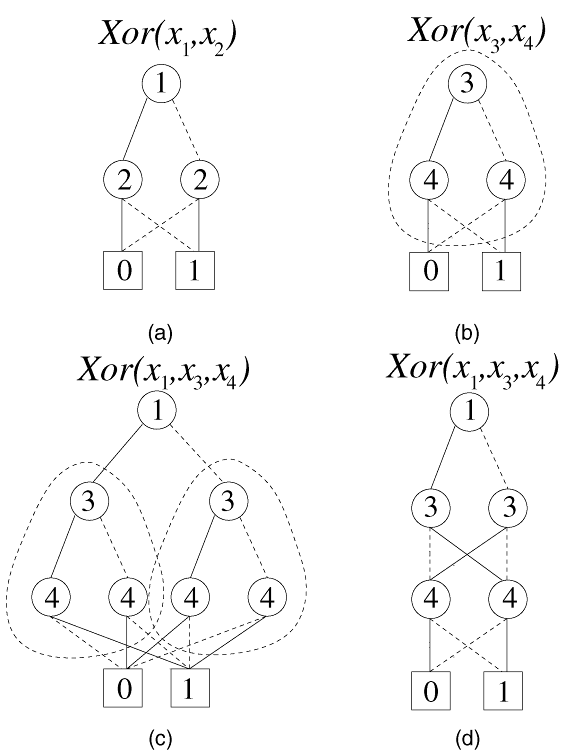

Fig. 2. Composition of ROBDDs resulting in a nonreduced OBDD.

2) "only if" part: We assume that $G_1(i \leftarrow G_2)$ is not reduced and show by transformations that then the conditions hold.

An OBDD is not reduced if either: a) It contains a vertex v with $low(v) = high(v)$, or if: b) It contains two distinct vertices v and u such that the subgraphs rooted by v and u are isomorphic [1, p. 679].

If G_1 and G_2 are reduced, then Case a) can never occur as a result of composition because each vertex $v \in G_1$ having the index i is replaced by a different copy of G_2 and all branches in G_2 going to the different terminal vertices are replaced by the branches to the different children of G_1. So, the vertices having different children in G_1 and G_2 will have different children in $G_1(i \leftarrow G_2)$. Therefore, if $G_1(i \leftarrow G_2)$ is not reduced, then b) holds, i.e., it has isomorphic subgraphs.

Let w_1 and w_2 be vertices in $G_1(i \leftarrow G_2)$ rooting two isomorphic subgraphs. By definition of isomorphism between two OBDDs [1, p. 679], these subgraphs match in both their structure and their attributes. So, $index(w_1) = index(w_2) = j$, for some $j \in (I_f - \{i\}) \cup I_g$, and the functions they represent are equivalent:

$$f_{w_1}|_{x_i=g} = f_{w_2}|_{x_i=g}. \quad (5)$$

There are three possibilities for the relative position of w_1 and w_2 in the graph:

- $j \in I_f - \{i\}$,
- $j \in I_g$, and both w_1 and w_2 are in the same copy of G_2,
- $j \in I_g$, and w_1, w_2 are in the different copies of G_2.

These three exhaust all possible cases.

Let $j \in I_f - \{i\}$. Suppose we decompose $G_1(i \leftarrow G_2)$ back into G_1 and G_2. Since (5) holds, there must be some vertices v_1 and v_2 in G_1 for which $f_{v_1} = f_{v_2}$. This implies that G_1 has isomorphic subgraphs, contradicting the assumption that G_1 is reduced.

Let $j \in I_g$ and both w_1 and w_2 be in the same copy of G_2. Then, the subgraphs rooted by w_1 and w_2 represent the following functions:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_2} f_{low(v_1)} + g_{u_2} f_{high(v_1)}, \end{aligned} \tag{6}$$

where $f_{low(v_1)}$ and $f_{high(v_1)}$ are the functions of the subgraphs rooted by low and $high$ children of the vertex $v_1 \in G_1$ in which G_2 was substituted, and g_{u_1}, g_{u_2} are the functions represented by the subgraphs rooted by some u_1 and u_2 in G_2. From (5) and (6), we can conclude that $g_{u_1} = g_{u_2}$ and, thus, G_2 has isomorphic subgraphs. This contradicts the assumption that G_2 is reduced.

The only case that remains is $j \in I_g$ and w_1, w_2 are in the different duplicate copies of G_2. Then, the subgraphs rooted by w_1 and w_2 represent the functions:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_2} f_{low(v_2)} + g_{u_2} f_{high(v_2)}, \end{aligned} \tag{7}$$

where $f_{low(v_1)}, f_{high(v_1)}, f_{low(v_2)}, f_{high(v_2)}$ are the functions of the subgraphs rooted by the children of the vertices v_1 and $v_2 \in G_1$ in which G_2 was substituted and g_{u_1}, g_{u_2} are the functions of the subgraphs rooted by some vertices $u_1, u_2 \in G_2$. Moreover, since $index(w_1) = index(w_2) = j$, we have $index(u_1) = index(u_2) = j$.

From (5) and (7), we can derive:

$$g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} = g'_{u_2} f_{low(v_2)} + g_{u_2} f_{high(v_2)}.$$

It is easy to show that the only two solutions satisfying the above equation are:

1. $f_{low(v_1)} = f_{low(v_2)}, f_{high(v_1)} = f_{high(v_2)}$ and $g_{u_1} = g_{u_2}$
2. $f_{low(v_1)} = f_{high(v_2)}, f_{high(v_1)} = f_{low(v_2)}$ and $g'_{u_1} = g_{u_2}$.

If 1 holds, then $f_{v_1} = f_{v_2}$ and G_1 has isomorphic subgraphs, which contradicts the assumption. Therefore, 2 holds, directly giving us the second condition of the theorem. Since G_1 is reduced, from $f_{low(v_1)} = f_{high(v_2)}, f_{high(v_1)} = f_{low(v_2)}$ we can also conclude that $low(v_1) = high(v_2)$, $high(v_1) = low(v_2)$. Thus, the first condition holds, too. □

3 Conclusion

In this paper, we show that the composition of two ROBDDs may result in a nonreduced OBDD and summarize necessary and sufficient conditions for this.

It should be noted that most typical BDD packages use hash tables to check for existing vertices before creating new ones, in this way ensuring that the obtained graph is reduced [8], [9]. However, in theoretical investigations, overlooking the conditions given by Theorem 1 may lead to incorrect conclusions about the size of the resulting OBDD (like in the statement of Lemma 1 from [3, p. 8] and in the proof of Lemma 3.4 from [4, p. 58]), as well as about its canonicity.

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