On Disjoint Covers and ROBDD Size

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Abstract

The relation between the number of nodes in a ROBDD and the number of implicants in the disjoint cover of the function represented by that ROBDD is studied. We identify a class of functions for which there are disjoint covers such that a cover of a larger size can be represented by a ROBDD with a smaller number of nodes. This shows that the size of a ROBDD is not a monotonically increasing function of the size of the disjoint cover.

1. Introduction

Reduced Ordered Binary Decision Diagrams (ROBDDs) are a graphical data structure for the representation of Boolean functions [1]. A function is represented by a directed acyclic graph. An ROBDD has two terminal nodes labeled "0" and "1", respectively. The ROBDD has a single root node. Each nonterminal node (including the root) is labeled by an input variable and has two outgoing edges labeled "0" and "1". A ROBDD is *reduced* as (i) it has no redundant nodes, *i.e.* no node whose outgoing edges point to the same node, and (ii) it has no isomorphic subgraphs, *i.e.* no two subgraphs that represent the same function. In an ROBDD, each input variable appears at most once on any path from the root node to a terminal node, and the variables appear subject to a fixed *ordering*.

A major concern regarding ROBDDs is that the size (number of nodes) of the graph varies for different variable orderings. For some functions, the size is highly sensitive to the ordering. For example, the size of an ROBDD representing an adder is exponential in the number of variables in the worst case but linear in the best case. While for some functions, *e.g.* multiplication, all variable orderings yield a ROBDD with an exponential number of nodes, most practical functions do have a variable ordering with size linear, or polynomial, in the number of variables. Hence, selecting an ordering for the function variables, minimizing the number of nodes in the graph is critical to the effective use of ROBDDs.

The problem of determining a best variable ordering is co NP-complete [3], and therefore exact algorithms are feasible only for functions of a small number of variables.

A number of heuristic procedures have been developed, using various strategies to produce a good ordering within a reasonable time, including [4]-[10]. Many of the ordering heuristics analyse the structure of a logic circuit implementing the function under consideration and use its underlying topology to determine a good ordering. However, if there is no circuit to refer to, as is the case in synthesis applications or finding the set of reachable states of a finite state machine, finding a good order is more difficult. But, the problem is a crucial one as one must avoid explosion of the ROBDDs and exceeding the peak memory limit in the course of computation when many intermediate ROBDDs are often required.

When there is no circuit to refer to, properties of the function can potentially be used to guide the selection of a good ordering. In this paper, we study whether the size of a disjoint cover of a function can be used to guide the choice of a good ordering. The paths from the root node of an ROBDD to the terminal node labeled by "1" represent the implicants of a disjoint cover of the function represented by the ROBDD. Different orderings of variables normally encode different paths in the graph which in turn correspond to different disjoint covers of the function. The efficiency of ROBDDs comes from the fact that they can *potentially* encode an exponential number of implicants with a linear number of nodes. However, the relation between the orderings of the variables and the implicants from the disjoint

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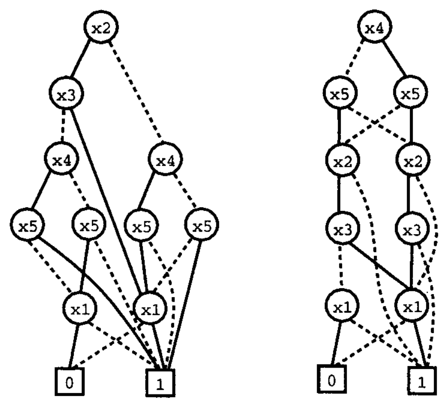


(a) (b)

Figure 1. ROBDDs for the orderings (a) $\langle x_2, x_3, x_4, x_5, x_1\rangle$ and (b) $\langle x_4, x_5, x_2, x_3, x_1\rangle$ for the function. $f(x_1, \ldots, x_5) = x_1(x_4 \oplus x_5)' + x_2'(x_4 \oplus x_5) + x_1x_3 + x_1'x_2x_3'$.

cover encoded by these orderings is not well understood.

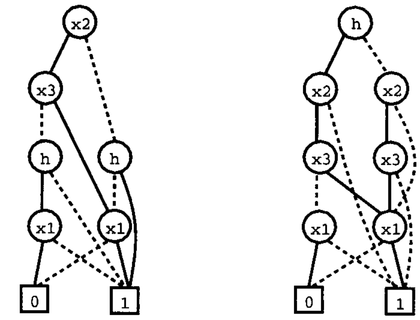
Intuitively, one would expect that more nodes will be needed to encode more paths, *i.e.* that the number of nodes in a ROBDD is a monotonically increasing function of the number of implicants in a disjoint cover represented by the paths of the ROBDD. If this were the case, then covers of minimal size could be used as guides in selecting a best ordering of the variables. However, in this paper we show that the function is not monotonically increasing. In the next section, we identify a class of functions for which there are disjoint covers such that a cover of a larger size can be represented by a ROBDD with a smaller number of nodes.

2. The relation between the number of nodes and the number of paths in a ROBDD

Let f be a Boolean function of type $\{0,1\}^n \to \{0,1\}$ with arguments $x_1, \ldots, x_n, n \geq 1$. We use $G(f)$ to denote the ROBDD for f.

An *ordering* of the variables in $G(f)$ is a vector, identifying the variables in order from the root to the terminals in the ROBDD. We use $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ to denote the set of all orderings π_i for $G(f)$. We use N_{π_i} to denote the number of nodes in $G(f)$ for the ordering π_i, and $P^0_{\pi_i}$ and $P^1_{\pi_i}$ to denote the number of paths in $G(f)$ for the ordering π_i, passing from the root node to the terminal nodes labeled by "0" and "1", respectively.

The paths from the root node to the terminal node labeled by "1" give the implicants from a disjoint cover of the function f. The following theorem shows that it is possible



(a) (b)

Figure 2. ROBDDs for the orderings (a) $\langle x_2, x_3, h, x_1\rangle$ and (b) $\langle h, x_2, x_3, x_1\rangle$ for the function $f(x_1, \ldots, x_n) = x_1h' + x_2'h + x_1x_3 + x_1'x_2x_3'$.

to represent a cover of a larger size with a ROBDD with a smaller number of nodes.

Theorem 1 *For any $n \geq 5$, there exists a function $f(x_1, \ldots, x_n)$ such that for at least one pair of orderings $\pi_i, \pi_j \in \Pi$ in $G(f)$, it holds that $P^1_{\pi_i} > P^1_{\pi_j}$ and $P^0_{\pi_i} > P^0_{\pi_j}$, but $N_{\pi_i} < N_{\pi_j}$.*

Proof: Consider the following function of 5 variables:

$$f(x_1, \ldots, x_5) = x_1(x_4 \oplus x_5)' + x_2'(x_4 \oplus x_5) + x_1x_3 + x_1'x_2x_3'$$

The ordering $\pi_1 = \langle x_2, x_3, x_4, x_5, x_1\rangle$ (Figure 1(a)) results in a ROBDD with $N_{\pi_1} = 12$, $P^1_{\pi_1} = 9$ and $P^0_{\pi_1} = 5$. On the other hand, the ordering $\pi_2 = \langle x_4, x_5, x_2, x_3, x_1\rangle$ (Figure 1(b)) yields a ROBDD with $N_{\pi_2} = 11$, $P^1_{\pi_2} = 12$ and $P^0_{\pi_2} = 8$. Therefore, the latter ordering encodes more paths to "0" and to "1" with a smaller number of nodes.

The phenomenon demonstrated by the above example holds for any n-variable function $f(x_1, \ldots, x_n)$, $n \geq 5$, of the type:

$$f(x_1, \ldots, x_n) = x_1h' + x_2'h + x_1x_3 + x_1'x_2x_3'$$

where h denotes a $(n-3)$-variable function $h(x_4, \ldots, x_n) = x_4 \oplus x_5 \oplus \ldots \oplus x_n$, and h' denotes its complement. To show this, consider the two ROBDDs in Figure 2(a) and (b). The former has two nodes labeled by h, while the latter has just one such node. Suppose we structurally replaced the nodes labeled by h by the ROBDD for $h(x_4, \ldots, x_n)$. Then, the number of nodes for the ordering $\pi_1 = \langle x_2, x_3, \langle h\rangle, x_1\rangle$ is $N_{\pi_1} = 4n - 8$, and for the ordering $\pi_2 = \langle\langle h\rangle, x_2, x_3, x_1\rangle$ is $N_{\pi_2} = 2n + 1$, where $\langle h\rangle$ denotes any ordering of the variables of $h(x_4, \ldots, x_n)$. Therefore, for $n \geq 5$, $N_{\pi_2} < N_{\pi_1}$.

On the other hand, the structural replacement of a node labeled with h by the ROBDD for $h(x_4, \ldots, x_n)$ multiplies the paths originating in h by 2^{n-4}. Therefore, we have $P^0_{\pi_1} = 1 + 2 \cdot 2^{n-4}, P^1_{\pi_1} = 1 + 4 \cdot 2^{n-4}$, and $P^0_{\pi_2} = 4 \cdot 2^{n-4}, P^1_{\pi_2} = 6 \cdot 2^{n-4}$. So, for $n \geq 5$, $P^1_{\pi_2} > P^1_{\pi_1}$ and $P^0_{\pi_2} > P^0_{\pi_1}$. Thus, the theorem holds for $n \geq 5$. □

In the next section we show that this result also holds when complemented edges [2] are allowed in the ROBDD.

3. ROBDDs with complemented edges

The size of a ROBDD can be reduced by up to a factor of two if *complemented edges* are allowed [2]. A complemented edge indicates that the edge points to the complement of the function associated with the node the edge points to. This enables a function and its complement to be represented by the same sub-graph in a ROBDD.

Let $G^c(f)$ denote the ROBDD with complemented edges for f. $N^c_{\pi_i}$ denotes the number of nodes in $G^c(f)$ for the ordering π_i. $P^{c0}_{\pi_i}$ and $P^{c1}_{pi_i}$ denote the number of paths in $G^c(f)$ for the ordering π_i, passing from the root node to the terminal nodes "o" and "1", respectively. Theorem 2 extends the result of Theorem 1 to ROBDDs with complemented edges.

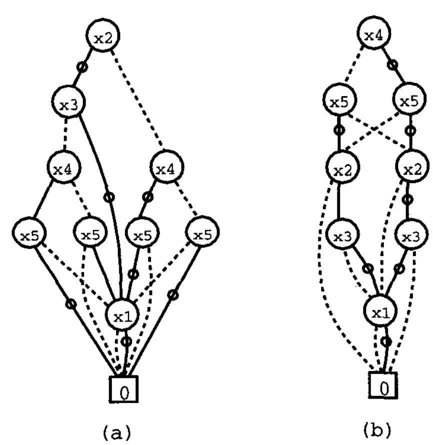


Figure 3. ROBDDs with complemented edges for the orderings (a) $\langle x_2, x_3, x_4, x_5, x_1\rangle$ and (b) $\langle x_4, x_5, x_2, x_3, x_1\rangle$ for the function $f(x_1, \ldots, x_5) = x_1(x_4 \oplus x_5)' + x_2'(x_4 \oplus x_5) + x_1 x_3 + x_1' x_2 x_3'$.

Theorem 2 *For any $n \geq 5$, there exists a function $f(x_1, \ldots, x_n)$ such that for at least one pair of orderings $\pi_i, \pi_j \in \Pi$ in $G^c(f)$, it holds that $P^c_{\pi_i} > P^c_{\pi_j}$, but $N^c_{\pi_i} < N^c_{\pi_j}$.*

Proof: Consider the 5-variable function from the proof of Theorem 1. The ordering $\pi_1 = \langle x_2, x_3, x_4, x_5, x_1\rangle$ (Figure 3(a)) results in a ROBDD with complemented edges with $N^c_{\pi_1} = 10$, $P^{c0}_{\pi_1} = 5$ and $P^{c1}_{\pi_1} = 9$. On the other hand, the ordering $\pi_2 = \langle x_4, x_5, x_2, x_3, x_1\rangle$ (Figure 3(b)) yields a graph with $N^c_{\pi_2} = 9$, $P^{c0}_{\pi_2} = 8$ and $P^{c1}_{\pi_2} = 12$. Therefore, the latter ordering encodes more paths to both "0" and "1" with a smaller number of nodes. We can extend this result to $n \geq 5$ in the same manner as above. and □

4. Conclusion

In this paper, we showed that the number of nodes in a ROBDD is not a monotonically increasing function of the number of implicants in a disjoint cover, represented by the paths of this ROBDD. A class of function showing this is the case for ROBDDs with or without complemented edges was presented. Hence, the size of a disjoint cover cannot be used as a direct guide in selecting a good variable ordering.

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