Composition of Reduced Ordered Binary Decision Diagrams

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Abstract

A widely accepted opinion is that, under a certain ordering restriction, the class of Reduced Ordered Binary Decision Diagrams (ROBDDs) is closed under composition. However, in this paper we show that this is not correct. We formulate necessary and sufficient conditions for the composition of two ROBDDs to result in a non-reduced OBDD. Ignoring these conditions may lead to an incorrect conclusion about the canonicity of the composed graph or to an inaccurate estimation of its size. We also prove that, on the other hand, under a certain ordering restriction, the class of ROBDDs with complemented edges is indeed closed under composition.

1 Introduction

It is quite common that a logic network describing a structured design contains repeating substructures. Bottom-up approaches exploit this regularity to produce a more economic logic description for such a network. The main facility of these approaches is functional composition. First, the subfunctions representing the individual substructures are derived, and then the complete function is composed from these subfunctions.

Let f and g be Boolean functions of type $\{0,1\}^n \rightarrow \{0,1\}$, of the arguments $x_1, \ldots, x_n, n \geq 1$. *Composition* is the operation of replacing some argument x_i, $i \in \{1, \ldots, n\}$, of f by function g, resulting in the following function

$$f|_{x_i=g}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n).$$

Using the Shannon expansion [1] of a function with respect to the variable x_i, namely

$$f(x_1, \ldots, x_n) = x_i' \cdot f|_{x_i=0} + x_i \cdot f|_{x_i=1}$$

we can derive the following expression for $f|_{x_i=g}$:

$$f|_{x_i=g}(x_1, \ldots, x_n) = g' \cdot f|_{x_i=0} + g \cdot f|_{x_i=1}. \tag{1}$$

Bryant [2] has presented an efficient algorithm $Compose$ for performing composition of two functions represented as ROBDDs. To compute (1), $Compose$ utilizes the ternary Boolean operation ITE (if-then-else), namely $ITE(a, b, c) = a \cdot b + a' \cdot c$. The algorithm has a worst-case complexity $O(|G_1|^2 \cdot |G_2|)$, where G_1 and G_2 are ROBDDs representing f and g. Furthermore, if the following ordering restriction holds:

$$\text{There are no } j \in I_f \text{ and } k \in I_g \text{ such that } i < j \leq k \text{ or } i > j \geq k \tag{2}$$

where I_f and I_g are *dependence sets* of f and g, respectively, defined by $I_f = \{i \mid f|_{x_i=0} \neq f|_{x_i=1}\}$, then composition can be performed in a simpler and more efficient way by substituting each vertex $v \in G_1$ having index i by a copy of G_2, replacing each branch to a terminal vertex 0 in G_2 by a branch to $low(v)$ and each branch to a terminal vertex 1 in G_2 by a branch to $high(v)$. It was stated in [2] that, provided G_1 and G_2 are reduced, the graph resulting from composition is also reduced [2, p. 686]. Many ROBDD-related works use this assumption, including [3, p. 8], [4, p. 58], [5]. However, we found that there are cases when composition results in a non-reduced OBDD. For example, consider the functions $f = x_1 \oplus x_2$ and $g = x_3 \oplus x_4$, where "$\oplus$" denotes XOR. Let G_1 be a ROBDD for f with the ordering $\langle x_1, x_2 \rangle$ and G_2 be a ROBDD for g with the ordering $\langle x_3, x_4 \rangle$. If G_2 is substituted in the vertices labeled by 2 in G_1, then we get the graph with 7 non-terminal vertices, which is not reduced.

In this paper we present necessary and sufficient conditions for the composition of two ROBDDs to result in a non-reduced graph. Similar conditions, formulated differently, were shown in [7] without proof[1]. We also prove that, for the case of ROBDDs with complemented edges [6], the resulting OBDD is always reduced.

2 Necessary and sufficient conditions

The following theorem presents necessary and sufficient conditions for the composition of two ROBDDS to result in a non-reduced OBDD. We use the notation $G_1(i \leftarrow G_2)$ for the graph representing $f|_{x_i=g}$, the indexed letters v, u and w to denote the vertices of the graphs G_1, G_2 and $G_1(i \leftarrow G_2)$, respectively, and the terms f_v, g_u and $f_w|_{x_i=g}$ for the functions represented by the subgraphs rooted by v, u and w, respectively.

Theorem 1 *Let G_1 and G_2 be ROBDDs for functions f and g, respectively. For any $i \in I_f$, if the ordering restriction (2) holds, then the OBDD $G_1(i \leftarrow G_2)$ for $f|_{x_i=g}$ is not reduced if and only if G_1 and G_2 satisfy the following two conditions:*

1. *$\exists v_1, v_2 \in G_1$ such that $low(v_1) = high(v_2)$, $high(v_1) = low(v_2)$ and $index(v_1) = index(v_2) = i$.*
2. *$\exists u_1, u_2 \in G_2$ such that $g_{u_1} = g'_{u_2}$ and $index(v_1) = index(v_2) = j$, for some $j \in I_g$.*

Proof: 1) "if" part: Suppose the conditions are satisfied. Then, by definition of ROBDDs [2, p. 679], the subgraphs of G_1 having root vertices v_1 and v_2 represent the following functions:

$$\begin{aligned} f_{v_1} &= x'_i f_{low(v_1)} + x_i f_{high(v_1)}, \\ f_{v_2} &= x'_i f_{low(v_2)} + x_i f_{high(v_2)} \\ &= x'_i f_{high(v_1)} + x_i f_{low(v_1)} \qquad \{\text{by condition } 1\} \end{aligned} \tag{3}$$

Consider the graph $G_1(i \leftarrow G_2)$, obtained after replacement of vertices v_1 and $v_2 \in G_1$ by G_2. Let $w_1 \in G_1(i \leftarrow G_2)$ be a copy of the vertex $u_1 \in G_2$ obtained after replacement of G_2 in v_1, and $w_2 \in G_1(i \leftarrow G_2)$ be a copy of $u_2 \in G_2$ obtained after replacement of G_2 in v_2. That is, w_1 and w_2 belong to the different copies of G_2.

Since composition is performed by replacing each branch to terminal vertices in G_2 by branches to the correspondent children of the vertices being replaced, the subgraphs rooted by w_1 and w_2 represent the following functions:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_2} f_{high(v_1)} + g_{u_2} f_{low(v_1)} \\ &= g_{u_1} f_{high(v_1)} + g'_{u_1} f_{low(v_1)} \qquad \{\text{by condition } 2\} \end{aligned} \tag{4}$$

Thus, $f_{w_1}|_{x_i=g} = f_{w_2}|_{x_i=g}$. From the way $G_1(i \leftarrow G_2)$ is constructed it is easy to see that the subgraphs rooted by w_1 and w_2 match is their structure and their attributes. Thus, by definition [2, p. 679], they are isomorphic.

2) "only if" part: We assume that $G_1(i \leftarrow G_2)$ is not reduced and show by transformations that then the conditions hold.

An OBDD is not reduced either (a) it contains a vertex v with $low(v) = high(v)$, or (b) it contains distinct vertices v and u such that the subgraphs rooted by v and u are isomorphic [2, p. 679].

If G_1 and G_2 are reduced, (a) can nether happen as a result of composition, because each vertex $v \in G_1$ having index i is replaced by a different copy of G_2, and branches in G_2 going to the different terminal

[1] In a private communication, Luca Macchiarulo pointed out that a proof can be found in [8].

vertices are replaced by branches to different children of G_1. So, the vertices having different children in G_1 and G_2 with have different children in $G_1(i \leftarrow G_2)$. Therefore, if $G_1(i \leftarrow G_2)$ is not reduced, then (b) must hold, i.e. it has isomorphic subgraphs.

Let w_1 and w_2 be vertices in $G_1(i \leftarrow G_2)$, rooting two isomorphic subgraphs. By definition of isomorphism between two OBDDs [2, p. 679], they match in both their structure and their attributes, so $index(w_1) = index(w_2) = j$, for some $j \in (I_f - \{i\}) \cup I_g$, and the function they represent are equivalent:

$$f_{w_1}|_{x_i=g} = f_{w_2}|_{x_i=g}. \tag{5}$$

There are three possibilities for the relative position of w_1 and w_2 in the graph:

- $j \in I_f - \{i\}$,
- $j \in I_g$, and both w_1 and w_2 are in the same copy of G_2,
- $j \in I_g$, w_1 and w_2, and both w_1 and w_2 are in different copies of G_2.

These three exhaust all possible cases.

Let $j \in I_f - \{i\}$. Suppose we decomposed g back. Then, since (5) holds, there must be some vertices v_1, $v_2 \in G_1$, labeled by j, such that $f_{v_1} = f_{v_2}$. This implies that G_1 has isomorphic subgraphs, which contradicts the assumption that G_1 is reduced.

Let $j \in I_g$, and both w_1 and w_2 are in the same copy of G_2. Then the subgraphs rooted by w_1 and w_2 represent the following functions:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_2} f_{low(v_1)} + g_{u_2} f_{high(v_1)}. \end{aligned} \tag{6}$$

where $f_{low(v_1)}$ and $f_{high(v_1)}$ are children of some $v_1 \in G_1$, in which G_2 was substituted, and g_{u_1} and g_{u_2} are functions represented by the subgraphs rooted by some u_1 and $u_2 \in G_2$, which are labeled by j. From (6) and (5) we can conclude that $g_{u_1} = g_{u_2}$, and thus G_2 has isomorphic subgraphs. This contradicts the assumption that G_2 is reduced.

The only case that remains is $j \in I_g$, and w_1 and w_2 are in different duplicate copies of G_2. In this case, the subgraphs rooted by w_1 and w_2 represent the functions:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_2} f_{low(v_2)} + g_{u_2} f_{high(v_2)}. \end{aligned} \tag{7}$$

where $f_{low(v_1)}, f_{high(v_1)}, f_{low(v_2)}, f_{high(v_2)}$ are the functions of the subgraphs rooted by the children of some vertices v_1 and $v_2 \in G_1$, in which G_2 was substituted, and g_{u_1} and g_{u_2} are functions of the subgraphs rooted by some vertices $u_1, u_2 \in G_2$, which are labeled by j. Moreover, since $index(w_1) = index(w_2) = j$, we have $index(u_1) = index(u_2) = j$.

From (5) and (7), we can derive:

$$g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} = g'_{u_2} f_{low(v_2)} + g_{u_2} f_{high(v_2)}.$$

Clearly, the above equation is satisfied if either

(a) $f_{low(v_1)} = f_{low(v_2)}, f_{high(v_1)} = f_{high(v_2)}$ and $g_{u_1} = g_{u_2}$, or

(b) $f_{low(v_1)} = f_{high(v_2)}, f_{high(v_1)} = f_{low(v_2)}$ and $g'_{u_1} = g_{u_2}$.

If (a) holds, then $f_{v_1} = f_{v_2}$ and G_1 has isomorphic subgraphs, which contradicts the assumption. Therefore (b) holds, directly giving us condition 2. Since G_1 is reduced, from $f_{low(v_1)} = f_{high(v_2)}, f_{high(v_1)} = f_{low(v_2)}$ we can also conclude that $low(v_1) = high(v_2)$, $high(v_1) = low(v_2)$, and thus condition 1 holds, too.

□

3 ROBDDs with complemented edges

Next, we consider the case of ROBDDs with *complemented edges* [6]. A complemented edge indicates that the function associated with it is the complement of the function being pointed by the edge. We show that the class of ROBDDs with complemented edges, satisfying the ordering restriction (2), is closed under composition operation.

Theorem 2 *Let G_1 and G_2 be ROBDDs with complemented edges for f and g, respectively. For any $i \in I_f$, if the ordering restriction (2) holds, then the OBDD $G_1(i \leftarrow G_2)$ for $f|_{x_i=g}$ is reduced.*

Proof: By following the same steps as in "only if" part of the proof of Theorem 1, we can show that, for the ROBDDs with complemented edges, $G_1(i \leftarrow G_2)$ is not reduced only if the conditions 1 and 2 hold.

Suppose both conditions hold. Consider the condition 2. If $g_{u_1} = g'_{u_2}$, then $u_1 = u_2$, i.e. both functions are represented by the same subgraph in G_2. Therefore, the equations (4) become:

$$\begin{aligned} f_{w_1}|_{x_i=g} &= g'_{u_1} f_{low(v_1)} + g_{u_1} f_{high(v_1)} \\ f_{w_2}|_{x_i=g} &= g'_{u_1} f_{high(v_1)} + g_{u_1} f_{low(v_1)} \end{aligned}$$

Thus, if the conditions hold, $f_{w_1}|_{x_i=g} \neq f_{w_2}|_{x_i=g}$, and there are no isomorphic subgraphs in $G_1(i \leftarrow G_2)$. Since the conditions are proved to be necessary, $G_1(i \leftarrow G_2)$ cannot have isomorphic subgraphs.

□

4 Conclusion

In this paper we show that the composition of two ROBDDs may result in a non-reduced OBDD and summarize necessary and sufficient conditions for this. We also prove that the class of ROBDDs with complemented edges, satisfying the ordering restriction (2), is closed under composition.

It should be noted, that most typical BDD packages use hash tables to check for existing vertices before creating new ones, in this way ensuring that the obtained graph is reduced. However, in theoretical investigations, overlooking the conditions given by Theorem 1 may lead to incorrect conclusions about the canonicity of the resulting graph (like in [5]), or about the number of vertices in it (like in [3, p. 8], [4, p. 58]).

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