Upper Bound on the Number of Product-Terms in a Sum-of-Products Expansion of Multiple-Valued Functions

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Abstract

This paper considers the complexity of sum-of-products expansions of multiple-valued functions over a chain-based Post algebra. An upper bound on the number of product-terms in these expansions is derived. Such a bound provides a measure for estimating the maximal size of a Programmable Logic Array needed to implement a function of a fixed number of variables. It can also be used to evaluate the performance of heuristic logic minimizers, by being contrasted to their solutions. To derive this bound, a class of functions which are "worst-case" for the expansion is studied.

Keywords: multiple-valued function, sum-of-products expansion, upper bound.

1 Introduction

In this paper we consider the complexity of sum-of-products expansions of multiple-valued functions over a chain-based Post algebra $\mathcal{P} := \langle M;\ +, \cdot, J;\ 0, m-1\rangle$, where $M := \{0, 1, \ldots, m-1\}$ is a set whose elements form a totally ordered chain, "+" and "·" are the binary operations *maximum* (MAX) and *minimum* (MIN) respectively, and $J := \{J_0, J_1,\ \ldots, J_{m-1}\}$ is a set of *literal* operators, such that

$$J_i x\ := \begin{cases} m-1 & \text{if } x = i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

*Supported in part by Research Grant No 240-98-101 from the Swedish Research Council for Engineering Sciences and by a fellowship from the Knut and Alice Wallenbergs foundation of Sweden.

where x is a multiple-valued variable and $i \in M$ is a constant. For convenience, we write $J_i x$ as $\overset{i}{x}$. Under the operations "+" and "·" the chain forms a distributive lattice with the least element 0 and the greatest element $m-1$. This algebra is known to be functionally complete with constants [1], meaning that every multiple-valued function on M can be defined as a composition of its basic operations and constants.

Any m-valued n-variable function has a *canonical sum-of-products expansion* over $\mathcal{P}$ of the form

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{m^n-1} c_i \overset{i_1}{x_1} \overset{i_2}{x_2} \ldots \overset{i_n}{x_n} \qquad (2)$$

where $c_i \in M$ are constants, and $(i_1 i_2 \ldots i_n)$ is the m-ary representation of i with i_1 being the least significant digit.

While the canonical sum-of-products expansion of a function is unique, usually more than one non-canonical sum-of-products expansion of a function exists. For example, the 2-variable, 3-valued function shown in Figure 1 can be expressed as $f(x_1, x_2) = 1 \overset{1}{x_1}\overset{0}{x_2} + 1 \overset{1}{x_1}\overset{2}{x_2} + 2 \overset{1}{x_2}$, or as $f(x_1, x_2) = 1 \overset{1}{x_1} + 2 \overset{1}{x_2}$.

$x_2 \backslash x_1$	0	1	2
0	0	1	0
1	2	2	2
2	0	1	0

Figure 1: An example function.

As a measure of complexity of the expansion we use the total number of product-terms. This measure is common for sum-of-products expansions, because usually they are implemented by Programmable Logic Arrays (PLAs) [5]-[8], and the area of a PLA is proportional to the number of product-terms in the sum-of-products expansion [9]. Using this complexity measure, we define a *minimal* sum-of-products expansion as an expansion with the smallest number of product-terms. For example, for the function in Figure 1, the expansion $f(x_1, x_2) = 1 \overset{1}{x_1} + 2 \overset{1}{x_2}$ is the minimal one.

To characterize the "worst-case" among all functions of n variables, we derive an upper bound on the number of product-terms in minimal sum-of-products expansions over $\mathcal{P}$, which is a novel result. Some work has been done on deriving the upper bounds on the number of product-terms for other types of sum-of-products expansions. The expansion of multiple-valued functions over minimum, truncated sum and window literal has been considered in [17] and [18]. The *window literal* is an extension of the literal, defined by:

$$^{i}x^{j} := \begin{cases} m-1 & \text{if } i \leq x \leq j \\ 0 & \text{otherwise} \end{cases}$$

where $i, j \in M$ and $i \leq j$. The operation *truncated sum* is given by $\mathrm{TSUM}(x_1, x_2) := \mathrm{MIN}(x_1 + x_2, m-1)$, where "+" is the regular arithmetic addition. In [17], the upper bound on the number of product-terms in a minimal expansion of this type has been derived for the cases of 1-variable m-valued functions (for any $m > 1$) and of 2-variable m-valued functions (for $1 < m < 8$). This was extended in [18] to handle the case when output phase optimization is allowed, for 1-variable m-valued functions (for any $m > 1$) and for 2-variable 3-, 4- and 5-valued functions. In [19] an upper bound on the average number of product-terms in a minimal expansion of multiple-valued functions over minimum, maximum and window literal has been derived. In [20] the case of multiple-valued input two-valued output function has been considered. An interesting work, relating the complexity of an expansion over a given set of operations to the properties of the operations is [21].

Our result is important in several respects. First, it provides a measure for estimating the maximal size of a PLA needed to implement a function of a fixed number of variables. Second, the upper bound can be contrasted to the solutions computed by heuristic logic minimizers [10]-[16] in order to evaluate their performance. This is of special significance for multiple-valued functions for which the exact methods for minimization are few and inefficient.

To derive an upper bound, we identify a special class of m-valued n-variable functions and prove that, for the case $n \leq m-1$, they are "worst-case" for the sum-of-products expansion over $\mathcal{P}$. Therefore, the number of product-terms in minimal sum-of-products expansions of these functions gives the exact upper bound for $n \leq m-1$. Using this result, we derive an approximate upper bound for $n > m-1$. Finding the exact upper bound for $n > m-1$ seems to be a very hard combinatorial problem which remains open.

The paper is organized as follows. Section 2 gives the notation and definitions used in the sequel. In Section 3, a special type of multiple-valued functions is defined and studied. In Section 4, the number of product-terms in a minimal expansion of these functions is computed. In Section 5, these functions are used to derive an upper bound on the number of product-terms at the minimal sum-of-products expansions over $\mathcal{P}$. In the final section, some conclusions are drawn and a direction for further research is proposed.

2 Notation and definitions

We use the standard definitions and notation in the area of multiple-valued logic ([2]). The most important notions are briefly summarized in this section.

Let $M := \{0, 1, \ldots, m-1\}$ be a finite set of values. An *m-valued n-variable function* $f(x_1, \ldots, x_n)$ is a mapping $f : M^n \to M$. M^n is the *domain* of f and M is the *codomain*. A point in the domain of the function is called a *minterm*.

A *product-term* is MIN of one or more literals $\overset{i}{x}_j$, $i \in M, j \in \{1, \ldots, n\}$. A *sum-of-products expansion* of f is MAX of product-terms. We denote by $E^c(f)$ the canonical sum-of-products expansion (2) of f, and by $E^{min}(f)$ the minimal sum-of-products expansion of f. We use $N(E^c(f))$ and $N(E^{min}(f))$ to denote the number of product-terms in $E^c(f)$ and $E^{min}(f)$, respectively.

Let $\Omega_M(n)$ be the set of all n-variable functions on M. The exact upper bound on the number of product-terms in $E^{min}(f)$ for $f \in \Omega_M(n)$ is defined by [21]:

$$UB(n) := \max_{f \in \Omega_M(n)} N(E^{min}(f)). \tag{3}$$

Let p_1 and p_2 be two product-terms of at most n variables and $c_1, c_2 \in M$ be constants such that $c_1 < c_2$. Since $c_1 + c_2 = c_2$ if $c_1 < c_2$ for any $c_1, c_2 \in M$, therefore

$$c_1 \cdot p_1 + c_2 \cdot p_2 = c_1 \cdot (p_1 + p_2) + c_2 \cdot p_2. \tag{4}$$

The following rule is a special case of (4). It will often be used further in the proofs. Let $(a_1, \ldots, a_{j-1}, a_{j-1}, \ldots, a_n) \in M^{n-1}$ be some fixed $(n-1)$-tuple in M^{n-1}, $c_i \in M$ be constants such that $c_0 \leq c_i$, for all $i \in \{1, 2, \ldots, m-1\}$ and x_j be some variable, $j \in \{1, 2, \ldots, n\}$. Then:

$$\sum_{i \in M} c_i \overset{a_1}{x}_1 \ldots \overset{a_{j-1}}{x}_{j-1} \overset{i}{x}_j \overset{a_{j+1}}{x}_{j+1} \ldots \overset{a_n}{x}_n = c_0 \overset{a_1}{x}_1 \ldots \overset{a_{j-1}}{x}_{j-1} \overset{a_{j+1}}{x}_{j+1} \ldots \overset{a_n}{x}_n + \\ + \sum_{\substack{\forall k \in M \\ such\ that \\ c_k > c_0}} c_k \overset{a_1}{x}_1 \ldots \overset{a_{j-1}}{x}_{j-1} \overset{k}{x}_j \overset{a_{j+1}}{x}_{j+1} \ldots \overset{a_n}{x}_n \tag{5}$$

All product-terms with $c_i = c_0$, $i \in \{1, 2, \ldots, m-1\}$ get absorbed in the product-term $c_0 \overset{a_1}{x}_1 \ldots \overset{a_{j-1}}{x}_{j-1} \overset{a_{j+1}}{x}_{j+1} \ldots \overset{a_n}{x}_n$.

If $c_i = c_k, \forall i, k \in M$, then the rule (5) merges m product-terms of n variables into a single product-term of $n-1$ variables. We say that the simplification is carried of *with respect to* the variable x_j. E.g. if $m = 3$, then we can perform the following simplification with respect to x_1: $1 \overset{0}{x}_1 \overset{1}{x}_2 + 1 \overset{1}{x}_1 \overset{1}{x}_2 + 1 \overset{2}{x}_1 \overset{1}{x}_2 = 1 \overset{1}{x}_2$.

3 Generalized parity functions

It is well-known that the exact upper bound on the number of product-terms in a minimal sum-of-products expansion of an n-variable Boolean function in Boolean algebra over $\{0, 1\}$ is 2^{n-1}. For any n, there are two functions which have exactly 2^{n-1} product-terms in their minimal sum-of-products expansions, called *even-parity* and *odd-parity.* The even-parity (odd-parity) function has value 1 when an even (odd) number of its variables have value 1.

While it is easy to see that parity functions are the "worst-case" for the sum-of-products expansions of Boolean functions, it is much harder to recognize which m-valued functions are the "worst-case" for the sum-of-products expansions of multiple-valued functions over chain-based Post algebra.

In this section we identify n-variable m-valued functions which, as will be proved later, are "worst-case" for the sum-of-products expansion for $n \leq m-1$. The definition below shows the construction scheme. The number of product-terms in their minimal expansions will give us the exact upper bound for $n \leq m-1$. This bound will used as base for deriving an approximate upper bound for $n > m-1$. Finding a general construction scheme for the "worst-case" function for $n > m-1$ seems to be a very hard combinatorial problem which remains open.

We use the notation h_n instead of the conventional $h(x_1, \ldots, x_n)$ to denote an n-variable function h. Such an abbreviation simplifies the proofs and doesn't cause any confusion, because in our case it is the number of variables that matters, and not the variables themselves. A function h_n^r is defined inductively through the subfunctions h_{n-1}^r of $n-1$ variables.

Definition 1 *For a fixed $j \in M$, the function h_n^r is defined inductively by:*

1. $h_0^r := r$
2. $h_n^r := h_{n-1}^{r \ominus 1} \overset{j}{x}_n + \sum_{i \in M - \{j\}} h_{n-1}^r \overset{i}{x}_n$

where $r \in M$ and "$\ominus$" denotes subtraction modulo m.

As an example, consider the case of $m = 3$ and $n = 2$, and let $r = m-1$. Then, the defining tables for the functions h_2^2 for $j \in \{0, 1, 2\}$ are shown in Figure 2.

$j = 0$:

	0	1	2
0	0	1	1
1	1	2	2
2	1	2	2

$j = 1$:

	0	1	2
0	2	1	2
1	1	0	1
2	2	1	2

$j = 2$:

	0	1	2
0	2	2	1
1	2	2	1
2	1	1	0

Figure 2: The functions h_2^2, for a fixed $j \in \{0, 1, 2\}$ and $m = 3$.

For $m = 2$, h_n^1 corresponds to the odd-parity function when $j = 0$, and to the even-parity function when $j = 1$.

Now we prove a useful property of h_n^r functions which will be used further in the proofs.

Property 1 *Let* $n \geq 0$ *and* $r \in M$*. Then*

$$h_n^{r\ominus 1} = h_n^r \ominus \mathbf{1}$$

where "$\ominus$*" denotes subtraction modulo* m*, extended to functions as usual.*

Proof: By induction on n.

1) Obvious for $n = 0$.

2) Hypothesis: Assume the result holds for n. For a fixed $j \in M$ we have:

$$\begin{aligned}
h_{n+1}^{r\ominus 1} &= \sum_{i\in M-\{j\}} h_n^{r\ominus 1}\, \overset{i}{x}_{n+1} + h_n^{r\ominus 1\ominus 1}\, \overset{j}{x}_{n+1} \\
&\quad \{\text{Definition 1}\} \\
&= \sum_{i\in M-\{j\}} (h_n^r \ominus \mathbf{1})\, \overset{i}{x}_{n+1} + (h_n^{r\ominus 1} \ominus \mathbf{1})\, \overset{j}{x}_{n+1} \\
&\quad \{\text{by ind. hypothesis}\} \\
&= \sum_{i\in M-\{j\}} (h_n^r\, \overset{i}{x}_{n+1} \ominus \mathbf{1}\, \overset{i}{x}_{n+1}) + (h_n^{r\ominus 1}\, \overset{j}{x}_{n+1} \ominus \mathbf{1}\, \overset{j}{x}_{n+1}) \\
&\quad \{\text{distributivity of "}\cdot\text{" over "}\ominus\text{"}\} \\
&= (\sum_{i\in M-\{j\}} h_n^r\, \overset{i}{x}_{n+1} + h_n^{r\ominus 1}\, \overset{j}{x}_{n+1}) \ominus \mathbf{1} \cdot (\sum_{i\in M-\{j\}} \overset{i}{x}_{n+1} + \overset{j}{x}_{n+1}) \\
&\quad \{\text{distributivity of "}\ominus\text{" over "}+\text{"}\} \\
&= (\sum_{i\in M-\{j\}} h_n^r\, \overset{i}{x}_{n+1} + h_n^{r\ominus 1}\, \overset{j}{x}_{n+1}) \ominus \mathbf{1} \\
&\quad \{\text{by (1), sum of literals over M is } \mathbf{1}\} \\
&= h_{n+1}^r \ominus \mathbf{1} \\
&\quad \{\text{Definition 1}\}
\end{aligned}$$

□

As an example, consider the defining tables for the functions h_2^2 for $j \in \{0, 1, 2\}$, shown in Figure 2. The ith row in the defining table, $i \in \{0, 1, 2\}$, corresponds to the subfunction h_1^i. The value of j determines which row is decremented by 1 (modulo m) as compared to the other rows. E.g., for $j = 0$, the coefficients of the 0th row of the defining table are decremented by 1. Next, we prove a fundamental theorem showing that for the case of $r = m - 1$ a minimal sum-of-products expansion of h_n^r has the same number of product-terms as its canonical sum-of-products expansion.

Theorem 1

$$N(E^{min}(h_n^{m-1})) = N(E^c(h_n^{m-1})).$$

Proof: By induction on n.

1) Obvious for $n = 0$.

2) Hypothesis: Assume the result holds for n. No simplification, reducing the number of product-terms, can be carried out with respect to the variables $x_1, \ldots, x_n$, or otherwise $N(E^c(h_n^{m-1})) > N(E^{min}(h_n^{m-1}))$, which contradicts the hypothesis.

Consider the structure of the function h_{n+1}^r. By Property 1, it consists of $m-1$ identical subfunctions h_n^r and one subfunction $h_n^{r\ominus 1}$, which is different from h_n^r. Therefore, for all n-tuples $(a_1, \ldots, a_n) \in M^n$, we have $h_{n+1}^r(a_1, a_2, \ldots, a_n, i) = c$ for $i \in M - \{j\}$ and $h_{n+1}^r(a_1, a_2, \ldots, a_n, j) = c \ominus 1$ for some $j \in M$ and for some $c \in M$. The only simplification which can be applied to the canonical sum-of-product-term expansion of h_{n+1}^r with respect to the variable x_{n+1} is, by (5):

$$\begin{aligned} (c \ominus 1) \cdot \overset{a_1}{x}_1 \ldots \overset{a_n}{x}_n \overset{j}{x}_{n+1} + \sum_{i \in M - \{j\}} c \cdot \overset{a_1}{x}_1 \ldots \overset{a_n}{x}_n \overset{i}{x}_{n+1} = \\ = (c \ominus 1) \cdot \overset{a_1}{x}_1 \ldots \overset{a_n}{x}_n + \sum_{i \in M - \{j\}} c \cdot \overset{a_1}{x}_1 \ldots \overset{a_n}{x}_n \overset{i}{x}_{n+1} \end{aligned}$$

which eliminates the variable x_{n+1} from the first product-term, but does not reduce the number of product-terms in the expansion. No further simplification can be carried out with respect to x_{n+1}, so $N(E^{min}(h_{n+1}^{m-1})) = N(E^c(h_{n+1}^{m-1}))$.

□

4 The number of product-terms in the functions h_n^r

In this section, we derive the number of product-terms in a minimal sum-of-products expansion of a function h_n^r. In order to simplify the derivations below, we first introduce the following notation.

Definition 2 *In an m-valued system, P_n^r is defined inductively by:*

$$1. \quad P_0^r := \begin{cases} 1 & \textit{if } r \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$
$$2. \quad P_n^r := (m-1) \times P_{n-1}^r + P_{n-1}^{r \ominus 1}$$

where $r \in M$, $n > 0$, "$\ominus$" denotes subtraction modulo m, and "-", "+" and "$\times$" denote the regular arithmetic operations of subtraction, addition and multiplication, correspondently.

Notice that, for notational convenience, during this and next section, we use "+" to denote arithmetic addition. In the previous sections we used "+" for MAX.

Let $r \in M$ and $n \geq 0$. The following property shows that P_n^r gives the number of product-terms in the canonical sum-of-products expansion of h_n^r.

Property 2

$$N(E^c(h_n^r)) = P_n^r$$

Proof: Follows directly from Definitions 1 and 2.

□

As an example, consider the case $n = 3$ and $m = 3$. We can compute the number of product-terms in the canonical sum-of-products expansion of h_3^2 as follows:

1) $n = 0: \; P_0^2 = 1, P_0^1 = 1, P_0^0 = 0.$

2) $n = 1: \; P_1^2 = 2P_0^2 + P_0^1 = 3, P_1^1 = 2P_0^1 + P_0^0 = 2, P_1^0 = 2P_0^0 + P_0^2 = 1.$

3) $n = 2: \; P_2^2 = 2P_1^2 + P_1^1 = 8, P_2^1 = 2P_1^1 + P_1^0 = 5, P_2^0 = 2P_1^0 + P_1^2 = 5.$

4) $n = 3: \; P_3^2 = 2P_2^2 + P_2^1 = 21.$

Although it is possible to compute P_n^r directly from its definition, a more convenient way exists. Next, we prove a property showing that P_n^r can be obtained using the binomial coefficients C_n^k (or $\binom{n}{k}$). Recall, that these coefficients are defined for non-negative integers n and k as follows [22, p. 101]:

$$C_n^k := \begin{cases} \frac{n!}{k!(n-k)!} & \text{for } 0 \le k \le n \\ 0 & \text{for } 0 \le n < k. \end{cases} \tag{6}$$

Property 3

$$P_n^r = \sum_{i=0}^{n} C_n^i \times (m-1)^{n-i} \times P_0^{r \ominus i}.$$

where $r \in M$*,* $n \ge 0$ *and "*$\times$*" denotes arithmetic multiplication.*

Proof: By induction on n. We omit "$\times$" where obvious.

1) Let $n = 1$. Then

$$\begin{aligned} P_1^r &= (m-1)P_0^r + P_0^{r\ominus 1} && \{\text{Definition 2}\} \\ &= C_1^0\ (m-1)\ P_0^r + C_1^1\ P_0^{r\ominus 1} && \{C_1^0 = C_1^1 = 1 \text{ by (6)}\} \\ &= \sum_{i=0}^{1} C_1^i\ (m-1)^{n-i}\ P_0^{r\ominus i} && \{\text{reordering}\} \end{aligned}$$

2) Hypothesis: Assume the result holds for n. Then we have

$$\begin{aligned} P_{n+1}^r &= (m-1)\ P_n^r + P_n^{r\ominus 1} \\ &\quad \{\text{Definition 2}\} \\ &= (m-1) \sum_{i=0}^{n} C_n^i\ (m-1)^{n-i}\ P_n^{r\ominus i} + \sum_{i=0}^{n} C_n^i\ (m-1)^{n-i}\ P_n^{r\ominus i\ominus 1} \\ &\quad \{\text{by ind. hypothesis}\} \\ &= (m-1)\ C_n^0\ (m-1)^n\ P_n^r + (m-1) \sum_{i=1}^{n} C_n^i\ (m-1)^{n-i}\ P_n^{r\ominus i} + \\ &\quad + \sum_{i=0}^{n-1} C_n^i\ (m-1)^{n-i}\ P_n^{r\ominus(i+1)} + C_n^n\ (m-1)^0\ P_n^{r\ominus(n+1)} \\ &\quad \{\text{reordering, } r\ominus i\ominus 1 = r\ominus(i+1)\} \\ &= (m-1)^{n+1}\ P_n^r + \sum_{i=1}^{n} C_n^i\ (m-1)^{n-i+1}\ P_n^{r\ominus i} + \\ &\quad + \sum_{j=1}^{n} C_n^{j-1}\ (m-1)^{n-j+1}\ P_n^{r\ominus j} + P_n^{r\ominus(n+1)} \\ &\quad \{C_n^0 = C_n^n = 1,\ \text{substituting } j = i+1 \text{ in the third term}\} \end{aligned}$$

$$
\begin{aligned}
&= (m-1)^{n+1}\ P_n^r + \sum_{i=1}^{n} (C_n^i + C_n^{i-1})\ (m-1)^{n-i+1}\ P_n^{r\ominus i} + P_n^{r\ominus(n+1)} \\
&\quad \{C_n^0 = C_n^n = 1\} \\
&= (m-1)^{n+1}\ P_n^r + \sum_{i=1}^{n} C_{n+1}^i\ (m-1)^{n-i+1}\ P_n^{r\ominus i} + P_n^{r\ominus(n+1)} \\
&\quad \{(6)\} \\
&= C_{n+1}^0\ (m-1)^{n+1-0}\ P_n^{r\ominus 0} + \sum_{i=1}^{n} C_{n+1}^i\ (m-1)^{n-i+1}\ P_n^{r\ominus i} + \\
&\quad + C_{n+1}^{n+1}\ (m-1)^{n+1-(n+1)}\ P_n^{r\ominus(n+1)} \\
&\quad \{C_{n+1}^0 = C_{n+1}^{n+1} = 1\} \\
&= \sum_{i=0}^{n+1} C_{n+1}^i\ (m-1)^{n-i+1}\ P_n^{r\ominus i} \\
&\quad \{\text{reordering}\}
\end{aligned}
$$

□

5 Upper bound on the number of product-terms in sum-of-products expansions over $\mathcal{P}$

In this section we use the functions h_n^{m-1} to derive an upper bound on the number of product-terms in sum-of-products expansions over $\mathcal{P}$. First, we show that, for the case $n \leq m-1$, h_n^{m-1} are "worst-case" functions, giving the exact upper bound. Using this result, we then derive an approximate upper bound for the case $n > m-1$. Finding the exact upper bound for $n > m-1$ seems to be a very hard combinatorial problem which remains open.

Theorem 2 *For any $f \in \Omega_M(n)$, the upper bound on the number of product-terms in a minimal sum-of-products expansion of f over $\mathcal{P}$ is:*

1. *$UB(n) = m^n$, for $n < m-1$.*

2. *$UB(n) = m^n - 1$, for $n = m-1$.*

3. *$UB(n) \leq (m^{m-1} - 1) \times m^{n-(m-1)}$, for $n > m-1$.*

where $n \geq 0$, *and "-" and "×" denote arithmetic subtraction and multiplication, correspondently.*

Proof: 1) Let $n < m-1$. Obviously, no n-variable m-valued function can have more than m^n product-terms. We show that there exists a function which has m^n product-terms in its minimal sum-of-products expansion and that h_n^{m-1} is such a function.

On one hand, from Theorem 1 and Property 2, we can conclude that $N(E^{min}(h_n^{m-1}))$ $= P_n^{m-1}$. On the other hand:

$$\begin{aligned}
P_n^{m-1} &= \sum_{i=0}^{n} C_n^i \times (m-1)^{n-i} \times P_0^{(m-1)\ominus i} && \{\text{Property 3}\} \\
&= \sum_{i=0}^{n} C_n^i \times (m-1)^{n-i} && \{\text{Df. 2}, \forall\ 0 \leq i < m-1 : P_0^{(m-1)\ominus i} = 1\} \\
&= \sum_{i=0}^{n} C_n^i \times (m-1)^{n-i} \times 1^i && \{\forall i \geq 0 : 1^i = 1\} \\
&= ((m-1)+1)^n && \{\text{binomial expansion [22, p. 101]}\} \\
&= m^n
\end{aligned}$$

So, for $n < m-1$, there exist $f \in \Omega_M(n)$ such that $N(E^{min}(f)) = m^n$.

2) Let $n = m-1$. We first prove that $N(E^{min}(h_n^{m-1})) = m^n - 1$ and then show that $\forall f \in \Omega_M(n) :\ N(E^{min}(f)) \leq N(E^{min}(h_n^{m-1}))$.

Similarly to the case 1, $N(E^{min}(h_n^{m-1})) = P_n^{m-1}$. By Definition 2, for all $0 \leq i < m-2$ we have $P_0^{(m-1)\ominus i} = 1$, and for $i = m-1$ we have $P_0^{(m-1)\ominus(m-1)} = P_0^0 = 0$. Thus, we can conclude that:

$$\begin{aligned}
P_n^{m-1} &= \left(\sum_{i=0}^{n} C_n^i \times (m-1)^{n-i}\right) - C_n^n \times (m-1)^{(m-1)-n} \\
&= m^n - 1 \qquad \{C_n^n = 1 \text{ and } (m-1)^{(m-1)-n} = 1 \text{ for } n = m-1\}
\end{aligned}$$

So, for $n = m-1$, there exists $f \in \Omega_M(n)$ such that $N(E^{min}(f)) = m^n - 1$. We also know that $N(E^{min}(h_n^{m-1})) = N(E^c(h_n^{m-1}))$, therefore if there exists a function $f \in \Omega_M(n)$ with more than $m^n - 1$ product-terms in its minimal sum-of-products form, then $N(E^c(f)) > N(E^c(h_n^{m-1}))$, i.e. $N(E^c(f)) = m^n$.

Let f be any function in $\Omega_M(n)$ with $N(E^c(f)) = m^n$. We will prove that $N(E^c(f)) = m^n$ implies $N(E^{min}(f)) < N(E^c(f))$.

Since $N(E^c(f)) = m^n$, f has only non-zero values in its codomain. Let $c \in M - \{0\}$ be the smallest value in the codomain of f. Suppose f evaluates to c for the minterm $(a_{11}, a_{12}, \ldots, a_{1n}) \in M^n$, i.e. that $f(a_{11}, a_{12}, \ldots, a_{1n}) = c$. Then, for all other minterms, the value of f should be strictly larger then c, or otherwise we can subsequently apply the rule (5) and merge all minterms mapped by f to c into a single product-term (constant-c). This would reduce the number of product-terms in $E^{min}(f)$. So, in order to have $N(E^{min}(f)) = N(E^c(f))$, the following should hold:

$$f(a_{11}, a_{12}, \ldots, a_{1n}) < f(b_1, b_2, \ldots, b_n)$$

for all $(b_1, b_2, \ldots, b_n) \in M^n - \{(a_{11}, a_{12}, \ldots, a_{1n})\}$.

Next, consider any $(n-1)$-variable subfunction of f which doesn't have the value c in its codomain. If $a_{21} \neq a_{11}, a_{21} \in M$, then $f(a_{21}, x_2, \ldots, x_n)$ is one such subfunction. By making similar considerations as above, we can see that $f(a_{21}, x_2, \ldots, x_n)$ can evaluate to the minimal value only for a single $(n-1)$-tuple, say, $(a_{22}, \ldots, a_{2n}) \in M^{n-1}$, or otherwise we can apply the rule (5) and reduce the number of product-terms in $E^{min}(f)$. So, to have $N(E^{min}(f)) = N(E^c(f))$, the following should hold:

$$f(a_{11}, a_{12}, \ldots, a_{1n}) < f(a_{21}, a_{22}, \ldots, a_{2n}) < f(a_{21}, b_2, \ldots, b_n)$$

for all $(b_2, \ldots, b_n) \in M^{n-1} - \{(a_{22}, \ldots, a_{2n})\}$.

Continuing for $(n-2)$-variable subfunctions of $f(a_{21}, x_2, \ldots, x_n)$, and successively further down to the subfunctions of 0-variables, we finally get a necessary condition for $N(E^{min}(f)) = N(E^c(f))$ in the form:

$$f(a_{11}, a_{12}, \ldots, a_{1n}) < f(a_{21}, a_{22}, \ldots, a_{2n}) < f(a_{21}, a_{32}, \ldots, a_{3n}) < \ldots$$

$$\ldots < f(a_{21}, a_{32}, \ldots, a_{n(n-1)}, a_{nn}) < f(a_{21}, a_{32}, \ldots, a_{n(n-1)}, a_{(n+1)n})$$

$a_{21} \neq a_{11}, a_{32} \neq a_{22}, \ldots, a_{n(n-1)} \neq a_{(n-1)(n-1)}, a_{(n+1)n} \neq a_{nn}$. Obviously, we need to have at least $n+1$ different non-zero values in the codomain of f to satisfy this condition. On the other hand, since $n = m - 1$, we have only n non-zero values in M. Thus the necessary condition will be violated and some product-terms will merge. Therefore $N(E^c(f)) = m^n$ implies $N(E^{min}(f)) < N(E^c(f)$.

3) Let $n > m - 1$. By [2], any n variable function can be decomposed as:

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{m^{n-(m-1)}-1} x_1^{i_1} \ldots x_{n-(m-1)}^{i_{n-(m-1)}} \; f(i_1, \ldots, i_{n-(m-1)}, x_{n-(m-1)+1}, \ldots, x_n)$$

where $(i_1 \ldots i_{n-(m-1)})$ is the m-ary representation of i and $f(i_1, \ldots, i_{n-(m-1)}, x_{n-(m-1)+1}, \ldots, x_n)$ are subfunctions of $f(x_1, \ldots, x_n)$ obtained by fixing the first $n-(m-1)$ variables to the values $i_1, i_2, \ldots, i_{n-(m-1)}$. Obviously, there are $m^{n-(m-1)}$ such subfunctions for any choice of the $n - (m - 1)$ variables. By case 2, each of these subfunctions has at most $m^{m-1} - 1$ product-terms in its minimal form. Thus, $N(E^{min}(f)) \leq (m^{m-1} - 1) \times m^{n-(m-1)}$.

□

For $m = 2$, the upper bound given by Theorem 2 reduces to the familiar $UB(n) = 2^{n-1}$ for $n > 1$ and $UB(n) = 1$ for $n \leq 1$.

For the case $n > m-1$, the functions h_n^{m-1} are not the "worst-case" any longer. E.g., for $m = 3$ and $n = 3$, $N(E^{min}(h_3^2)) = 21$, however, the functions with more product-terms in there minimal sum-of-products expansion exist. For example, the function shown in Figure 3 has 24 product-terms in its minimal sum-of-products expansion. Notice, that $(m^{m-1} - 1) \times m^{n-(m-1)} = 24$, for $m = 3$ and $n = 3$, and therefore the upper bound given by the case 3 of Theorem 2 can be exact for some n and m.

6 Conclusion

In this paper we derive an upper bound on the number of product-terms in sum-of-products expansions of multiple-valued functions over a chain-based Post algebra. To obtain this result we identify multiple-valued functions which are the "worst-case" for the expansion for the case $n \leq m-1$ and compute the number of product-

	0			1			2		
	0	1	2	0	1	2	0	1	2
0	0	2	2	2	2	1	2	1	2
1	2	2	1	2	0	2	1	2	2
2	2	1	2	1	2	2	2	2	0

Figure 3: A function with $N(E^{min}(f)) = 24$.

terms in their minimal expansions. It gives the exact upper bound for $n \leq m-1$. This bound is used to obtain an approximate upper bound for $n > m-1$. Finding the exact upper bound for $n > m-1$ remains a challenging problem for further research.

Acknowledgment

The author is grateful to Dr. Dilian Gurov and Prof. Michael Miller for carefully reading and commenting on the manuscript.

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