Solutions to Assignment 2

1. The function f looks like

f :

		c	c, a	a
	-	0	1	1
b	0	1	1	1

   (a) **on-set:** i. $\{011, 100, 101, 110, 111\}$, or alternatively
   ii. $\{1--, -11\}$, or
   iii. $\{1--, 011\}$ or...
   off-set: $\{001, 010\}$
   dc-set: $\{000\}$

   (b) **on-set:** i. $\{(3,3), (4,4), (5,5), (6,6), (7,7)\}$, or alternatively
   ii. $\{(4,7), (3,7)\}$, or
   iii. $\{(4,7), (3,3)\}$, or...
   off-set: $\{(1,1), (2,2)\}$
   dc-set: $\{(0,0)\}$

   (c) **on-set:** i. $\{100101, 011010, 011001, 010110, 010101\}$, or alternatively
   ii. $\{011111, 110101\}$, or
   iii. $\{011111, 100101\}$, or...
   off-set: $\{101001, 100110\}$
   dc-set: $\{101010\}$

2. Using parallel encoding: $c_1 = (4,7)$, $c_2 = (3,7)$.
   OR_b and AND_b denote bitwise-OR and bitwise-AND respectively.

   (a) $c_1 \cap c_2 = (7,7)$
   $c_1 \cap c_2 = (min(c_1)\ \mathrm{OR}_b\ min(c_2), max(c_1)\ \mathrm{AND}_b\ max(c_2))$

   (b) $\mathrm{supercube}(c_1, c_2) = (0,7)$
   $\mathrm{supercube}(c_1, c_2) = (min(c_1)\ \mathrm{AND}_b\ min(c_2), max(c_1)\ \mathrm{OR}_b\ max(c_2))$

   (c) $c2 \subseteq c_1$? No.
   $c2 \subseteq c_1$ if, and only if, $c_1 \cap c_2 = c_2$
   $c_1 \cap c_2 = (7,7) \neq (3,7) = c_2$, so c_2 is not contained in c_1.

   (d) $\overline{c_1 \cup c_2} = \{(0,1), (2,2)\}$
   We use the fact that $\overline{c_1 \cup c_2} = \overline{c_1} \cap \overline{c_2}$.
   $\overline{c_1} = \{(0,3)\}$, $\overline{c_2} = \{(0,5), (2,6)\}$
   (See below how to calculate the complement of a cube.)
   Observe that $\overline{c_2}$ is not a single cube, but a set of two cubes. The intersection of two sets of cubes A and B is the union of the intersection

of all pairs of cubes in $A \times B$. In our case,

$$\begin{aligned}\overline{c_1} \cap \overline{c_2} &= \{(0,3)\} \cap \{(0,5),(2,6)\} \\ &= \{(0,3) \cap (0,5)\} \cup \{(0,3) \cap (2,6)\} \\ &= \{(0,1)\} \cup \{(2,2)\} = \{(0,1),(2,2)\}\end{aligned}$$

3. Using sequential encoding: $c_1 = 011111$, $c_2 = 110101$.
   OR_b and AND_b denote bitwise-OR and bitwise-AND respectively.

   (a) $c_1 \cap c_2 = 010101$
   $c_1 \cap c_2 = c_1 \text{ AND}_b\ c_2$

   (b) $\text{supercube}(c_1, c_2) = 111111$
   $\text{supercube}(c_1, c_2) = c_1 \text{ OR}_b\ c_2$

   (c) $c2 \subset c_1$? No. $c2 \subseteq c_1$ if, and only if, $c_1 \cap c_2 = c_2$
   $c_1 \cap c_2 = 010101 \neq 110101 = c_2$, so c_2 is not contained in c_1.

   (d) $\overline{c_1 \cup c_2} = \{101011, 101110\}$
   Similar to exercise 2)d).
   $\overline{c_1} = 101111$, $\overline{c_2} = \{111011, 111110\}$

$$\begin{aligned}\overline{c_1} \cap \overline{c_2} &= \{101111\} \cap \{111011, 111110\} \\ &= \{101111 \cap 111011\} \cup \{101111 \cap 111110\} \\ &= \{101011\} \cup \{101110\} = \{101011, 101110\}\end{aligned}$$

   (See below how to calculate the complement of a cube.)

How to Calculate the Complement of a Cube

Parallel encoding: Let c be a cube in parallel encoding.
Make n pairs p_i by making

1. all bits of $min(p_i)$ are 0, except the ith bit, that is the ith bit of $min(c)$ inverted.
2. all bits of $max(p_i)$ are 1, except the ith bit, that is the ith bit of $max(c)$ inverted.

Delete all pairs that do not represent cubes, i.e. the pairs p_i in which $min(p_i) > max(p_i)$. The remaining pairs form the complement of c.
For example, if $c = (3,7)$ we obtain the pairs $(4,3)$, $(0,5)$, $(2,6)$. We delete $(4,3)$ since it does not represent a cube. $\bar{c} = \{(0,5),(2,6)\}$

Sequential encoding: Let c be a cube in parallel encoding.
Make n sequences s_i by making all bit-pairs of s_i equal to 11, except the ith bit-pair, that is the ith bit-pair of c inverted. Delete all sequences that do not represent cubes, i.e. the sequences s_i in which there is a 00 bit-pair. The remaining sequences form the complement of c.
For example, if $c = 11\,01\,01$ we obtain the sequences $00\,11\,11$, $11\,10\,11$, $11\,11\,10$. We delete $00\,11\,11$ since it does not represent a cube. We obtain $\bar{c} = \{111011, 111110\}$.

(4) **1 point** Prove that $a+1=1$, $\forall a \in B$, for the Boolean algebra $(B;+,\cdot,';0,1)$, where $B=\{0,1\}$.

First, we prove two supplementary properties:
$P6_1$: $a \overset{A4_1}{=} a1 \overset{A5_2}{=} a(a+a') \overset{A3_1}{=} (aa)+(aa') \overset{A5_1}{=} (aa)+0 \overset{A4_2}{=} aa$.

$P6_2$: $a \overset{A4_2}{=} a+0 \overset{A5_1}{=} a+(aa') \overset{A3_2}{=} (a+a)(a+a') \overset{A5_2}{=} (a+a)1 \overset{A4_1}{=} a+a$.

Now we prove that $P3_2 : a+1=1$ as follows:
$P3_2$: $a+1 \overset{A4_1}{=} (a+1)1 \overset{A5_2}{=} (a+1)(a+a') \overset{A3_1}{=} ((a+1)a)+((a+1)a') \overset{A3_1}{=} ((aa)+(1a)+(aa')+(1a') \overset{P6_1,A4_1}{=} (a+a)+((aa')+(1a')) \overset{A5_1,A4_1}{=} (a+a)+(0+a') \overset{P6_2,A4_2}{=} a+a' \overset{A5_2}{=} 1$.

(5) **1 point** Prove that $a+a\cdot b=a$, $\forall a,b \in B$, for the Boolean algebra $(B;+,\cdot,';0,1)$, where $B=\{0,1\}$.

$P5_2$: $a+(ab) \overset{A4_1}{=} (a1)+(ab) \overset{A3_1}{=} a(1+b) \overset{P3_2}{=} a1 \overset{A4_1}{=} a$.

(6) **1 point** Check whether the following algebra $(B;+,\cdot,';\mathbf{0},\mathbf{1})$ is a Boolean algebra:

$B=\{0,1,2,3\}$
"$+$" is the maxim (the largest of two values)
"$\cdot$" is the minimum (the smallest of two values)
"$'$" is the complement dened by: $x'=(x+1) mod\ 4$, i.e. $0'=1, 1'=2, 2'=3, 3'=0$.
0 = 0
1 = 3

No. For example, Axiom $A5_1 : a\cdot a'=0$ does not hold for the value $a=1$, since $1\cdot 1'=1\cdot 2=1$.