Rigorous studies of anyons

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based on work in collaborations with
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Outline of Talk

1. Fractional statistics and 2D anyons
2. General energy bounds
3. Anyons in a harmonic trap
4. Average Field Approximation for almost-bosonic anyons

Identical particles and statistics in 2D

Particle exchange in 2D: $\Psi \in L^2((\mathbb{R}^2)^N) \cong \bigotimes^N L^2(\mathbb{R}^2)$

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi} \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$

$e^{i\alpha\pi} \in U(1)$ **any** phase
$\alpha = 0$: bosons
$\alpha = 1$: fermions

anyons: fractional-statistics quasiparticles in confined systems; conjectured to play a role in the FQHE

Streater & Wilde 1970 (QFT) ... Leinaas & Myrheim '77; Goldin, Menikoff & Sharp '81; Wilczek '82 ...

Reviews by Fröhlich '90, Wilczek '90, Iengo & Lechner '92, Lerda '92, Myrheim '99, Khare '05, Ouvry '07, ...

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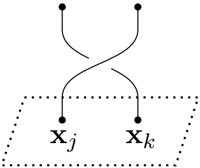

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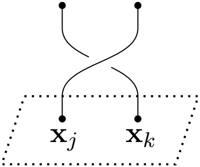

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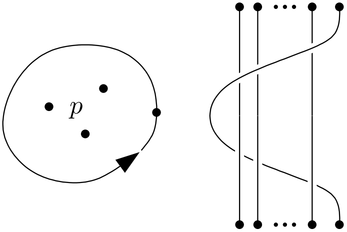
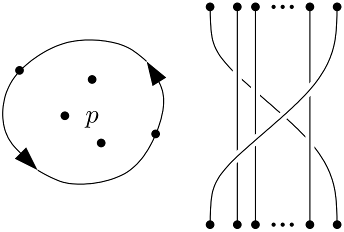
Modelling anyons rigorously



$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$

Think: free kinetic energy $\hat{T}_0 = \frac{1}{2m}\sum_{j=1}^{N}(-\Delta_j)$ acting on multivalued

$$\Psi_\alpha := U^\alpha\,\Psi_0, \qquad U := \prod_{j<k} e^{i\phi_{jk}} = \prod_{j<k}\frac{z_j - z_k}{|z_j - z_k|}.$$

Modelling anyons rigorously

Bosons ($\Psi \in L^2_{\mathsf{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_\alpha := \frac{1}{2m}\sum_{j=1}^{N} D_j^2, \quad D_j = -i\nabla_j + \alpha \mathbf{A}_j, \quad \mathbf{A}_j(x) = \sum_{k\neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

Precise definition in magnetic gauge: (DL, Solovej, 2013/'14)

$$D : L^2_{\mathsf{sym}}(\mathbb{R}^{2N}) \to \mathscr{D}'(\mathbb{R}^{2N} \setminus \triangle; \mathbb{C}^{2N}), \quad \int_{\mathbb{R}^{2N}} |D\Psi|^2 < \infty$$

Def. & Theorem: $\hat{T}_{\alpha\in\mathbb{R}} := \frac{1}{2m}(D_{\mathsf{min}})^* D_{\mathsf{min}} = \frac{1}{2m}(D_{\mathsf{max}})^* D_{\mathsf{max}}$

$$\Rightarrow \begin{array}{lcl} \mathrm{Dom}(\hat{T}_{\alpha=2n}) & = & U^{-2n} H^2_{\mathsf{sym}}(\mathbb{R}^{2N}) \\ \mathrm{Dom}(\hat{T}_{\alpha=2n+1}) & = & U^{-(2n+1)} H^2_{\mathsf{asym}}(\mathbb{R}^{2N}) \end{array}$$

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Energy bounds: Lieb-Thirring inequalities for anyons

Theorem ([DL-Solovej'13] Kinetic energy inequality for anyons)

Let Ψ be an N-anyon wavefunction on $\mathbb{R}^2$ with any $\alpha \in \mathbb{R}$. Then

$$\langle \Psi, \hat{T}_\alpha \Psi \rangle \;\geq\; C_{\alpha,N}^2\, C \int_{\mathbb{R}^2} \rho_\Psi(\mathbf{x})^2\, d\mathbf{x},$$

for a constant $10^{-4} \leq C \leq \pi$.

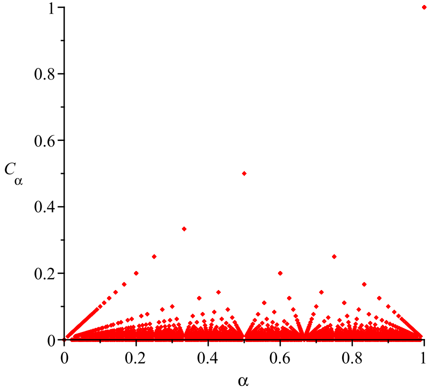
Corollary (Lieb-Thirring inequality for anyons)

Let V be a real-valued potential on $\mathbb{R}^2$. Then

$$\hat{T}_\alpha + \hat{V} \;\geq\; -C_{\alpha,N}^{-2}\, C' \int_{\mathbb{R}^2} |V_-(\mathbf{x})|^2\, d\mathbf{x},$$

for a positive constant C'.

Energy bounds: A local exclusion principle



$$C_{\alpha,N} := \min_{p\in\{0,1,\ldots,N-2\},\, q\in\mathbb{Z}} |(2p+1)\alpha - 2q|$$

$$\xrightarrow[N\to\infty]{} \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Ground state energy of the ideal anyon gas

Ground state energy per unit area for a gas of "free" anyons with **odd** numerator fractional statistics parameter $\alpha = \frac{\mu}{\nu}$, confined to an area L^2:

$$\frac{C}{\nu^2}\,\bar{\rho}^2 \leq \frac{E_0}{L^2} \leq \tilde{C}\,\bar{\rho}^2, \quad \text{with density } \bar{\rho} := \frac{N}{L^2}.$$

Anyons in a harmonic trap

Harmonic oscillator Hamiltonian:

$$\hat{H}_N = \hat{T}_\alpha + \hat{V} = \sum_{j=1}^{N} \left(\frac{1}{2m}(-i\nabla_j + \alpha \mathbf{A}_j)^2 + \frac{m\omega^2}{2}|\mathbf{x}_j|^2 \right).$$

Rigorous bounds for the ground-state energy $E_0 := \inf \operatorname{spec} \hat{H}_N$:

$$\hat{H}_N|_{\text{ang. mom. } L} \ \geq \ \omega \left(N + \left| L + \alpha \frac{N(N-1)}{2} \right| \right)$$ (Chitra, Sen, 1992)

$$\frac{1}{3}\sqrt{\frac{8C}{\pi}} C_{\alpha,N} \ \leq \ E_0/(\omega N^{\frac{3}{2}}) \ \leq \ \tilde{C} \quad \forall \alpha, N$$ (DL, Solovej, 2013)

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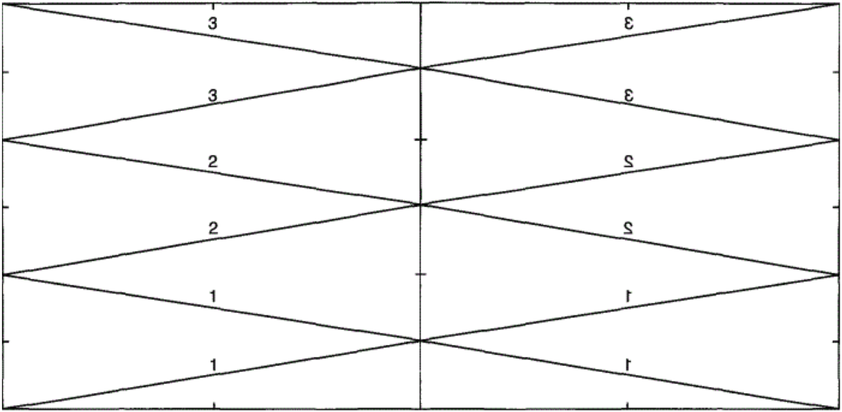
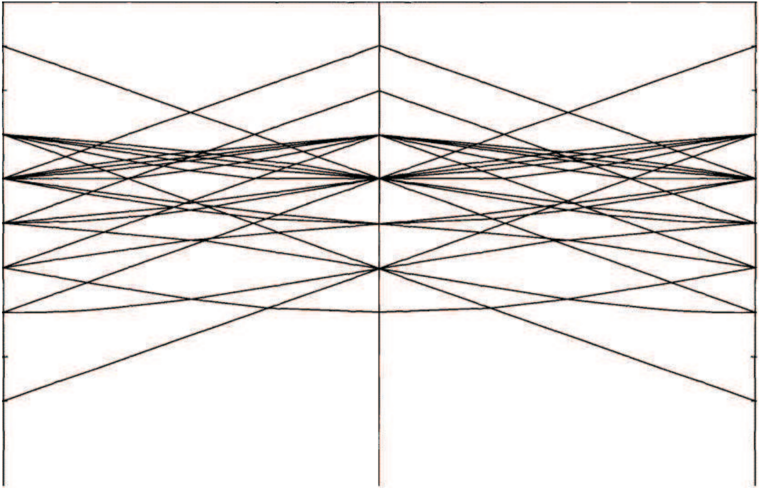


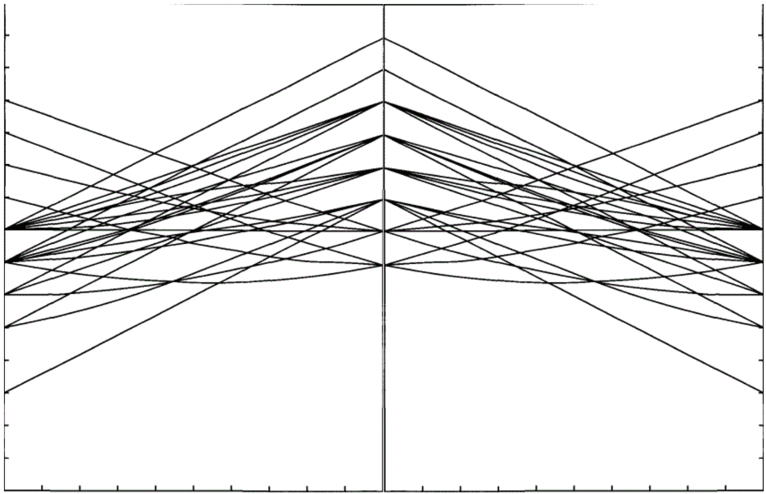
Exact $N = 2$ spectrum: Leinaas, Myrheim, 1977

Anyons in a harmonic trap



Numerical $N = 3$ spectrum: Murthy, Law, Brack, Bhaduri, 1991; Sporre, Verbaarschot, Zahed, 1991

Anyons in a harmonic trap



Numerical $N = 4$ spectrum: Sporre, Verbaarschot, Zahed, 1992

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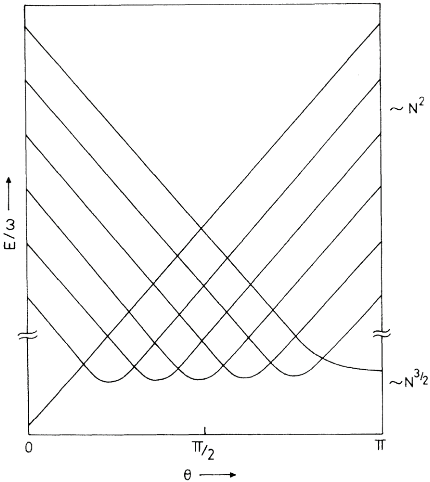
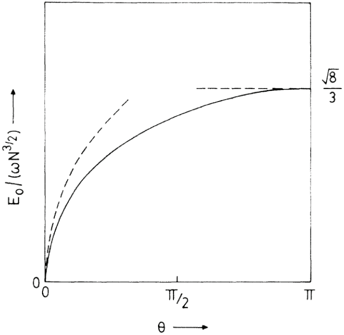
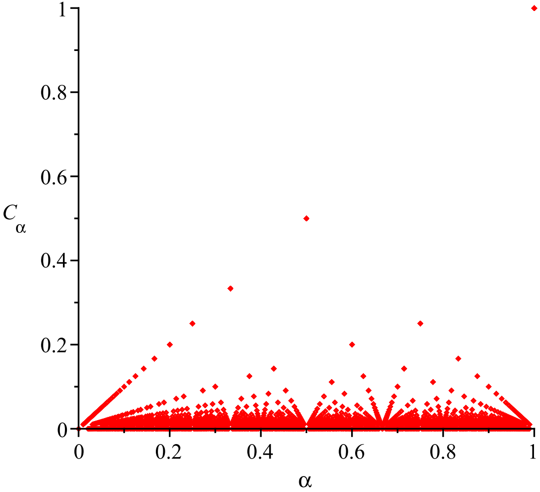




Schematic $N \to \infty$ spectrum: Chitra, Sen, 1992 $(\theta = \alpha\pi)$

Anyons in a harmonic trap

$$\frac{E_0}{\omega N^{\frac{3}{2}}} \gtrsim$$



Rigorous lower bound: DL, Solovej, 2013/'14

Average Field Approximation: (see e.g. Wilczek 1990 review)

$$\mathbf{A}_j = \nabla^\perp \log|\,.\,| * \sum_{k\neq j} \delta_{\mathbf{x}_k} \quad \approx \quad \mathbf{A}[\rho] := \nabla^\perp \log|\,.\,| * \rho$$

$$\hat{H}_N \approx \sum_{j=1}^{N} \left((\mathbf{p}_j + \alpha \mathbf{A}[\rho])^2 + V(\mathbf{x}_j) \right)$$

$\Rightarrow$ If $\Psi = u^{\otimes N}$, $\|u\|_2 = 1$, $\rho = N|u|^2$:

$$\langle \Psi, \hat{H}_N \Psi \rangle \approx N\mathcal{E}^{\mathrm{af}}[u], \quad \text{with}$$

$$\mathcal{E}^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \left(\left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right), \quad \beta \sim N\alpha$$

$$\mathrm{curl}\, \mathbf{A}[|u|^2] = 2\pi |u|^2$$

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Average Field Approximation for almost-bosonic anyons

We consider R-extended anyons: $R > 0$

$$\mathbf{A}_j^R := \sum_{k \neq j} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k), \quad w_R(\mathbf{x}) := \log|\,.\,| * \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(\mathbf{x}),$$

$$\begin{aligned}\hat{H}_N^R &:= \sum_{j=1}^N \left((-i\nabla_j + \alpha \mathbf{A}_j^R)^2 + V(\mathbf{x}_j)\right) \\ &= \sum_{j=1}^N \left(\mathbf{p}_j^2 + V(\mathbf{x}_j)\right) \quad + \ \alpha^2 \sum_{j \neq k} |\nabla w_R(\mathbf{x}_j - \mathbf{x}_k)|^2 \\ &\quad + \alpha \sum_{j \neq k} \Big(\mathbf{p}_j \cdot \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) + \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \cdot \mathbf{p}_j\Big) \\ &\quad + \alpha^2 \sum_{j \neq k \neq \ell} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \cdot \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_\ell)\end{aligned}$$

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&= \sum_{j=1}^{N} \left(\mathbf{p}_j^2 + V(\mathbf{x}_j) \right) \quad + \; \alpha^2 \sum_{j \neq k} |\nabla w_R(\mathbf{x}_j - \mathbf{x}_k)|^2 \\
&\quad + \alpha \sum_{j \neq k} \Big(\mathbf{p}_j \cdot \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) + \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \cdot \mathbf{p}_j \Big) \\
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\end{aligned}$$

Average Field Approximation for almost-bosonic anyons

Theorem ([DL-Rougerie'15] Validity of the average field approx.)

Assume we have N extended anyons with radius $R \sim N^{-\eta}$, $0 < \eta < \frac{1}{4}\left(1 + \frac{1}{s}\right)^{-1}$, and statistics parameter $\alpha = \beta/N$, $\beta \in \mathbb{R}$, in a confining potential $V(\mathbf{x}) \sim |\mathbf{x}|^s$. Then, in the limit $N \to \infty$,

$$\frac{E_0^R(N)}{N} \to E^{\mathrm{af}} := \inf_{\|u\|_2=1} \mathcal{E}^{\mathrm{af}}[u].$$

Moreover, if Ψ_N is a sequence of ground states for $\hat{H}_N^R$, with associated reduced density matrices $\gamma_N^{(k)}$, then (up to a subseq.)

$$\gamma_N^{(k)} \to \int_{\mathcal{M}^{\mathrm{af}}} |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\mu(u),$$

$$\mathcal{M}^{\mathrm{af}} := \{u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} = 1,\ \mathcal{E}^{\mathrm{af}}[u] = E^{\mathrm{af}}\}.$$

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