Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics

Douglas Lundholm
IHES / IHP / CARMIN

joint work with
Jan Philip Solovej,
University of Copenhagen

Variational & Spectral Methods in Quantum Mechanics
Paris, June 20, 2013

Outline of Talk

1. Identical particles and statistics in lower dimensions
2. Local exclusion principle
3. New Lieb-Thirring type inequalities
4. Applications
5. Outlook

Identical particles and statistics: standard theory

N-particle wavefunction $\psi \in L^2((\mathbb{R}^d)^N)$

$$\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_N) = \pm\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_N)$$

$+$: **bosons**: $\psi \in \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^d)$

$-$: **fermions**: $\psi \in \bigwedge^N L^2(\mathbb{R}^d)$

$\Leftrightarrow$ Pauli's exclusion principle: $\varphi \wedge \varphi = 0, \;\; \varphi \in L^2(\mathbb{R}^d)$

This classification is valid for elementary particles in $d = 3$ spatial dimensions

Identical particles and statistics: standard theory

N-particle wavefunction $\psi \in L^2((\mathbb{R}^d)^N)$

$$\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_N) = \pm\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_N)$$

$+$: **bosons**: $\psi \in \bigotimes_{\mathrm{sym}}^N L^2(\mathbb{R}^d)$

$-$: **fermions**: $\psi \in \bigwedge^N L^2(\mathbb{R}^d)$

$\Leftrightarrow$ Pauli's exclusion principle: $\varphi \wedge \varphi = 0, \ \varphi \in L^2(\mathbb{R}^d)$

This classification is valid for elementary particles in $d = 3$ spatial dimensions

Identical particles and statistics: generalized theory

Streater, Wilde, 1970 (QFT) ... Leinaas, Myrheim, 1977; Goldin, Menikoff, Sharp, 1981; Wilczek, 1982 ...

Fractional statistics quasiparticles in $d = 2$ — **anyons**:

$$\psi(\ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \ldots) = e^{i\alpha\pi}\psi(\ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_j, \ldots), \quad e^{i\alpha\pi} \in U(1)$$

Note: **continuous** interchange, $\pi_1(\mathbb{R}^{2N} \setminus \triangle / S_N) = B_N$

Intermediate/fractional statistics in $d = 1$:

$$x_1 < x_2 < \ldots < x_N$$

Particle collision $\Rightarrow$ b.c. for ψ at $r = x_{j+1} - x_j \searrow 0$

$$\partial_r \psi = \eta \psi \qquad \text{or} \qquad \psi(r) \sim r^{\alpha}$$

Identical particles and statistics: generalized theory

Streater, Wilde, 1970 (QFT) ... Leinaas, Myrheim, 1977; Goldin, Menikoff, Sharp, 1981; Wilczek, 1982 ...

Fractional statistics quasiparticles in $d = 2$ — **anyons**:

$$\psi(\ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \ldots) = e^{i\alpha\pi}\psi(\ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_j, \ldots), \quad e^{i\alpha\pi} \in U(1)$$

Note: **continuous** interchange, $\pi_1(\mathbb{R}^{2N} \setminus \triangle / S_N) = B_N$

Intermediate/fractional statistics in $d = 1$:

$$x_1 < x_2 < \ldots < x_N$$

Particle collision $\Rightarrow$ b.c. for ψ at $r = x_{j+1} - x_j \searrow 0$

$$\partial_r \psi = \eta \psi \qquad \text{or} \qquad \psi(r) \sim r^{\alpha}$$

Identical particles and statistics: generalized theory

Streater, Wilde, 1970 (QFT) . . . Leinaas, Myrheim, 1977; Goldin, Menikoff, Sharp, 1981; Wilczek, 1982 . . .

Fractional statistics quasiparticles in $d = 2$ — **anyons**:

$$\psi(\ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \ldots) = e^{i\alpha\pi}\psi(\ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_j, \ldots), \quad e^{i\alpha\pi} \in U(1)$$

Note: **continuous** interchange, $\pi_1(\mathbb{R}^{2N} \setminus \triangle / S_N) = B_N$

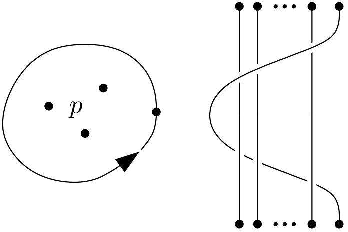
Intermediate/fractional statistics in $d = 1$:

$$x_1 < x_2 < \ldots < x_N$$

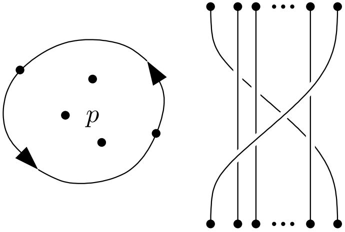
Particle collision $\Rightarrow$ b.c. for ψ at $r = x_{j+1} - x_j \searrow 0$

$$\partial_r \psi = \eta\psi \qquad \text{or} \qquad \psi(r) \sim r^{\alpha}$$

Modelling anyons



$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$

Bosons ($\psi \in L^2_{\mathsf{sym}}$) in $\mathbb{R}^2$ with Aharonov-Bohm magnetic interactions:

$$\hat{T}_{\mathrm{A}} := \frac{1}{2m}\sum_{j=1}^{N} D_j^2, \quad D_j = -i\nabla_j + \boldsymbol{A}_j, \quad \boldsymbol{A}_j(x) = \alpha \sum_{k\neq j} \frac{(\boldsymbol{x}_j - \boldsymbol{x}_k)I}{|\boldsymbol{x}_j - \boldsymbol{x}_k|^2}$$

Modelling anyons

Morally, free kinetic energy $\hat{T} = -\frac{1}{2m}\sum_j \Delta_j$ acting on multivalued

$$\psi_{\mathrm{A}} = U^{\alpha}\,\psi, \qquad U := \prod_{j<k} e^{i\phi_{jk}} = \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|}.$$

Precise definition in magnetic gauge: (DL, Solovej, 2013)

$$D\colon L^2_{\mathsf{sym}}(\mathbb{R}^{2N}) \to \mathscr{D}'(\mathbb{R}^{2N} \setminus \triangle; \mathbb{C}^{2N})$$

$$\hat{T}_{\mathrm{A}}^{(\alpha\in\mathbb{R})} := \frac{1}{2m}(D_{\mathsf{min}})^* D_{\mathsf{min}} = \frac{1}{2m}(D_{\mathsf{max}})^* D_{\mathsf{max}}$$

$$\Rightarrow \begin{array}{rcr} \mathrm{Dom}(\hat{T}_{\mathrm{A}}^{(\alpha=2n)}) & = & U^{-2n} H^2_{\mathsf{sym}}(\mathbb{R}^{2N}) \\ \mathrm{Dom}(\hat{T}_{\mathrm{A}}^{(\alpha=2n+1)}) & = & U^{-(2n+1)} H^2_{\mathsf{asym}}(\mathbb{R}^{2N}) \end{array}$$

Modelling 1D intermediate statistics

Bosons on $\mathbb{R}$ with pairwise interaction potential, singular at the diagonals (boundary) $\triangle$

- Schrödinger-type quantization (η) $\Rightarrow$ **Lieb-Liniger**

$$\hat{T}_{\rm LL} := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2\eta\sum_{j<k}\delta(x_k - x_j)$$

- Heisenberg-type quantization (α) $\Rightarrow$ **Calogero-Sutherland**

$$\hat{T}_{\rm CS} := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{j<k}\frac{\alpha(\alpha-1)}{(x_k - x_j)^2} = \frac{1}{2}Q_\alpha^* Q_\alpha$$

Leinaas, Myrheim, 1977; 1988; 1993; Polychronakos, 1989; Aneziris, Balachandran, Sen, 1991; Isakov, 1992; Myrheim, 1999 (review); DL, Solovej, 2013

Modelling 1D intermediate statistics

Bosons on $\mathbb{R}$ with pairwise interaction potential, singular at the diagonals (boundary) $\triangle$

- Schrödinger-type quantization (η) $\Rightarrow$ **Lieb-Liniger**

$$\hat{T}_{\mathrm{LL}} := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2\eta\sum_{j<k}\delta(x_k - x_j)$$

- Heisenberg-type quantization (α) $\Rightarrow$ **Calogero-Sutherland**

$$\hat{T}_{\mathrm{CS}} := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{j<k}\frac{\alpha(\alpha-1)}{(x_k - x_j)^2} = \frac{1}{2}Q_\alpha^* Q_\alpha$$

Leinaas, Myrheim, 1977; 1988; 1993; Polychronakos, 1989; Aneziris, Balachandran, Sen, 1991; Isakov, 1992; Myrheim, 1999 (review); DL, Solovej, 2013

Modelling 1D intermediate statistics

Bosons on $\mathbb{R}$ with pairwise interaction potential, singular at the diagonals (boundary) $\triangle$

- Schrödinger-type quantization (η) $\Rightarrow$ **Lieb-Liniger**

$$\hat{T}_{\mathrm{LL}} := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2\eta\sum_{j<k}\delta(x_k - x_j)$$

- Heisenberg-type quantization (α) $\Rightarrow$ **Calogero-Sutherland**

$$\hat{T}_{\mathrm{CS}} := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{j<k}\frac{\alpha(\alpha-1)}{(x_k - x_j)^2} = \frac{1}{2}Q_\alpha^* Q_\alpha$$

Leinaas, Myrheim, 1977; 1988; 1993; Polychronakos, 1989; Aneziris, Balachandran, Sen, 1991; Isakov, 1992; Myrheim, 1999 (review); DL, Solovej, 2013

Pauli exclusion and energy inequalities

Pauli exclusion: say $q \in \mathbb{N}$ particles allowed in each one-particle state of $\hat{H}_1 = -\frac{1}{2}\Delta_{\mathbb{R}^d} + V(\boldsymbol{x})$

$\Rightarrow$ Lieb-Thirring inequality: (Lieb, Thirring, 1975)

$$\hat{H}_N = \hat{T} + \hat{V} = \sum_{j=1}^{N} \left(-\frac{1}{2}\Delta_j + V(\boldsymbol{x}_j) \right)$$
$$\geq \ -q \sum_{k=0}^{\infty} |\lambda_k| \ \geq \ -q\, C_d \int_{\mathbb{R}^d} |V_-(\boldsymbol{x})|^{1+\frac{d}{2}}\, d\boldsymbol{x}$$

$\Leftrightarrow$ kinetic energy inequality:

$$T = \frac{1}{2} \int_{\mathbb{R}^{dN}} \sum_{j=1}^{N} |\nabla_j \psi|^2\, dx \ \geq \ \frac{C_d'}{q^{2/d}} \int_{\mathbb{R}^d} \rho(\boldsymbol{x})^{1+\frac{2}{d}}\, d\boldsymbol{x}$$

Bosons: $q = N \to \infty$ $\Rightarrow$ trivial bounds

Local exclusion for fermions

cp. Dyson, Lenard, 1967

Lemma (Local exclusion for fermions in $d = 3$)

Let $\psi \in \bigwedge^n L^2(\mathbb{R}^3)$ be a wavefunction of n fermions and let Ω be a ball of radius ℓ. Then

$$\int_{\Omega^n} \sum_{j=1}^{n} |\nabla_j \psi|^2 \, dx \;\geq\; (n-1)\frac{\xi^2}{\ell^2} \int_{\Omega^n} |\psi|^2 \, dx,$$

where $\xi \approx 2.082$ is the smallest positive root of the equation

$$\frac{d^2}{dx^2} \frac{\sin x}{x} = 0.$$

Local exclusion for anyons

Lemma (Local exclusion for anyons)

Let ψ be a wavefunction of n anyons with $\alpha \in \mathbb{R}$ and let $\Omega \subseteq \mathbb{R}^2$ be either a disk or a square, with area $|\Omega|$. Then

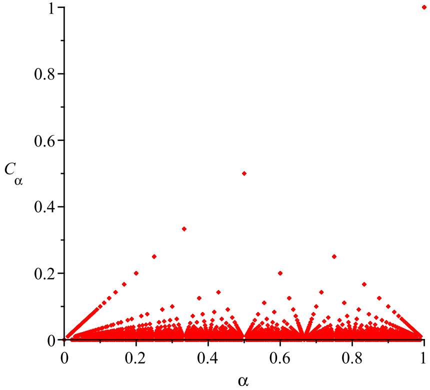
$$\int_{\Omega^n} \sum_{j=1}^{n} |D_j \psi|^2 \, dx \;\geq\; (n-1) \frac{c_\Omega C_{\alpha,n}^2}{|\Omega|} \int_{\Omega^n} |\psi|^2 \, dx,$$

where c_Ω is a constant which satisfies $c_\Omega \geq 0.169$ for the disk and $c_\Omega \geq 0.112$ for the square, and

$$C_{\alpha,n} := \min_{p \in \{0,1,\ldots,n-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|.$$

Idea of proof: pairwise relative magnetic Hardy inequality

Local exclusion for anyons



$$C_{\alpha,N} \quad := \min_{p\in\{0,1,\ldots,N-2\},\, q\in\mathbb{Z}} |(2p+1)\alpha - 2q|$$

$$\xrightarrow[N\to\infty]{} \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Local exclusion for 1D intermediate statistics

Lemma (Local exclusion in 1D)

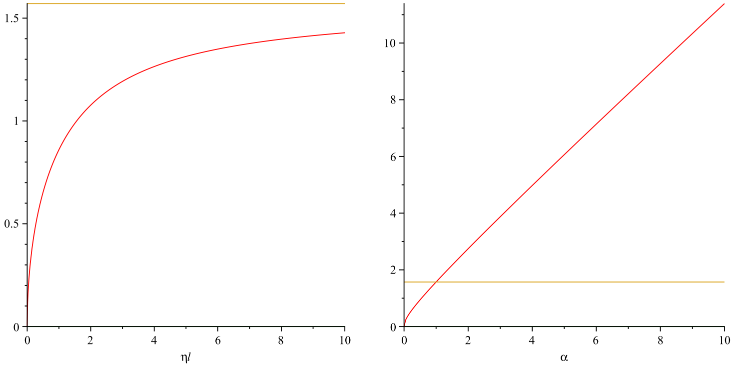
Let ψ be a symmetric wavefunction of n particles on $\mathbb{R}$ and let Ω be an interval of length ℓ. Then for $\eta \geq 0$ resp. $\alpha \geq 1$

$$\int_{\Omega^n} \left(\frac{1}{2} \sum_{j=1}^{n} |\partial_j \psi|^2 + \sum_{j<k} V_{\mathrm{LL/CS}}(x_j - x_k) |\psi|^2 \right) dx \geq (n-1) \frac{\xi_{\mathrm{LL/CS}}^2}{\ell^2} \int_{\Omega^n} |\psi|^2 \, dx,$$

where $\xi_{\mathrm{LL/CS}} = \xi_{\mathrm{LL}}(\eta\ell)$ resp. $\xi_{\mathrm{CS}}(\alpha)$ is the smallest positive root of the equation

$$x \tan x = \eta\ell \qquad \textit{resp.} \qquad \frac{d}{dx} \sqrt{x} J_{\alpha - \frac{1}{2}}(x) = 0.$$

Local exclusion for 1D intermediate statistics



$\xi_{\mathrm{LL}}(\eta\ell)$ resp. $\xi_{\mathrm{CS}}(\alpha)$ as a function of $\eta\ell$ resp. α

Lieb-Thirring inequalities for anyons

Theorem (Kinetic energy inequality for anyons)

Let ψ be an N-anyon wavefunction on $\mathbb{R}^2$ with any $\alpha \in \mathbb{R}$. Then

$$T_{\mathrm{A}} \ \geq \ C_{\alpha,N}^2 \, C_{\mathrm{A}} \int_{\mathbb{R}^2} \rho(\boldsymbol{x})^2 \, d\boldsymbol{x},$$

for a constant $10^{-4} \leq C_{\mathrm{A}} \leq \pi$.

Corollary (Lieb-Thirring inequality for anyons)

Let V be a real-valued potential on $\mathbb{R}^2$. Then

$$\hat{T}_{\mathrm{A}} + V \ \geq \ -C_{\alpha,N}^{-2} \, C'_{\mathrm{A}} \int_{\mathbb{R}^2} |V_-(\boldsymbol{x})|^2 \, d\boldsymbol{x},$$

for a positive constant C'_{A}.

Lieb-Thirring inequalities for 1D statistics

Theorem (Kinetic energy inequality for 1D Lieb-Liniger)

Let ψ *be a symmetric* N*-particle wavefunction on* $\mathbb{R}$*. Then for* $\eta \geq 0$

$$T_{\mathrm{LL}} \;\geq\; C_{\mathrm{LL}} \int_{\mathbb{R}} \xi_{\mathrm{LL}}(2\eta/\tilde{\rho}(x))^2 \rho(x)^3 \, dx,$$

for a constant $3 \cdot 10^{-5} \leq C_{\mathrm{LL}} \leq 2/3$*.*

Theorem (Kinetic energy inequality for 1D Calogero-Sutherland)

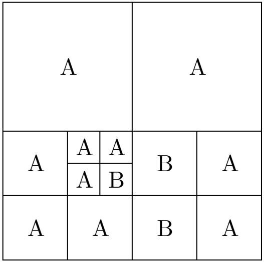
Let ψ *be a symmetric* N*-particle wavefunction on* $\mathbb{R}$*. Then for* $\alpha \geq 1$

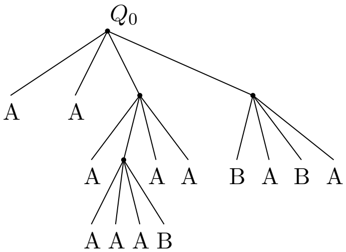
$$T_{\mathrm{CS}} \;\geq\; \xi_{\mathrm{CS}}(\alpha)^2 \, C_{\mathrm{CS}} \int_{\mathbb{R}} \tilde{\rho}(x)^3 \, dx,$$

for a constant $1/32 \leq C_{\mathrm{CS}} \leq 2/3$*.*

$\tilde{\rho}$ is a local approximation of ρ: $\tilde{\rho}|_Q := \frac{\int_Q \rho}{|Q|}$.

Proofs using splitting algorithm





Split recursively until each box contains approximately 2 particles (B) or ~ 0 particles (A). Apply local uncertainty on every box with non-constant density. Apply local exclusion on B's, which also cover for A's with $\sim$constant density.

Application: Ground state energy of an ideal gas

Energy per unit area of a gas of "free" anyons with **odd** numerator fractional statistics parameter $\alpha = \frac{\mu}{\nu}$, confined to an area L^2, is bounded below by

$$\frac{T_{\mathrm{A}}}{L^2} \geq C_{\mathrm{A}} \frac{\bar{\rho}^2}{\nu^2}, \quad \text{with density } \bar{\rho} := N/L^2.$$

In the 1D cases, with density $\bar{\rho} := N/L$:

$$\frac{T_{\mathrm{LL}}}{L} \geq C_{\mathrm{LL}}\, \xi_{\mathrm{LL}}(2\eta/(\gamma\bar{\rho}))^2\, \bar{\rho}^3, \qquad \text{if } \rho(x) \leq \gamma\bar{\rho},$$

resp.

$$\frac{T_{\mathrm{CS}}}{L} \geq C_{\mathrm{CS}}\, \xi_{\mathrm{CS}}(\alpha)^2\, \bar{\rho}^3.$$

For $N \to \infty$ cp. Lieb, Liniger, 1963; resp. Sutherland, 1971

Application: Ground state energy of an ideal gas

Energy per unit area of a gas of "free" anyons with **odd** numerator fractional statistics parameter $\alpha = \frac{\mu}{\nu}$, confined to an area L^2, is bounded below by

$$\frac{T_{\mathrm{A}}}{L^2} \geq C_{\mathrm{A}} \frac{\bar{\rho}^2}{\nu^2}, \quad \text{with density } \bar{\rho} := N/L^2.$$

In the 1D cases, with density $\bar{\rho} := N/L$:

$$\frac{T_{\mathrm{LL}}}{L} \geq C_{\mathrm{LL}}\, \xi_{\mathrm{LL}}(2\eta/(\gamma\bar{\rho}))^2\, \bar{\rho}^3, \qquad \text{if } \rho(x) \leq \gamma\bar{\rho},$$

resp.

$$\frac{T_{\mathrm{CS}}}{L} \geq C_{\mathrm{CS}}\, \xi_{\mathrm{CS}}(\alpha)^2\, \bar{\rho}^3.$$

For $N \to \infty$ cp. Lieb, Liniger, 1963; resp. Sutherland, 1971

Application: External potentials

Harmonic oscillator Hamiltonian:

$$\hat{H} = \hat{T} + \hat{V}_{\text{ext}} = \hat{T} + \sum_{j=1}^{N} \frac{\omega^2}{2} |\boldsymbol{x}_j|^2.$$

By minimization of the corresponding quadratic forms we obtain

$$\hat{H}_{\mathrm{A}} \geq \frac{1}{3}\sqrt{\frac{8C_{\mathrm{A}}}{\pi}}\, C_{\alpha,N}\, \omega N^{\frac{3}{2}} \quad \text{resp.} \quad \hat{H}_{\mathrm{CS}} \geq \frac{\sqrt{3}}{8\pi}\, \xi_{\mathrm{CS}}(\alpha)\, \omega N^2$$

cp. with fermions in 2D: $E_0 \sim \frac{\sqrt{8}}{3}\omega N^{\frac{3}{2}}$ as $N \to \infty$

resp. exact C-S ground state: $E_0 = \frac{1}{2}\omega N(1 + \alpha(N-1))$

Outlook: Anyon trial states

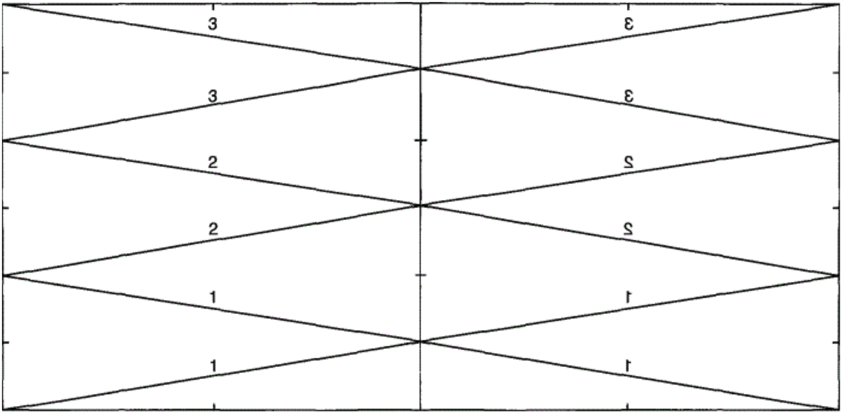
What is the true ground state energy E_0 as a function of α?
Is it even true that $E_0(\alpha) \leq E_0(\alpha = 1)$?

General bound for anyons in a harmonic oscillator: (Chitra, Sen, 1992)

$$\hat{H}_{\mathrm{A}} \geq \omega \left(N + \left| L + \alpha \frac{N(N-1)}{2} \right| \right)$$

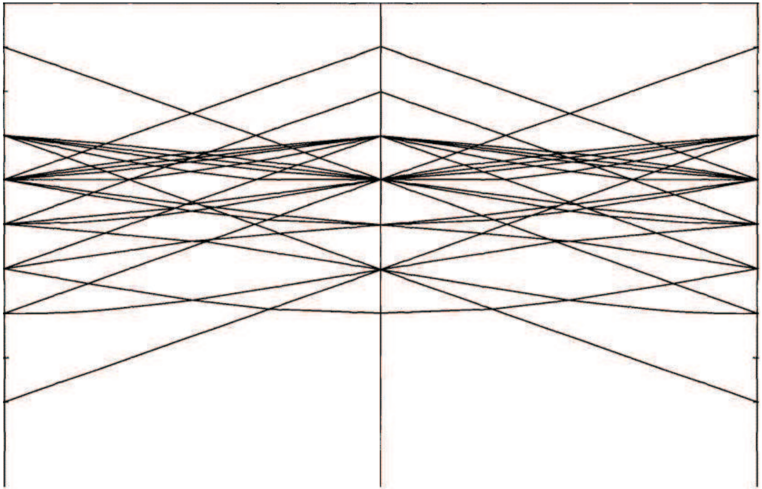
$\Rightarrow$ angular momentum for ground states ψ s.t. $E_0 \sim N^{3/2}$ must be $L = -\alpha\binom{N}{2} + O(N^{3/2})$

Outlook: Anyon trial states



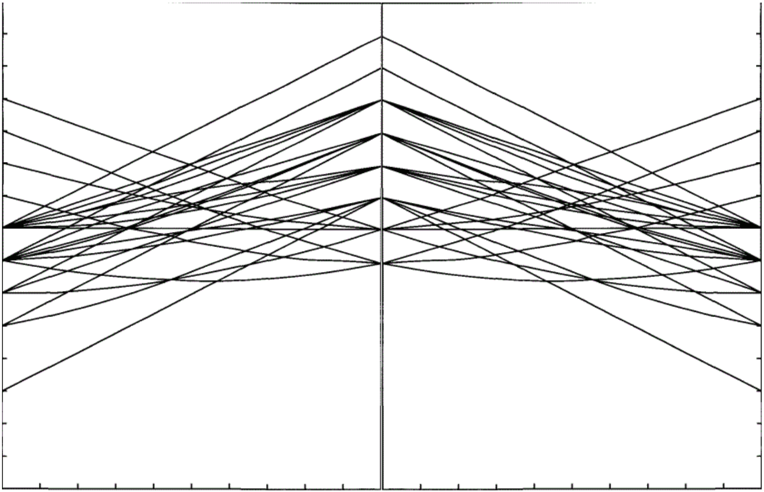
$N = 2$: Leinaas, Myrheim, 1977

Outlook: Anyon trial states



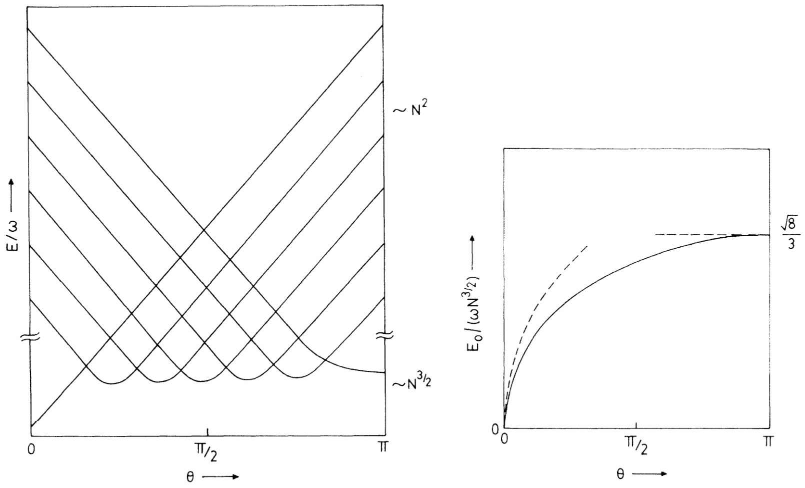
$N = 3$: Murthy, Law, Brack, Bhaduri, 1991; Sporre, Verbaarschot, Zahed, 1991

Outlook: Anyon trial states



$N = 4$: Sporre, Verbaarschot, Zahed, 1992

Outlook: Anyon trial states



$N \to \infty$: Chitra, Sen, 1992

Outlook: Anyon trial states

Consider $N = K\nu$ particles arranged into ν disjoint complete graphs (V_q, E_q). For $\alpha = \mu/\nu$ with **even** numerator:

$$\psi_\alpha(z) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k)\in E_q} (\bar{z}_{jk})^{\mu} \right] \prod_{k=1}^{N} \varphi_0(z_k)$$

For $\alpha = \mu/\nu$ with **odd** numerator:

$$\psi_\alpha(z) := \prod_{j<k} |z_{jk}|^{-\alpha} \, \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k)\in E_q} (\bar{z}_{jk})^{\mu} \bigwedge_{k=0}^{K-1} \varphi_k \left(z_{j\in V_q} \right) \right]$$

These have $L = -\alpha\binom{N}{2} + O(N)$ and yield cancellations in $D_j\psi$

References

D. L., J.P. Solovej, *Hardy and Lieb-Thirring inequalities for anyons*, arXiv:1108.5129, to appear in CMP

D. L., J.P. Solovej, *Local exclusion for intermediate and fractional statistics*, arXiv:1205.2520

D. L., J.P. Solovej, *Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics*, arXiv:1301.3436, to appear in AHP

See also
R.L. Frank, R. Seiringer, *Lieb-Thirring Inequality for a Model of Particles with Point Interactions*, JMP 53, 095201 (2012).