Zero-energy states in supermembrane matrix models

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Outline of Talk

1. Introduction (membrane)
2. Spectrum and ground state conjecture
3. Recent approaches to the study of ground states
4. Outlook

Supermembrane matrix models

$$\mathcal{Z}_{d,N} = \int_{X_{j=1,\ldots,d} \in \mathfrak{su}(N)} \exp\left(\sum_{j<k} \operatorname{tr}\,[X_j, X_k]^2 \right) \prod_{j=1}^{d} dX_j$$

- Extremal supermembranes in $\mathbb{R}^{1,d+1}$
- Super-Yang-Mills in $\mathbb{R}^{1,d}$
- D0-branes / M-theory

W. Taylor, *M(atrix) Theory: Matrix Quantum Mechanics as a Fundamental Theory*, Rev. Mod. Phys., 2001

D. L., *Zero-energy states in supersymmetric matrix models*, Ph.D. thesis, KTH, 2010

Extremal bosonic membrane in $\mathbb{R}^{1,1+d}$

World-volume topology: $\mathbb{R} \times \Sigma$,
Σ fixed 2D compact manifold (Riemann surface)

Embedding coordinate functions: $(x_{j=1,\ldots,d}) : \mathbb{R} \times \Sigma \to \mathbb{R}^d$
(*light-front coordinates*)

Hamiltonian: $H = \int_\Sigma \left(\sum_{j=1}^{d} \dot{x}_j^2 + \sum_{1 \leq j < k \leq d} \{x_j, x_k\}_\Sigma^2 \right)$

Canonical Poisson bracket on Σ: $\{f, g\}_\Sigma = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$

Matrix regularization

Infinite-dimensional Poisson algebra of zero-average real-valued functions x_j	$\rightarrow$	(N^2-1)-dimensional algebra of traceless hermitian $N \times N$ matrices X_j
$\{x_j, x_k\}_\Sigma$	$\rightarrow$	$\frac{1}{i}[X_j, X_k]$
$\int_\Sigma$	$\rightarrow$	tr

(convergence of structure constants $f_{ABC}^{(N)}$)

Respects symmetries:

Diffeomorphism invariance	$\rightarrow$	$\mathrm{SU}(N)$ invariance

$\Rightarrow$ Hamiltonian: $H = \mathrm{tr}\left(\sum_{j=1}^{d} \dot{X}_j^2 - \sum_{1\leq j<k\leq d} [X_j, X_k]^2\right)$

Quantization

Schrödinger representation on $L^2(\mathbb{R}^d \otimes \mathbb{R}^{N^2-1})$:

$$\begin{aligned} X_j &\rightarrow \hat{X}_j = x_{jA} T_A, && x_{jA} \text{ coordinate multiplication operators} \\ \dot{X}_j &\rightarrow \hat{P}_j = p_{jA} T_A, && p_{jA} = -i\partial_{x_{jA}} \end{aligned}$$

Hamiltonian: $H_B = -\Delta_{\mathbb{R}^d \otimes \mathbb{R}^{N^2-1}} + \frac{1}{2} \sum_{A,j,k} (f_{ABC} x_{jB} x_{kC})^2$

Physical Hilbert space: $\mathrm{SU}(N)$-invariant states Ψ

$$f_{ABC} x_{jB} p_{jC} \Psi = 0$$

J. Goldstone, unpublished; J. Hoppe, MIT Ph.D. thesis, 1982

Supersymmetry

Add spin degrees of freedom → supermembrane → SUSY QM

Clifford algebras:

Over $\mathbb{R}^d$: $\{\gamma^j, \gamma^k\} = 2\delta^{j,k}$ real irrep: $\mathbb{R}^{\mathcal{N}_d}$

Over $\mathbb{R}^{\mathcal{N}_d} \otimes \mathbb{R}^{N^2-1}$: $\{\boldsymbol{\theta}_{\alpha A}, \boldsymbol{\theta}_{\beta B}\} = 2\delta_{\alpha,\beta}\delta_{A,B}$ irrep: $\mathcal{F} = \mathbb{C}^{2^{\frac{1}{2}\mathcal{N}_d(N^2-1)}}$

Hamiltonian:

$$H = p_{jA}p_{jA} + \frac{1}{2}\sum_{A,j,k}(f_{ABC}x_{jB}x_{kC})^2 + \frac{i}{2}x_{jC}f_{CAB}\gamma^j_{\alpha\beta}\,\boldsymbol{\theta}_{\alpha A}\,\boldsymbol{\theta}_{\beta B}$$

Supercharges: $\mathcal{Q}_{\alpha=1,\ldots,\mathcal{N}_d} = \left(p_{jA}\gamma^j_{\alpha\beta} + \frac{1}{2}f_{ABC}x_{jB}x_{kC}\gamma^{jk}_{\alpha\beta}\right)\boldsymbol{\theta}_{\beta A}$

s.t. $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 2\delta_{\alpha\beta}H + 4\gamma^j_{\alpha\beta}x_{jA}J_A$

Requirement: $d = 2, 3, 5$, or $9 \Rightarrow \mathcal{N}_d = 2(d-1) = 2, 4, 8$, or 16

Supersymmetry (cont.)

Full Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^d \otimes \mathbb{R}^{N^2-1}) \otimes \mathcal{F}$

Physical Hilbert space $\mathcal{H}_{\text{phys}}$: $J_A \Psi = 0$, where

$$J_A = f_{ABC}\Big(x_{jB} p_{jC} - \frac{i}{4} \boldsymbol{\theta}_{\alpha B} \boldsymbol{\theta}_{\alpha C} \Big)$$

$\mathrm{Spin}(d)$-symmetry:

$$J_{jk} = x_{jA} p_{kA} - x_{kA} p_{jA} - \frac{i}{8} \gamma^{jk}_{\alpha\beta} \boldsymbol{\theta}_{\alpha A} \boldsymbol{\theta}_{\beta A}$$

M. Baake, P. Reinicke, V. Rittenberg, *Fierz identities for real Clifford algebras and the number of supercharges*, J. Math. Phys., 1985; Claudson, Halpern, 1985; Flume, 1985

B. de Wit, J. Hoppe, H. Nicolai, *On the quantum mechanics of supermembranes*, Nucl. Phys. B, 1988

Spectrum: Classical model

Potential: $V = \sum_{1 \le j < k \le d} \mathrm{tr}\, (i[X_j, X_k])^2 \quad \ge 0$

Toy model in $\mathbb{R}^2$: $V_{\mathsf{toy}} = x^2 y^2$

Flat directions $\Rightarrow$ unconfined

Spectrum: Quantum mechanical model

Scalar Schrödinger operator: $H_B = -\Delta + V(x)$

Toy model: $H_{B,\text{toy}} = -\partial_x^2 - \partial_y^2 + x^2y^2$

Purely discrete spectrum:

$$\begin{aligned} H_{B,\text{toy}} &= \frac{1}{2}(-\partial_x^2 - \partial_y^2) + \frac{1}{2}\underbrace{(-\partial_x^2 + y^2x^2)}_{\geq |y|} + \frac{1}{2}\underbrace{(-\partial_y^2 + x^2y^2)}_{\geq |x|} \\ &\geq \frac{1}{2}(-\Delta + |x| + |y|) \end{aligned}$$

M. Lüscher, NPB 1983; B. Simon, Ann. Phys. 1983

Garcia del Moral et. al., NPB, 2007; 2010 (BLG/ABJM type)

Spectrum: Supersymmetric quantum mechanical model

Matrix Schrödinger operator: $H = (-\Delta + V(x))1 + x_{jA} M_{jA}$
s.t. $H = \mathcal{Q}_\alpha^2 \geq 0$ on $\mathcal{H}_{\text{phys}}$

Toy model: $H_{\text{toy}} = (-\Delta_{\mathbb{R}^2} + x^2 y^2)1 + \underbrace{x\sigma_1 + y\sigma_2}_{\geq -\sqrt{x^2+y^2}} = Q_{\text{toy}}^2 \geq 0$

Theorem (dW-L-N)

For any $\lambda \geq 0$ there exists a sequence Ψ_t of rapidly decaying smooth $SU(N)$-invariant functions s.t. $\|\Psi_t\| = 1$ and $\|(H - \lambda)\Psi_t\| \to 0$ as $t \to \infty$. Hence, $\sigma(H) = [0, \infty)$.

For toy model: $\Psi_t(x, y) := \chi_t(x)\phi_x(y)\xi$

B. de Wit, M. Lüscher, H. Nicolai, *The supermembrane is unstable*, NPB, 1989

Ground state conjecture

BFSS Conjecture

$d = 9$: *Unique normalizable zero-energy ground state for all* N

$d = 2, 3, 5$: *No normalizable zero-energy state for any* N

T. Banks, W. Fischler, S. Shenker, L. Susskind, *M Theory As A Matrix Model: A Conjecture*, Phys. Rev. D, 1997

$d = 1$ degenerate model ($V = 0$):
Plane-wave (non)normalizable zero-energy state for any N

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$d = 1$ degenerate model ($V = 0$):
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Ground state conjecture (cont.)

Conjecture supported by:

- Rigorous proof for $d = 2, N = 2$

  J. Fröhlich, J. Hoppe, *On Zero-Mass Ground States in Super-Membrane Matrix Models*, CMP, 1998

- Asymptotics (necessary decay known for $N = 2$)

  M. B. Halpern, C. Schwartz, *Asymptotic Search for Ground States of SU(2) Matrix Theory*, Int. J. Mod. Phys. A, 1998

  J. Fröhlich, G. M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, *Asymptotic form of zero energy wave functions in supersymmetric matrix models*, NPB, 2000

- Witten index calculations

  P. Yi, *Witten Index and Threshold Bound States of D-Branes*, NPB, 1997
  S. Sethi, M. Stern, *D-Brane Bound States Redux*, CMP, 1998

  Green, Gutperle, 1998; Krauth, Nicolai, Staudacher, 1998; Kac, Smilga, 2000; Moore, Nekrasov, Shatashvili, 2000

  Caution! Imbimbo, Mukhi, 1984; Staudacher, 2000; Jaffe, 2000

Recent approaches to the study of ground states

I. Construction by recursive methods

J. Hoppe, D.L., M. Trzetrzelewski, *Construction of the Zero-Energy State of SU(2)-Matrix Theory: Near the Origin*, NPB, 2009

Hynek, Trzetrzelewski, 2010; Michishita, 2010; 2011

II. Deformation

J. Hoppe, D.L., M. Trzetrzelewski, *Octonionic twists for supermembrane matrix models*, Ann. Henri Poincaré, 2009

III. Averaging w.r.t. symmetries

J. Hoppe, D.L., M. Trzetrzelewski, *Spin(9) Average of SU(N) Matrix Models I. Hamiltonian*, J. Math. Phys., 2009

IV. Weighted spaces and index theory

D.L., *Weighted Supermembrane Toy Model*, Lett. Math. Phys., 2010; Ph.D. thesis, 2010

arXiv:0911.3386; arXiv:1101.2653

I. Construction by recursive methods

Consider the structure of a possible ground state $\Psi(x)$ around $x = 0$:

$$\Psi(x) = \psi^{(0)} + x_{jA}\psi^{(1)}_{jA} + \frac{1}{2}x_{jA}x_{kB}\psi^{(2)}_{jA,kB} + \dots,$$

where $\gamma^j_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(1)}_{jA} = 0, \quad \gamma^j_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(2)}_{jA,kB} = 0,$

$$\gamma^j_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(3)}_{jA,kB,lC} + if_{ABC}\gamma^{kl}_{\beta\alpha}\,\boldsymbol{\theta}_{\alpha A}\,\psi^{(0)} = 0, \quad \text{etc.}$$

Theorem (JH-DL-MT)

For $d = 9$, $N = 2$ we have
(where $\mathcal{F} = \otimes^3\mathcal{F}_{256}$ and $\mathcal{F}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ under $\mathrm{Spin}(9)$)

$$\psi^{(0)} \propto (44 \otimes 44 \otimes 44)_{\mathsf{sym}} + \frac{13}{36}(44 \otimes 84 \otimes 84)_{\mathsf{sym}}$$

I. Construction by recursive methods (cont.)

$$
\begin{aligned}
(44 \otimes 44 \otimes 44)_{\mathsf{sym}} &:= |jl\rangle_1 |kl\rangle_2 |jk\rangle_3 \\
(44 \otimes 84 \otimes 84)_{\mathsf{sym}} &:= |jk\rangle_1 |jlm\rangle_2 |klm\rangle_3 \\
&\quad + |klm\rangle_1 |jk\rangle_2 |jlm\rangle_3 \\
&\quad + |jlm\rangle_1 |klm\rangle_2 |jk\rangle_3
\end{aligned}
$$

Michishita recently found and proved uniqueness also of $\psi^{(1)}$ (for $N = 2$)

II. Deformation

A conjugation of a combination of supercharges:

$$Q(\mu) := e^{\mu g(x)} \frac{1}{\sqrt{2}} (\mathcal{Q}_8 + i\mathcal{Q}_{16}) e^{-\mu g(x)},$$

with

$$g(x) = \frac{1}{6} f_{ABC} x_{jA} x_{kB} x_{lC} \gamma^{jkl}_{8,16},$$

leads to a family of new models $H(\mu) := \{Q(\mu)^\dagger, Q(\mu)\} \geq 0$ with $G_2 \times \mathrm{U}(1) \times \mathrm{SU}(N)$ symmetry:

$$H(\mu) = -\Delta_{1\ldots7} + (\mu - 1)^2 V_{1\ldots7} + H_D + (\mu - 1) x_{1\ldots7} \cdot M_1 + x_{89} \cdot M_2$$

cp. M. Porrati, A. Rozenberg, NPB, 1998

II. Deformation (cont.)

Consider $\tilde{H} := H(\mu = 1)$, which is a truncation of H

Theorem (JH-DL-MT)

$\sigma(\tilde{H}) = \sigma(H) = [0, \infty)$

Deformation approach has been successful for simpler models

L. Erdős, D. Hasler, J. P. Solovej, *Existence of the D0 - D4 bound state: A Detailed proof*, Ann. Henri Poincaré, 2005

III. Averaging w.r.t. Spin(9)

Coordinate split: $\mathbb{R}^9 = \mathbb{R}^7 \times \mathbb{R}^2$

Truncated Hamiltonian
$H_D = -\Delta_{89} + x_{89} \cdot S(x_{1\ldots7})x_{89} + x_{1\ldots7} \cdot M$

Interpretation: 2D SUSY $\mathrm{SU}(N)$ matrix model with 7D space of parameters

Simple spectrum: set of $2(N^2-1)$ SUSY harmonic oscillators

III. Averaging w.r.t. Spin(9) (cont.)

Slightly modified operator:

$$H'_D := -\frac{9}{2}\Delta_{89} + \frac{18}{7}x_{89} \cdot S(x_{1\ldots 7})x_{89} + \frac{36}{7}x_{1\ldots 7} \cdot M$$

still simple spectrum, rescaled frequencies

Theorem (JH-DL-MT)

The average of the operator H'_D w.r.t. $\mathrm{Spin}(9)$ is equal to the full Hamiltonian H.

IV. Weighted spaces and index theory

Asymptotic analysis suggests to allow for more slowly decaying ground states (cp. also $d = 1$ model).

Weighted Hilbert space: $\mathcal{H}_w = L^2(\mathbb{R}^{d(N^2-1)}, \rho_\alpha(x)dx) \otimes \mathcal{F}$, with $\rho_\alpha(x) = (1 + |x|^2)^{-\alpha/2}$.
$\Rightarrow \quad \langle \Phi, \Psi \rangle_w = \langle \Phi, \rho_\alpha \Psi \rangle$

Self-adjoint Hamiltonian H_w defined by Friedrichs extension of:
$\langle \Psi, H_w \Psi \rangle_w := \langle \Psi, H\Psi \rangle = \|Q\Psi\|^2 \geq 0, \ \ \Psi \in C_0^\infty$.

Ground state correspondence:

$$\Psi \in \ker_{\mathcal{H}} H \quad \Rightarrow \quad \Psi \in \ker_{\mathcal{H}_w} H_w \quad \Rightarrow \quad \Psi \in C^\infty \text{ and } Q\Psi = 0$$

IV. Weighted spaces and index theory (cont.)

Spectral relation:

$$\langle \Psi, (H_w - \lambda)\Psi \rangle_w = \langle \Psi, (H - \lambda\rho_\alpha)\Psi \rangle \quad \Rightarrow \quad N(H_w - \lambda)_w = N(H - \lambda\rho_\alpha)$$

Hence, if H_w has a discrete spectrum in $\mathcal{H}_w$
($\Leftrightarrow H - \lambda\rho_\alpha$ in $\mathcal{H}$ has finitely many negative eigenvalues $\forall\lambda$),
then

$\ker_{\mathcal{H}_w} H_w \neq 0 \quad \Leftrightarrow \quad H - \lambda\rho_\alpha$ has a negative eigenvalue $\forall\lambda > 0$

Theorem (DL)

For the toy model we have for $\alpha > 2$

$$N(H_{\text{toy}} - \lambda\rho_\alpha) \leq o(\lambda^{\frac{3}{2}}),$$

and hence discrete spectrum of H_w.

IV. Weighted spaces and index theory (cont.)

We have $H_w = Q_w^* Q_w, \;\; Q_w = \rho_\alpha^{-1/2} Q, \;\; Q_w^* = \rho_\alpha^{-1} Q \rho_\alpha^{1/2}$
Consider $H_w' := Q_w Q_w^*$

Weighted index:

$$I_w := \mathrm{tr}_{\mathcal{H}_w} e^{-\beta H_w} - \mathrm{tr}_{\mathcal{H}_w} e^{-\beta H_w'} = \dim \ker_{\mathcal{H}_w} H_w - \dim \ker_{\mathcal{H}} H,$$

independent of $\beta > 0$ whenever H_w, H_w' have discrete spectra

Works fine for free line model and $d = 1$ model for sufficient α

Toy model?

Outlook

I. Continued construction at $x \sim 0$ and $x \to \infty$

II. Zero-energy states for the deformed operator $\tilde{H}$?

III. Averaging of eigenstates of H_D resp. H'_D

IV. Discreteness of H'_w, and weighted index for toy model?
$d = 2, 3, 5, 9$ SMM? Physical relevance of weighted states?

Thank you!

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Thank you!