Geometric Algebra and Its Applications to Physics

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Abstract

In this report for the seminar course in theoretical physics at KTH I give a short introduction to Geometric Algebra and one of its powerful developments, Geometric Calculus. I also discuss one of the most successful applications to physics, the spacetime algebra, which finally results in a compact formulation of Maxwell's equations in the setting of relativistic electrodynamics.

Contents

1 Introduction

Geometric Algebra (GA) was invented by William Kingdon Clifford in the 19th century, building on a foundation laid by Hermann Günter Grassmann in 1844. Although Clifford himself chose to use the name Geometric Algebra, it is now more widely known as *Clifford Algebra*. It is sometimes advantageous, however, to view GA more as a whole package of definitions and geometric interpretations of a Clifford algebra over the real numbers. In this view Clifford Algebra is the underlying algebraic structure which can be defined for scalars in arbitrary fields or rings.

The application of GA to spacetime, called the *spacetime algebra*, was first developed by David Hestenes in the 1960s. He also pioneered the field of *Geometric Calculus* (GC) in the 1980s.

Today GA has been successfully applied to many areas of physics and mathematics. Many existing algebras and concepts have been found to have clear geometric representations within GA, such as the Pauli and Dirac matrix algebras of quantum mechanics, the quaternions and even the algebra of differential forms, just to mention a few. Some people even claim GA to be a universal language for almost all of mathematics and physics.

In this short report the focus is on introducing the concepts of GA and GC and showing how we can reformulate physics in this language. As indicated above, we always use real numbers as scalars. One might think that an algebra over the complex numbers is more general, but in fact most, if not all, applications in physics are handled efficiently with real numbers as scalars. As we will see, complex structures appear naturally within the real algebra and often this provides a clearer geometrical picture of what is going on.

Before we start, let us just say some words about the notation: We sum over a matching upper and lower index. A check ˇ means the corresponding term is removed from the expression. The Kronecker delta is denoted by δ_{ij} or δ^i_j. We use a number of different products, so in order to get rid of a lot of brackets, we use the convention of prioritizing $A \wedge B$ before $A \cdot B$ before AB.

2 A short introduction to Geometric Algebra

In this section we introduce GA by generalizing the concept of a vector to other linear subspaces. We define the geometric product and observe how it simplifies common linear transformations.

2.1 The concept of blades

Let $\mathcal{V}$ be an n-dimensional vector space with an inner product $a \cdot b$, which is allowed to have any (non-degenerate) signature. Consider a vector in this space. It is normally illustrated by an arrow (see figure 1) and as such can be described as a one-dimensional subspace with an associated orientation[1] and a magnitude (length). We can easily generalize this concept to higher-dimensional subspaces in $\mathcal{V}$ by defining an *r-blade* A_r to be an r-dimensional linear subspace, fitted with an orientation and a (positive) magnitude. For example, a 2-blade can

[1]The direction in which the arrow is pointing, i.e. 'forwards' or 'backwards' with respect to some reference.

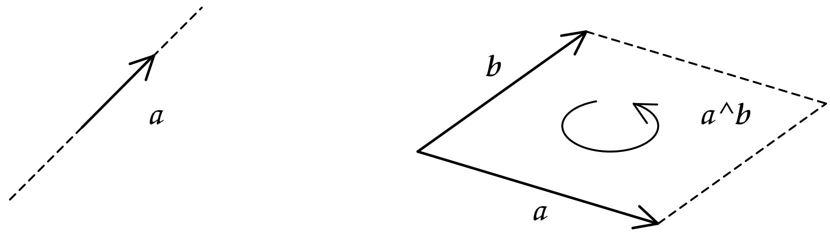

Figure 1: A vector $a \in \mathcal{V}$ and a 2-blade $a \wedge b \in \mathcal{V}^2$.

be visualized as a plane with a curled arrow describing the orientation and an area describing the magnitude. One can also define blades algebraically by introducing an *outer product* $\wedge$ of vectors[2]. Thus, for vectors $a, b \in \mathcal{V}$ we let $a \wedge b$ denote the 2-blade described by the plane these vectors span, with the orientation of first moving in the direction of a and then b, and the area of the parallelogram formed by a and b. Note that the area is zero if the vectors are parallel, and we identify the resulting blade with the zero vector.

The 3-blade $a \wedge b \wedge c$ is defined as $\mathrm{Span}_{\mathbb{R}}\{a, b, c\}$ with the orientation given by the order and direction of these vectors, and a magnitude given by the volume of the parallelepiped formed by $\{a, b, c\}$.

We can continue in this manner to form higher blades, but when we reach the n-blade $I = a_1 \wedge a_2 \wedge \ldots \wedge a_n$ there are no more linearly independent vectors available in $\mathcal{V}$, which means that the outer product of I with any vector is zero.

Now we have a method of constructing r-blades in $\mathcal{V}$, but we can also define an addition operation for these blades. If we have a basis $\{e_i\}$ in $\mathcal{V}$, we can view $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}\}_{i_1 < i_2 < \ldots < i_r}$ as forming a basis of $\mathcal{V}^r$, the linear space[3] of r-blades. We also assert that the outer product is bilinear[4], i.e.

$$(\alpha A + \beta B) \wedge C = \alpha A \wedge C + \beta B \wedge C \qquad \forall\ \alpha, \beta \in \mathbb{R},\ A, B \in \mathcal{V}^r,\ C \in \mathcal{V}^s. \quad (1)$$

An element in $\mathcal{V}^r$ is called an *r-vector*[5] and is said to have *grade r*.

Finally, we also consider the scalars $\mathbb{R}$ as 0-vectors and define $\alpha \wedge A := \alpha A$.

2.2 Examples: 2d and 3d Euclidean spaces

With a basis $\{e_1, e_2\}$ in $\mathbb{R}^2$ we obtain the following blade bases in $\mathcal{V}^0$, $\mathcal{V}^1$, $\mathcal{V}^2$:

$$\{1\}, \qquad \{e_1, e_2\}, \qquad \{e_1 \wedge e_2\}, \quad (2)$$

whereas with a basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ we have

$$\{1\}, \qquad \{e_1, e_2, e_3\}, \qquad \{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}, \qquad \{e_1 \wedge e_2 \wedge e_3\}. \quad (3)$$

Note that the dimension of $\mathcal{V}^r$ is given by $\binom{n}{r}$.

[2] Pronounced 'wedge'. The outer product was originally introduced by Grassmann.

[3] Note that we also have a natural multiplication by scalars that scales the magnitude and reverses the orientation of the blade (if the scalar is negative).

[4] Note that from the definition of the outer product it follows that it is associative and antisymmetric.

[5] 2-vectors are often called *bivectors* and 3-vectors *trivectors*.

2.3 The geometric product

The spaces of r-vectors can be combined to form a *geometric algebra* $\mathcal{G}(\mathcal{V})$ of $\mathcal{V}$, i.e.

$$\mathcal{G}(\mathcal{V}) := \mathcal{V}^0 \oplus \mathcal{V}^1 \oplus \ldots \oplus \mathcal{V}^n, \tag{4}$$

and we note that $\dim \mathcal{G}(\mathcal{V}) = 2^n$. An element $A \in \mathcal{G}(\mathcal{V})$ is called a *multivector* and the r-vector-part of A is denoted by $\langle A \rangle_r$.

For $a, b \in \mathcal{V}$ we define the *geometric product* $\mathcal{V} \times \mathcal{V} \to \mathcal{G}(\mathcal{V})$ by

$$ab := a \cdot b + a \wedge b, \tag{5}$$

thus combining both the inner and outer products into a single expression. The geometric product is the most important concept in GA and it has a number of interesting properties. For example, any vector squares to a scalar, orthogonal vectors anticommute, and it is *invertible* for non-null vectors ($a^2 \neq 0$). More explicitly

$$a^{-1} := \tfrac{1}{a^2} a \qquad \Rightarrow \qquad aa^{-1} = a^{-1}a = 1. \tag{6}$$

Also note that

$$a \cdot b = \tfrac{1}{2}(ab + ba), \qquad a \wedge b = \tfrac{1}{2}(ab - ba). \tag{7}$$

The geometric product can be extended to act on the whole of $\mathcal{G}(\mathcal{V})$ by asserting that it is linear in each argument and associative[6]. The result is that (see [1] or [4]) for $A_r \in \mathcal{V}^r, B_s \in \mathcal{V}^s$

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \ldots + \langle A_r B_s \rangle_{r+s} \tag{8}$$

with the special case $aA_r = \langle aA_r \rangle_{r-1} + \langle aA_r \rangle_{r+1} = a \cdot A_r + a \wedge A_r$, where we define $A_r \cdot B_s := \langle A_r B_s \rangle_{|r-s|}$ for $r, s > 0$ and realize that $A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}$.

Another result is that we can write r-blades as geometric products of orthogonal vectors, $A_r = a_1 a_2 \ldots a_r$, so that these also square to a scalar

$$\begin{aligned} A_r^2 &= a_1 a_2 \ldots a_r a_1 a_2 \ldots a_r = (-1)^{r(r-1)/2} a_r a_{r-1} \ldots a_1 a_1 a_2 \ldots a_r \\ &= (-1)^{r(r-1)/2} a_1^2 a_2^2 \ldots a_r^2. \end{aligned} \tag{9}$$

From this follows that a non-null[7] blade also has an inverse given by $A^{-1} = \frac{1}{A^2} A$.

A unit n-blade is called a *pseudoscalar* and is denoted by I. We usually also fix the orientation of I according to the order of some basis, which results in I being unique for the space $\mathcal{V}$ under consideration. Using (8) we observe that $A_r I = \langle A_r I \rangle_{n-r}$, which represents the orthogonal complement of A_r in $\mathcal{V}$ and is called the *dual* of A_r.

Using these products it is easy to state geometric properties algebraically, e.g. $a \wedge A_r = 0$ means the vector a lies in the hyperplane represented by A_r and $a \cdot A_r = 0$ means that a is orthogonal to every vector in that hyperplane.

[6]Actually, a rigorous way to define GA is to start with an associative algebra (with the geometric product) and some more axioms and define the inner and outer products from (7). There are other ways of defining GA and Clifford algebras in general, some of which are discussed in [5].

[7]An r-blade A_r is *null* if $A_r^2 = 0$, i.e. at least one of the a_i in (9) is null.

2.4 Projections, reflections and rotors

We now look at some of the elementary transformations in vector spaces and the advantages of using GA to describe them. As a start, let A be a non-null r-blade. For any $a \in \mathcal{V}$ we then have

$$a = aAA^{-1} = \underbrace{a \cdot AA^{-1}}_{a_\parallel} + \underbrace{a \wedge AA^{-1}}_{a_\perp}. \tag{10}$$

Since $a_\parallel \wedge A = \langle a_\parallel A \rangle_{r+1} = \langle a \cdot A \rangle_{r+1} = 0$, we note that $a_\parallel$ lies in the subspace $\bar{A} \subseteq \mathcal{V}$ represented by A. Similarly $a_\perp \cdot A = 0$, so $a_\perp \in \bar{A}^\perp$, and thus

$$P_A(x) := x \cdot AA^{-1} \tag{11}$$

is just the the orthogonal projection of x onto $\bar{A}$.

Next, consider a reflection along a non-null vector a (i.e. in the hyperplane orthogonal to a). Regardless of the signature of $\mathcal{V}$ this can be written as

$$x \mapsto \sigma_a(x) := x - 2\frac{x \cdot a}{a^2}a. \tag{12}$$

However, using (7), we can rewrite this as

$$\sigma_a(x) = x - 2x \cdot aa^{-1} = xaa^{-1} - (xa + ax)a^{-1} = -axa^{-1} \tag{13}$$

and due to the (anti-)commutativity of parallel (orthogonal) vectors, this clearly behaves like the reflection we started from.

Let us now assume we have two *orthonormal* vectors e_1, e_2 and consider a rotation in the e_1e_2-plane defined by

$$e_1 \mapsto \mathsf{R}(e_1) := \cos\theta \; e_1 + \sin\theta \; e_2. \tag{14}$$

Note that $(e_1e_2)^2 = e_1e_2e_1e_2 = -e_2e_1e_1e_2 = -1$, which means that e_1e_2 behaves just like a complex imaginary unit and we can rewrite (14) as[8]

$$\mathsf{R}(e_1) = \cos\theta \; e_1 + \sin\theta \; e_1^2e_2 = e_1(\cos\theta + \sin\theta \; e_1e_2) = e_1e^{\theta e_1e_2}. \tag{15}$$

This looks neat, but the fact that e_1 and e_2 anticommute leads us to an even more general result, namely

$$\begin{aligned} e_1e^{\theta e_1e_2} &= e_1e^{\frac{\theta}{2}e_1e_2}e^{\frac{\theta}{2}e_1e_2} = e_1(\cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2} \; e_1e_2)e^{\frac{\theta}{2}e_1e_2} \\ &= (\cos\tfrac{\theta}{2} - \sin\tfrac{\theta}{2} \; e_1e_2)e_1e^{\frac{\theta}{2}e_1e_2} = e^{-\frac{\theta}{2}e_1e_2}e_1e^{\frac{\theta}{2}e_1e_2}. \end{aligned} \tag{16}$$

The brilliance in this emerges when we consider a vector e_3 orthogonal to the e_1e_2-plane, which then is unaffected by the rotation and commutes with e_1e_2, i.e.

$$\mathsf{R}(e_3) = e_3 = e_3e^{-\frac{\theta}{2}e_1e_2}e^{\frac{\theta}{2}e_1e_2} = e^{-\frac{\theta}{2}e_1e_2}e_3e^{\frac{\theta}{2}e_1e_2}. \tag{17}$$

For *any* vector $x \in \mathcal{V}$ the rotation R can then be written

$$x \mapsto \mathsf{R}(x) = e^{-\frac{\theta}{2}e_1e_2}xe^{\frac{\theta}{2}e_1e_2} = RxR^{-1}. \tag{18}$$

[8] We define $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$ as usual.

The multivector $R := e^{-\frac{\theta}{2}e_1e_2}$ is called a *rotor* and it is one of the most powerful features of GA. The transformation rule (18) is surprisingly general and, as we will see, applies to mixed signature spaces as well in the form of Lorentz transformations. Moreover, rotors also do a great job in representing spinors in quantum mechanics.

Finally, in order to work efficiently with spacetime we consider the properties of frames. Assume we have a basis frame $\{e_i\}$ in $\mathcal{V}$, which need not be orthogonal. Since the inner product is non-degenerate, we can always find a *reciprocal frame* $\{e^i\}$ defined by

$$e^i \cdot e_j = \delta^i_j. \tag{19}$$

Actually, GA enables us to find a simple expression for the reciprocal frame

$$e^i = (-1)^{i-1} e_1 \wedge \cdots \wedge \check{e}_i \wedge \cdots \wedge e_n I^{-1}. \tag{20}$$

A nice derivation of this can be found in [4]. Except for the sign it is geometrically clear that this expression does the job, since it is the dual of a hyperplane spanned by the vectors $\{e_1, \ldots, \check{e}_i, \ldots, e_n\}$.

Using the basis frame and reciprocal frame we can express any vector $a \in \mathcal{V}$ as

$$a = a^i e_i = a_i e^i, \tag{21}$$

where

$$a^i = e^i \cdot a, \qquad a_i = e_i \cdot a. \tag{22}$$

From (21) and (22) we also obtain the useful identity

$$a = e^i e_i \cdot a = e_i e^i \cdot a. \tag{23}$$

3 An application to Minkowski spacetime

Let us illustrate the power of GA in physical applications with an example called the *spacetime algebra*. We introduce a set of basis vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ with the properties

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} \quad \text{(where } \eta \text{ is the Minkowski metric).}^9 \tag{24}$$

According to the last section, we also have an associated reciprocal frame $\{\gamma^\mu\}$ such that $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$. In this case it is simply $\gamma^0 = \gamma_0,\ \gamma^i = -\gamma_i$ $(i = 1, 2, 3)$. The pseudoscalar in this space is defined as $I := \gamma_0\gamma_1\gamma_2\gamma_3$. We also introduce a useful notation for the timelike 2-blades

$$\boldsymbol{\sigma}_i := \gamma_i\gamma_0 \quad (i = 1, 2, 3). \tag{25}$$

The bivectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ then have the properties

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \tfrac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \delta_{ij} \qquad \text{and} \qquad \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = I, \tag{26}$$

and so define a 3-dimensional Euclidean algebra called the algebra of *relative space*. In order to observe the usefulness of this, consider a spacetime event

[9] Note that from (7) we can also write (24) in the form $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$, which means that the Dirac matrix algebra is just another representation of this spacetime algebra.

$x \in \mathcal{V}$, which in the $\{\gamma_\mu\}$-frame can be expressed as (the index i is summed from 1 to 3)

$$x = x^\mu \gamma_\mu = x^0\gamma_0 + x^i\gamma_i = x^0\gamma_0 + x^i\gamma_i\gamma_0\gamma_0 = (x^0 + x^i\boldsymbol{\sigma}_i)\gamma_0. \tag{27}$$

On the other hand, suppose we have another observer with 4-velocity $v = \frac{dx}{d\tau}$, where τ is the proper time so that $v^2 = 1$. We can introduce a relative frame $\{e_\mu\}$ for this observer such that $e_0 = v$, $\{e_i\}$ are spacelike and $e_\mu \cdot e_\nu = \eta_{\mu\nu}$. The same event $x = x'^\mu e_\mu$ is then

$$x = xvv = (x \cdot v + x \wedge v)v = (t + \boldsymbol{x})v, \tag{28}$$

where $t := x \cdot v = x \cdot e^0 = x'^0$ is the time coordinate and $\boldsymbol{x} := x \wedge v = x'^i e_i e_0$ is the relative space vector in the $\{e_\mu\}$-frame. The operation of projecting out relative time and space vectors from an invariant 4-vector is called a *spacetime split*. From t and $\boldsymbol{x}$ we can also obtain the invariant interval

$$x^2 = xvvx = (x \cdot v + x \wedge v)(v \cdot x + v \wedge x) = (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2. \tag{29}$$

The method of spacetime splits can also be applied to the proper velocity $u(\tau) = \frac{dx(\tau)}{d\tau}$ of some particle or observer

$$u \cdot v + u \wedge v = uv = \frac{d}{d\tau}(xv) = \frac{d}{d\tau}(t + \boldsymbol{x}) = \frac{dt}{d\tau} + \frac{d\boldsymbol{x}}{d\tau}, \tag{30}$$

where v is assumed to be constant. We note that the relative velocity $\boldsymbol{u}$ as measured by an observer with velocity v is simply

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{x}}{d\tau}\frac{d\tau}{dt} = \frac{u \wedge v}{u \cdot v}. \tag{31}$$

Let us now consider a Lorentz transformation boost of velocity $|\boldsymbol{v}| = \tanh\alpha$ in the γ_1-direction (or the $\boldsymbol{\sigma}_1$-direction in relative space). The transformed frame $\{e_\mu\}$ can be written as[10]

$$e_0 = \cosh\alpha\ \gamma_0 + \sinh\alpha\ \gamma_1,\ e_1 = \sinh\alpha\ \gamma_0 + \cosh\alpha\ \gamma_1,\ e_2 = \gamma_2,\ e_3 = \gamma_3. \tag{32}$$

Following the procedure for rotations in the previous section, with the adjustment that

$$\boldsymbol{\sigma}_i^2 = \gamma_i\gamma_0\gamma_i\gamma_0 = 1 \quad \Rightarrow \quad e^{\alpha\boldsymbol{\sigma}_i} = \cosh\alpha + \sinh\alpha\ \boldsymbol{\sigma}_i, \tag{33}$$

we obtain the much simpler rotor expression

$$e_\mu = e^{\frac{\alpha}{2}\boldsymbol{\sigma}_1}\gamma_\mu e^{-\frac{\alpha}{2}\boldsymbol{\sigma}_1}. \tag{34}$$

4 A short introduction to Geometric Calculus

We will now introduce differentiation into our geometric algebra $\mathcal{G}(\mathcal{V})$. The synthesis of GA and vector calculus is commonly referred to as *Geometric Calculus*.

[10] The transformation is usually given in terms of the coordinates, but the frame vectors satisfy the inverse relation.

Let $F : \mathcal{V} \to \mathcal{G}(\mathcal{V})$ be a smooth[11] multivector-valued function. For $a \in \mathcal{V}$ we define the *directional derivative* $a \cdot \nabla F$ by

$$a \cdot \nabla F(x) := \lim_{\epsilon \to 0} \tfrac{1}{\epsilon}\big(F(x + \epsilon a) - F(x)\big). \tag{35}$$

Assuming we have a basis $\{e_i\}$ in $\mathcal{V}$, we can expand $x \in \mathcal{V}$ as $x = x^i e_i$. Thus, we can also write $\partial_i := \partial/\partial x^i := e_i \cdot \nabla$. The *vector derivative* is then defined by

$$\nabla := e^i e_i \cdot \nabla = e^i \partial_i. \tag{36}$$

This has the algebraic properties of a vector and the analytic properties of a differential operator. Specifically, just as any other vector in a geometric product, it lowers and raises the grade of the object it acts on. The respective parts are conveniently named $\nabla \cdot F = \langle \nabla F \rangle_{r-1}$ and $\nabla \wedge F = \langle \nabla F \rangle_{r+1}$ for $F = \langle F \rangle_r$.

As a first example, consider a scalar valued function $\phi : \mathcal{V} \to \mathbb{R}$. Acting on it with the vector derivative ∇ we find

$$\nabla \phi = e^i e_i \cdot \nabla \phi = e^i \partial_i \phi, \tag{37}$$

which is just the *gradient* of ϕ.

Next, consider a vector-valued function $f : \mathcal{V} \to \mathcal{V}$, which also can be expanded as $f = f_i e^i = f^i e_i$. The scalar and bivector parts of ∇f are

$$\nabla \cdot f = e^i \cdot (\partial_i f) = \partial_i f^i \qquad \text{and} \qquad \nabla \wedge f = e^i \wedge (\partial_i f) = e^i \wedge e^j \partial_i f_j. \tag{38}$$

These are also called div$f$ and curl$f$ and we see that the vector derivative combines the *divergence* and *curl*[12] of f in a single multivector object

$$\nabla f = \nabla \cdot f + \nabla \wedge f. \tag{39}$$

One can proceed along these lines and define a *multivector* derivative, a calculus on vector manifolds which incorporates the calculus of differential forms, and also a theory of directed integration. These topics cannot be covered here, but good introductions are found in the references[13].

5 An application to relativistic electrodynamics

In order to obtain a glimpse of the utility of GC in physics, we consider an application to relativistic electrodynamics. We return to the setting of a Minkowski spacetime algebra with basis frame $\{\gamma_\mu\}$. In the relative space corresponding to this frame we may have electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. These relative space vectors satisfy Maxwell's equations in their standard vector formulation. To reformulate those equations within the spacetime algebra framework, we introduce the *Faraday bivector*

$$F := \boldsymbol{E} + I\boldsymbol{B} = E^i \gamma_i \gamma_0 - B^1 \gamma_2 \gamma_3 - B^2 \gamma_3 \gamma_1 - B^3 \gamma_1 \gamma_2. \tag{40}$$

[11] We assume smoothness here for simplicity.

[12] In three dimensions the curl is traditionally defined as the dual of the expression in (38).

[13] [1] gives an intuitive introduction suitable for physicists, while [4] is the original work by Hestenes and Sobczyk. For a mathematically solid introduction, see [5].

This actually gives a Lorentz-invariant object and the bivector components

$$F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = \gamma^\nu \wedge \gamma^\mu \cdot F \tag{41}$$

are exactly those of the electromagnetic field strength tensor in standard formulations of relativistic electrodynamics. The source-part of Maxwell's equations can in such formulations be written $\partial_\mu F^{\mu\nu} = J^\nu$. Using (23) this can be restated in terms of GC as

$$J = \gamma_\nu J^\nu = \gamma_\nu \partial_\mu F^{\mu\nu} = \partial_\mu \gamma_\nu \gamma^\nu \cdot (\gamma^\mu \cdot F) = \partial_\mu \gamma^\mu \cdot F = \nabla \cdot F. \tag{42}$$

Similarly, the remaining equations have the correspondence

$$\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0 \quad \Leftrightarrow \quad \nabla \wedge F = 0. \tag{43}$$

Adding (42) and (43) we end up with a single equation which contains all the information in Maxwell's equations

$$\nabla F = J. \tag{44}$$

As a final note (see [1] for a derivation), given a Faraday bivector field F, the Lorentz force law for a particle with velocity $u(\tau)$, charge q and mass m can be simply stated as the rotor equation

$$u = R\gamma_0 R^{-1}, \quad \text{where} \quad \frac{dR}{d\tau} = \frac{q}{2m} FR. \tag{45}$$

6 Summary

We have seen how one can construct a geometric algebra over a finite-dimensional vector space by generalizing the concept of a vector to other linear subspaces. The introduction of a geometric product further enhanced the structure of this algebra by combining the inner and outer products into a new product with many useful properties. Using the introduced concepts we were able to reformulate a number of standard linear transformations to a compact and easily interpreted form.

The applicability of GA to physics was illustrated with examples in spacetime. The concept of rotors was seen to be equally well suited for rotations in Euclidean space as for Lorentz transformations in mixed signature spaces. It is interesting to contrast the rotor formulation of rotations with the corresponding matrix formulation. Not only are the relations easier to state in the rotor language, but it is also much easier to handle the dynamics of rotating objects using rotors. Another great advantage of the rotor formulation is that blades and general multivectors actually transform according to the exact same rule as for vectors.

We finally proceeded with introducing the concept of a vector derivative and saw how it could be used to unify Maxwell's equations into a single geometric equation. It is interesting to note that this equation is stated without reference to any specific frame or set of coordinates. Thus, GA enables us to focus on the physical content of equations and leave out specific frame-dependency which in some cases can block the view.

All these important properties clearly make GA a suitable language for describing physics and in my opinion it therefore deserves more attention from the physics community than it has currently received.

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