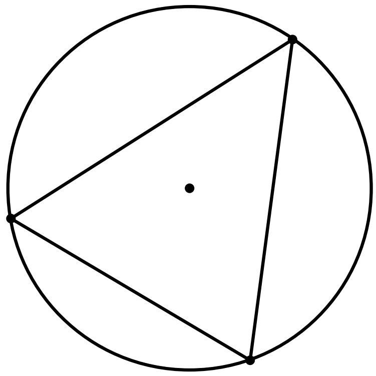
Triangles in particle interactions and applications of Clifford algebra



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Abstract

In this thesis we have utilized Clifford algebra to examine connections between geometry and two-dimensional magnetically interacting particles. We consider n particles in a plane in $\mathbb{R}^3$, each generating a disk-shaped constant magnetic field, where the field is perpendicular to the particle plane. Writing down the quantum mechanical Hamiltonian for the system, we go on to investigate the diamagnetic contribution.

Splitting the diamagnetic term into a two-particle interaction and a three-particle interaction, we show a number of results regarding the latter. This interaction turns out to be repulsive (i.e. greater for particles close to each other) and related to the geometry of triangles and their circumscribed circles.

Three particle positions form a triangle which has a circumscribed circle, i.e. the unique circle passing through all three vertices. We prove, using Clifford algebra, that the three-particle interaction is proportional to the squared inverse of the radius of this circle, for large enough particle separation. Furthermore, we show that this interaction is a repulsion, as mentioned above, among other results. Finally, we present and discuss some open questions and ideas that could be interesting to study further.

Contents

1 Introduction

Clifford algebra[1] was developed in the 1870's by English mathematician William Kingdon Clifford [1, 2] as a unification of Hamilton's quaternions [3] and Grassmann's algebra [4, 5]. It provides a rich mathematical framework for multilinear algebra and calculus on manifolds along with wide applications in physics, engineering and more.

The mathematician Hermann Grassmann introduced the outer product (also called the wedge product) defined on any vector space, in 1844. A vector can be thought of as a one-dimensional subspace, i.e. a line, equipped with magnitude and orientation. In Clifford algebra, the outer product is used to generalize this notion to subspaces of any dimension. The outer product of two vectors is called a 2-blade, of three vectors a 3-blade etc. With this notation a k-blade represents a linear k-dimensional subspace with magnitude and orientation. The general elements of the Clifford algebra form a vector space and are called multivectors.

The outer product of two vectors[2] a and b, denoted $a \wedge b$, is, as stated above, in a sense an oriented two-dimensional subspace with a magnitude. It encodes an oriented parallelogram, with sides a and b with an area equal to the magnitude of the blade. The outer product enables us to write an expression for the triangle area as half the magnitude of the blade generated by two of its sides. These kinds of geometric interpretations will become useful in later sections where Clifford algebra is used to investigate relationships between geometry and particle interactions.

In quantum mechanics we are used to having two kinds of particles, fermions and bosons. In three (or more) spatial dimensions, every particle falls into these categories. In two or one dimensions on the other hand, it is possible to have particles that are neither bosons nor fermions [6]. We will consider n of these (two-dimensional) anyons in a plane in $\mathbb{R}^n$. Each particle generates a disk-shaped constant magnetic field, where the field is perpendicular to the particle plane. In the Hamiltonian of such a system of particles one finds a diamagnetic term. This term is then split into a two-particle interaction and a three-particle interaction. The latter turns out to be repulsive (i.e. larger for particles close to each other) and related to the geometry of triangles and their circumscribed circles.

1.1 Purpose and contributions

The main purpose of this thesis is to collect and present results regarding the anyons mentioned above, in the language of Clifford algebra. It also serves as an accessible introduction to Clifford algebra and its application to geometry. The geometric connections and inequalities that we present in Section 5 have been studied in [7] and [8].

We begin, in Section 2, by reviewing some basic geometry related to triangles. Then, in Section 3 we introduce the Clifford algebra of $\mathbb{R}^n$ and derive some results that will be needed in later sections. In Section 4 we present the anyon-related physics that gives rise to the problems investigated in Section 5. In Section 6 we discuss the results and further directions.

[1] Also called geometric algebra.

[2] We will not use any special notation for vectors, such as bold symbols, vector arrows or bars.

2 Basic Geometry

Prior to analyzing the particle interactions in Sections 4 and 5 we shall lay out some geometric properties of triangles and their circumscribed circles.

2.1 The circumscribed circle of a triangle

The circumscribed circle of a triangle (also called the circumcircle) is the unique circle that passes through each of the three vertices of the triangle. The radius of this circle is referred to as the circumradius or just the radius. Let the triangle vertices be vectors x, y and z in $\mathbb{R}^n$. Furthermore let r be the circumradius, $d = 2r$ and A be the area of the triangle $\triangle xyz$. We denote the sides of the triangle as a, b and c with opposite angles α, β and γ respectively as in Figure 1.

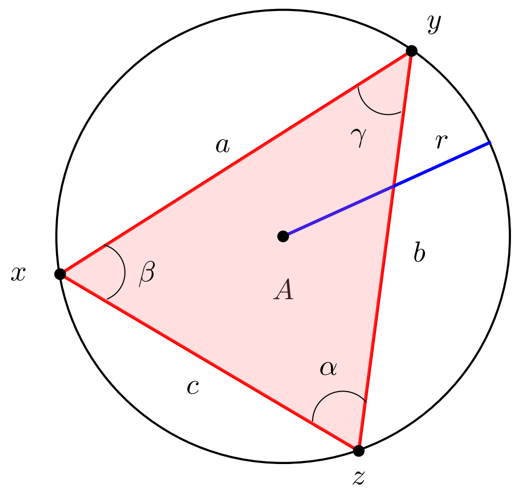


Figure 1: The triangle $\triangle xyz$ with a circumcircle and diameter $d = 2r$. The side lengths of the triangle are denoted a, b and c with opposite angles α, β, γ and area A.

The main result of this section is the following relation between the triangle area and the circumradius.

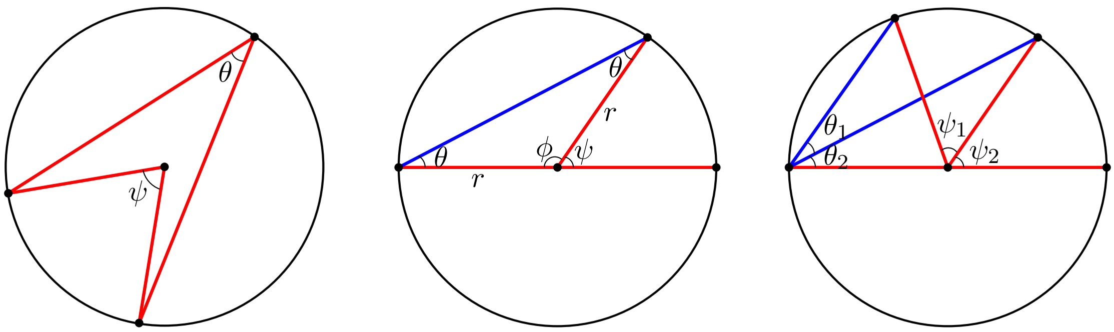
Proposition 2.1. For a triangle with sides a, b, c and area A, the circumradius is given by

$$r = \frac{abc}{4A}. \tag{1}$$

In order to prove this, we first need to review the central angle theorem which then gives us a useful extended version of the law of sines.

Proposition 2.2 (Central angle theorem)**.** An inscribed angle in a circle, subtending an arc on the circle, is half of the central angle subtending the same arc.

Proof. Consider Figure 2a and let the inscribed angle be θ with its vertex on the circumference of the circle. Denote the central angle ψ with vertex on the center of the circle subtending the same arc. Consider the special case in Figure 2b in which one side of the inscribed angle is the



(a) Central angle theorem with inscribed angle, θ, and central angle, ψ, subtending the same arc.

(b) Special case with one side being the diameter and the supplementary angle ϕ.

(c) Generalized central angle theorem.

Figure 2

diameter. The angles are the same as denoted above. Noting that ϕ is the supplementary angle to ψ we have the relation

$$\begin{aligned} \phi = 180^\circ - 2\theta &= 180^\circ - \psi \\ \implies \theta &= \frac{1}{2}\psi. \end{aligned}$$

This can be generalized by considering Figure 2c and by using the fact that

$$\theta_2 = \frac{1}{2}\psi_2,$$

along with

$$\theta_1 + \theta_2 = \frac{1}{2}\left(\psi_1 + \psi_2\right).$$

This ultimately yields

$$\theta_1 = \frac{1}{2}\psi_1.$$

For the case in Figure 2a we use the results above by inserting a diameter which gives the modfied situation seen in Figure 3. With the results concluded from Figure 2b we once again get the relations $\theta_1 = \frac{1}{2}\psi_1$ and $\theta_2 = \frac{1}{2}\psi_2$ yielding

$$\theta_1 + \theta_2 = \frac{1}{2}\underbrace{(\psi_1 + \psi_2)}_{\psi} = \frac{1}{2}\psi \iff \theta = \frac{1}{2}\psi.$$

□

We are now ready to prove the law of sines.

Proposition 2.3 (Law of sines)**.** $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = d$, where a,b and c are the side lengths of the triangle with opposite angles α, β, γ and d is the diameter of the circumcircle see Figure 1.

Proof. Consider Figure 4 and let w denote the diametrically opposed point of z, on the circumcircle. Construct the triangle $\triangle zwy$ and let the angle at w be δ. The angle $\angle wyz$ is right and $\delta = \beta$ by the central angle theorem (Proposition 2.2). Thus $\sin\beta = \frac{b}{d}$ so that $d = \frac{b}{\sin\beta}$. Proceeding analogously we obtain the corresponging equalities for a and c. □

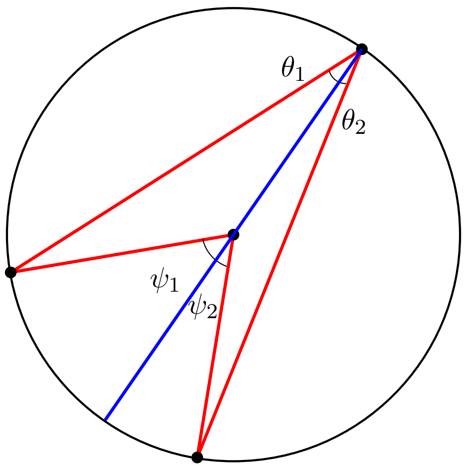


Figure 3: Central angle theorem with inscribed angle, $\theta = \theta_1 + \theta_2$, and central angle, $\psi = \psi_1 + \psi_2$, subtending the same arc.

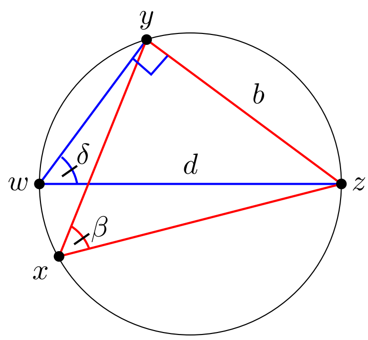


Figure 4: The triangle $\triangle xyz$ with its circumcircle as well as a right triangle constructed using the point w diametrically opposed to z.

We are now ready to prove Proposition 2.1.

Proof. The area of a triangle is half of its base times its height

$$2A = ab\sin\gamma = bc\sin\alpha = ca\sin\beta.$$

Summing over these yields

$$\begin{aligned}
\Rightarrow \frac{6A}{abc} &= \frac{1}{c}\sin\gamma + \frac{1}{a}\sin\alpha + \frac{1}{b}\sin\beta \\
&= \{\text{law of sines (prop. 2.3)}\} \\
&= \frac{3}{d} \\
\frac{4A}{abc} &= \frac{1}{r}.
\end{aligned}$$

□

With this proof completed, we will proceed to introduce the basics of Clifford algebra. This will enable us to solve the geometric problem of particle interaction algebraically.

3 Clifford Algebra

In this section we introduce the Clifford algebra (sometimes referred to as the geometric algebra) of Euclidian space. We prove some basic results which will be used in Section 5. For a more general and thorough introduction to Clifford algebra, see for example [9], [10] or [11].

3.1 Definitions

For our purposes it will suffice to consider the Clifford algebra on $\mathbb{R}^n$. Take an orthonormal (ON) basis $E = \{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. Let $\mathcal{B}_E$ be the set of lists of elements in E subject to the reduction rules

$$e_i e_i = e_i^2 = 1$$
$$e_i e_j = -e_j e_i, \quad i \neq j.$$

The set $\mathcal{B}_E$ also includes the empty list.

Definition 3.1. The *Clifford algebra* $\mathcal{C}l(\mathbb{R}^n)$ of $\mathbb{R}^n$ is $\operatorname{Span}_{\mathbb{R}} \mathcal{B}_E$, i.e. formal sums of elements in $\mathcal{B}_E$ with coefficients in $\mathbb{R}$, along with the geometric product. The *geometric product* is defined by concatenation of elements in $\mathcal{B}_E$ subject to the reduction rules above and extended linearly to all of $\operatorname{Span}_{\mathbb{R}} \mathcal{B}_E$. Elements in the Clifford algebra are called *multivectors*. The empty list in $\mathcal{B}$ is the identity of the geometric product. We will write $\mathcal{C}l = \mathcal{C}l(\mathbb{R}^n)$ when the dimension is implied.

Example 3.2. Consider the geometric product of the elements $e_1e_3e_4$ and e_2e_3. We then get

$$e_1e_3e_4 \underbrace{e_2e_3}_{-e_3e_2} = -e_1e_3 \underbrace{e_4e_3}_{-e_3e_4} e_2 = e_1 \underbrace{e_3e_3}_{1} e_4e_2 = e_1e_4e_2 = -e_1e_2e_4.$$

Proposition 3.3. $\mathcal{C}l(\mathbb{R}^n)$ is independent on the choice of ON-basis E.

For a proof of this proposition we direct the reader to [9].

We introduce a multi-index notation for sets of indices $A = \{i_1, i_2, \dots, i_k\}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The elements of $\mathcal{B}_E = \{e_A\}$ are called *basis blades*, where $\{e_A\}$ is the product $e_{i_1}e_{i_2}\cdots e_{i_k}$. Furthermore, let the symmetric difference of the sets A and B be denoted, $A \triangle B := A \cup B \setminus A \cap B$.

Proposition 3.4. For basis blades e_A, e_B the geometric product behaves like

$$e_A e_B = \pm e_{A \triangle B}.$$

Proof. This follows from the anticommutativity of the geometric product (each commutation of distinct basis vectors yields a factor -1) on E and the fact that $e_i^2 = 1$. $\square$

Definition 3.5. For a proposition P, let (P) evaluate as 1 if P is true and 0 if it is false. We define

$$\begin{aligned}
e_A \wedge e_B &:= (A \cap B = \varnothing) e_A e_B && \textit{outer product} \\
e_A * e_B &:= (A = B) e_A e_B && \textit{scalar product} \\
e_A \llcorner e_B &:= (A \subseteq B) e_A e_B && \textit{left inner product} \\
e_A \lrcorner e_B &:= (A \supseteq B) e_A e_B && \textit{right inner product} \\
\langle e_A \rangle_k &:= (|A| = k) e_A && \textit{projection on grade } k \\
{e_A}^\star &:= (-1)^{|A|} e_A && \textit{grade involution} \\
{e_A}^\dagger &:= (-1)^{\binom{|A|}{2}} e_A = e_{i_k} e_{i_{k-1}} \cdots e_{i_1} && \textit{reversion}
\end{aligned}$$

and extend linearly to the rest of $\mathcal{C}l$.

Note that this definition of the scalar product coincides with the usual one on vectors. Moreover the above definition of reversion of a basis blade is equivalent to reversing the order of its constituent basis vectors

$$(e_{i_1} e_{i_2} \cdots e_{i_k})^\dagger = e_{i_k} e_{i_{k-1}} \cdots e_{i_1},$$

since every commutation yields a factor of -1 and every basis vector needs to commute with every other basis vector in the basis blade exactly once. The number of possible pairs of basis vectors in the basis blade is $\binom{k}{2}$, which gives the right sign.

The outer product is of special interest for us and two of its properties are presented next.

Proposition 3.6. The outer product is associative.

Proof. Since the outer product is linear by definition, it suffices to prove associativity on basis blades. Let $e_A, e_B, e_C \in \mathcal{B}_E$.

$$\begin{aligned}
e_A \wedge (e_B \wedge e_C) &= (B \cap C = \varnothing) e_A \wedge (e_B e_C) \\
&= (B \cap C = \varnothing)(A \cap (B \triangle C) = \varnothing) e_A e_B e_C \\
&= (B \cap C = \varnothing)(A \cap B = \varnothing)(A \cap C = \varnothing) e_A e_B e_C \\
&= (A \cap B = \varnothing)(C \cap (A \triangle B) = \varnothing) e_A e_B e_C \\
&= (e_A \wedge e_B) \wedge e_C.
\end{aligned}$$

□

Proposition 3.7. For $v, w \in \mathbb{R}^n$ we have

$$v \wedge w = -w \wedge v.$$

Proof. This follows from anticommutativity on the basis, linearity and the fact that $e_i \wedge e_i = 0$. □

3.2 The scalar product and vectors

Proposition 3.8. For $v \in \mathbb{R}^n$ and $x \in \mathcal{C}l$ we have

$$\begin{aligned}
vx &= v \llcorner x + v \wedge x \\
xv &= x \lrcorner v + x \wedge v.
\end{aligned}$$

Proof. Noting that $e_i e_A = (i \in A) e_i e_A + (i \notin A) e_i e_A$ the proposition follows from linearity. $\square$

Proposition 3.9. For $v, w \in \mathbb{R}^n$ we have

$$v \llcorner w = v * w = v \lrcorner w.$$

Proof. This follows immediately from the definitions above and linearity. $\square$

Corollary 3.10. $vw = v * w + v \wedge w$

Proposition 3.11. For $v, w \in \mathbb{R}^n$ we have $vw = wv + 2v \wedge w$

Proof. The proposition follows from the anticommutativity of the outer product on vectors and the commutativity of the scalar product, since that yields $wv = v * w - v \wedge w$. $\square$

Proposition 3.12. For any two multivectors x and y, the scalar product can be written as

$$x * y = \langle xy \rangle_0,$$

where $\langle \cdot \rangle_0$ is projection on grade zero (i.e. the scalars) as in Definition 3.5.

Corollary 3.13. $\langle xy \rangle_0 = \langle yx \rangle_0$.

3.3 Blades

Let outer products of k vectors $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ be called *k-blades* and denote the set of all blades by $\mathcal{B} = \{a_1 \wedge a_2 \wedge \ldots \wedge a_k \mid a_i \in \mathbb{R}^n\}$. The blades can be interpreted geometrically with the following results.

Proposition 3.14. The vectors $\{v_1, \ldots, v_k\}$ are linearly independent if and only if

$$v_1 \wedge \cdots \wedge v_k \neq 0.$$

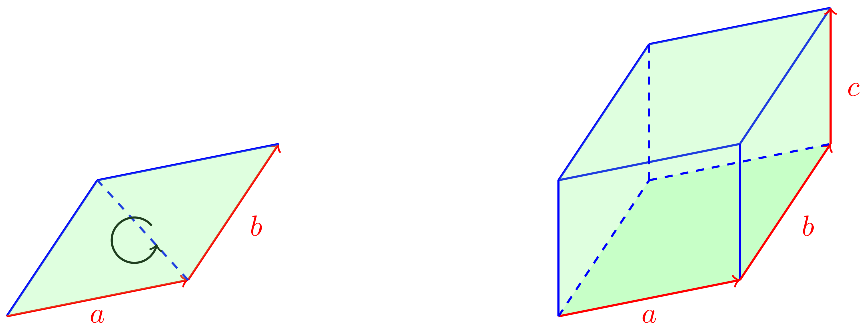
For a proof of this, please refer to [9], for example.

Corollary 3.15. Let $B = v_1 \wedge \cdots \wedge v_k \neq 0$ and $v \in \mathbb{R}^n$. Then

$$v \wedge B = 0 \iff v \in \operatorname{Span}\{v_1, \ldots, v_k\}.$$

This allows us to associate a k-blade $a_1 \wedge \cdots \wedge a_k$ with a k-dimensional subspace $\operatorname{Span}\{a_1, \ldots, a_k\}$ together with an orientation and magnitude. Blades are a geometric generalization of vectors in the sense that a vector is represented exactly by a one-dimensional subspace along with a length and a sign. For example a 2-blade $a \wedge b$ of vectors a and b can be visualized as directed parallelogram with its area equal to its magnitude. By the same token the 3-blade $a \wedge b \wedge c$ of vectors a, b and c can be visualized as a parallelepiped, this is seen in figure 5. Moreover, arbitrary multivectors are, by definition, linear combinations of basis blades with the geometric interpretation of being linear combinations of orthogonal subspaces.

Proposition 3.16. Any k-blade can be written as a scalar times a geometric product of k basis vectors.



(a) A 2-blade $a \wedge b$ with magnitude and orientation.

(b) A 3-blade $a \wedge b \wedge c$ with magnitude and orientation.

Figure 5

Proof. Consider a k-blade $B = a_1 \wedge \cdots \wedge a_k$. Take an ON-basis $\{\epsilon_1, \ldots, \epsilon_k\}$ of $\mathrm{Span}\{a_1, \ldots, a_k\}$, e.g. by Gram-Schmidt. The blade can thus be written as $B = \alpha \epsilon_1 \wedge \cdots \wedge \epsilon_k$, where $\alpha \in \mathbb{R}$ fixes the sign (i.e. orientation) and magnitude. Since $\{\epsilon_i\}_i$ is ON, $\epsilon_i \wedge \epsilon_j = \epsilon_i \epsilon_j$ by Proposition 3.9. Thus $B = \alpha \epsilon_i \cdots \epsilon_k$. □

Proposition 3.17. For any blade $B \in \mathcal{B}$, $B^2 \in \mathbb{R}$.

Proof. By prop. 3.16, we can write $B = \beta e_{i_1} \cdots e_{i_k}$ where $\beta \in \mathbb{R}$. Then

$$B^2 = \beta^2 e_{i_1} \cdots e_{i_k} e_{i_1} \cdots e_{i_k}.$$

Repeatedly swapping places using the anticommutativity property of the geometric product, the square becomes

$$B^2 = \pm \beta^2 e_{i_1}^2 \cdots e_{i_k}^2 = \pm \beta^2.$$

□

This proposition implies that every non-zero blade B is invertible with $B^{-1} = \frac{1}{B^2} B$.

Proposition 3.18. For a 2-blade $B = a \wedge b$, the area of the parallellogram spanned by a and b is given by the magnitude $\sqrt{B^\dagger B} = \sqrt{B B^\dagger} =: |B|^2$ of B.

Proof. By a change of basis, we can write

$$\begin{aligned} a &= a_1 e_1 \\ b &= b_1 e_1 + b_2 e_2, \end{aligned}$$

which makes the area of the parallellogram to be $|a_1 b_1|$. On the other hand,

$$B = a \wedge b = a_1 b_2 e_1 e_2 \implies B B^\dagger = a_1^2 b_2^2 e_1 e_2 e_2 e_1 = a_1^2 b_2^2.$$

Hence $\sqrt{B B^\dagger} = |a_1 b_2|$ and we are done. □

3.4 Summary

We defined the geometric product on ON basis vectors as juxtaposition, subject to two rules: $e_i e_i = 1$ and $e_i e_j = -e_j e_i$ if $i \neq j$. This was used this to define the Clifford algebra on Euclidean space as linear combinations of the lists or words of basis vectors together with the geometric product. We went on to define important operations and products and we also presented a number of propositions, some extra notation and interpretations. In the following sections, we will utilize the framework developed here to investigate links between geometry and the physics of a certain kind of particle.

4 Particle Interactions

In this section, we consider a system of two-dimensional particles (called anyons, see [12, 13, 14, 7]). We will investigate the quantum mechanical Hamiltonian of the system and show that it gives rise to a term with surprising geometric connections that we will investigate later using Clifford algebra.

Consider n particles at positions $x_1, x_2, \dots x_n \in \mathrm{Span}\{e_1, e_2\} \subset \mathbb{R}^3$. Denote the disk in $\mathrm{Span}\{e_1, e_2\}$ of radius R centered at p as $D_R(p)$. Let each particle generate a constant magnetic field parallel to e_3, in the disk $\mathrm{Span}\{e_1, e_2\}$ of radius R as seen in Figure 6.

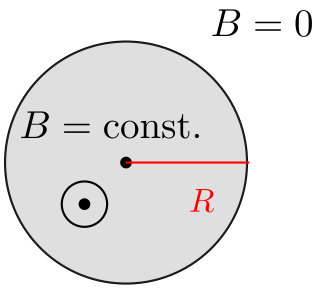


Figure 6: Each particle generates a constant magnetic field in a disk around it.

While this specific form of magnetic interaction might seem arbitrary it is actually well motivated from physics, see [12, 13]. It is also mathematically studied from the perspective of effective mean-field theory in [7].

Define the indicator function for a set M by $\mathbf{1}_M(x) := (x \in M)$, where the proposition in parenthesis takes on value 1 if true and 0 otherwise, as in Section 3. The magnetic field around particle i can then be written

$$B(x) = C\mathbf{1}_{D_R(x_i)}(x)e_3 = C\mathbf{1}_{D_R(0)}(x - x_i)e_3$$

and we will normalize it so that $C = 1$.

4.1 The vector potential of B

In order to investigate the quantum mechanical Hamiltonian, the electromagnetic vector potential is needed. We will start by introducing further notation in order to write down the vector potential, A.

Let $I_2 = e_1e_2$. For a vector $x \in \mathrm{Span}\{e_1, e_2\}$, define $x^\perp = xI_2$ with the geometric interpretation of x being rotated by a right angle. To realize this, note that it is linear and acts on the e_1, e_2 by sending them to $e_1I_2 = e_2$ and $e_2I_2 = -e_1$. We will also write $x^{-\perp}$, when $x \neq 0$, meaning $\frac{1}{x^2}x^\perp$. The following lemma is geometrically evident.

Lemma 4.1. $x^\perp * y^\perp = x * y$ for any $x, y \in \mathrm{Span}\{e_1, e_2\}$.

Define the following regularization

$$x_R^{-1} := \frac{1}{|x|_R^2}x, \tag{2}$$

where

$$|x|_R := \max\{|x|, R\},$$

for $x \in \mathbb{R}^n$.

We are now ready to introduce the magnetic vector potential corresponding to the magnetic field B above.

Proposition 4.2. $A(x) = (x-p)_R^{-\perp} := \frac{(x-p)^\perp}{|x-p|_R^2} \implies \nabla \times A \propto \mathbf{1}_{D_R(p)} e_3$

Proof. It is enough to show that $\nabla \times x_R^{-\perp} \propto \mathbf{1}_{D_R(0)}(x) e_3$. We shall write

$$
\begin{aligned}
x &= x_1 e_1 + x_2 e_2, \quad x_1, x_2 \in \mathbb{R} \\
\nabla &= e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}.
\end{aligned}
$$

The scalar coordinates x_1, x_2 in this proof are not to be confused with the vector positions x_i above. Now split into two cases. Case 1: $|x| \le R$.

$$
\begin{aligned}
\nabla \times x_R^{-\perp} &= (e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}) \times \frac{x_1 e_2 - x_2 e_1}{R^2} \\
&= \frac{1}{R^2}(1+1)e_3 = \frac{2}{R^2} e_3.
\end{aligned}
$$

Case 2: $|x| > R$.

$$
\begin{aligned}
\nabla \times x_R^{-\perp} &= (e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}) \times \frac{x_1 e_2 - x_2 e_1}{x_1^2 + x_2^2} \\
&= \left(\frac{\partial}{\partial x_1} \frac{x_1}{x_1^2 + x_2^2} + \frac{\partial}{\partial x_2} \frac{x_2}{x_1^2 + x_2^2} \right) e_3 \\
&= \left(\frac{x_1^2 + x_2^2 - 2x_1^2 + x_1^2 + x_2^2 - 2x_2^2}{(x_1^2 + x_2^2)^2} \right) e_3 = 0.
\end{aligned}
$$

Thus $\nabla \times x_R^{-\perp} = \frac{2}{R^2} \mathbf{1}_{D_R(0)}(x) e_3$. □

Since we are primarily interested in the vector potential, we will renormalize everything so that $A = x_R^{-\perp}$.

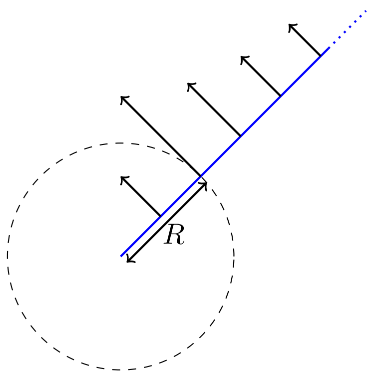


Figure 7: Visualization of the vector potential A, increasing linearly inside the radius R and as the inverse of the distance outside.

4.2 The Hamiltonian

The quantum Hamiltonian for a point charge at x in a magnetic field can be written as

$$\hat{H} = \frac{1}{2m}(\hat{p} - qA(x,t))^2 + q\phi(x,t),$$

where q is the charge, x is the position of the charge, $\hat{p}$ is the usual momentum operator, A is the vector potential and ϕ the scalar potential. Discarding the $q\phi$ term (i.e. assuming no external electric field) and expanding the square we obtain

$$\hat{p}^2 - q\hat{p} * A - qA * \hat{p} + q^2 A^2.$$

The first term is just the kinetic energy and the second two are paramagnetic. The last term, $q^2 A^2$, which is a diamagnetic contribution, is what we shall examine closer.

In our case, we have n charged particles. Thus, the corresponding diamagnetic term will be a sum of squares of the vector potentials felt by each particle. Each particle is affected by the vector potential from every other particle, and we will denote the total vector potential felt by the j:th particle as $A_j(x_j)$. Hence, the A-quadratic term of our system becomes

$$\begin{aligned}
\sum_j A_j(x_j) * A_j(x_j) &= \sum_j \sum_{k \neq j} \sum_{l \neq j} (x_j - x_k)_R^{-\perp} * (x_j - x_l)_R^{-\perp} && (3)\\
&= \sum_j \sum_{k \neq j} \sum_{l \neq j} (x_j - x_k)_R^{-1} * (x_j - x_l)_R^{-1} && (4)\\
&= \sum_j \sum_{k \neq j} |x_j - x_k|_R^{-2} + \sum_j \sum_{k \neq j} \sum_{l \neq j,k} (x_j - x_k)_R^{-1} * (x_j - x_l)_R^{-1}. && (5)
\end{aligned}$$

The first term is a two-particle interaction and represents a kind of repulsive force between pairs of particles, since it is positive and (mostly) decreases as their separation increases. Note that for the second term of (5), we can look at every choice of three particles independently. Let us therefore restrict ourselves to $n = 3$ and denote the particle positions as x, y and z. The second term of (5) then becomes

$$2\big((x-y)_R^{-1} * (x-z)_R^{-1} + (y-z)_R^{-1} * (y-x)_R^{-1} + (z-x)_R^{-1} * (z-y)_R^{-1}\big).$$

4.3 The tree-particle interaction, v_R

In light of the three-particle interaction that emerged from the diamagnetic contribution above, define

$$v_R(x,y,z) := \sum_{\text{cyclic in } x,y,z} (x-y)_R^{-1} * (x-z)_R^{-1}. \tag{6}$$

Later on, we will see that the three-particle interaction is, in many ways, analagous to the two-particle repulsion. Three particles at x, y, z naturally form a triangle with sides $x - y$, $y - z$ and $z - x$. In the next section, we will, for example, show that when all sides of this triangle are larger than R, then $v_R(x,y,z)$ is proportional to the squared inverse of the circumradius of this triangle. It will also be shown that the three-particle interaction is a purely repulsive term. This fact is not evident merely from inspection even though the sum of the interactions, A^2, is of course positive. Note that, unlike in classical physics, the particles affect each other magnetically even when they are not in each other's magnetic fields. This effect, of seeing magnetic fields remotely via the vector potential, is known as the Ehrenberg-Siday-Aharonov-Bohm effect [15, 16]

5 Results

In this section we will use the tools presented in previous sections to investigate triangles in particle interactions with applications of Clifford algebra. The connection between the potential v_R and the circumradius will be shown as well as inequalities involving v_R in regard to the magnetic field radius R.

We will now examine the potential

$$v_R(x,y,z) := \sum_{\text{cyclic in } x,y,z} (x-y)_R^{-1} * (x-z)_R^{-1}, \tag{7}$$

which we defined in Section 4. Even though we originally defined $v_R(x,y,z)$ for $x,y,z \in \mathbb{R}^3$, we will now let x,y,z be in $\mathbb{R}^n$ for generality. Note that v_R is

$$v_R(x,y,z) = \sum_{\text{cyclic in } x,y,z} \frac{(x-y)}{|x-y|_R^2} * \frac{(x-z)}{|y-z|_R^2}, \tag{8}$$

by definition, see Section 4. We will see that this potential is, when the particles are separated enough, positive and decreasing with distance and therefore purely repulsive.

5.1 The connection between v_R and the circumradius

Proposition 5.1. Let $x,y,z \in \mathbb{R}^n$ and $R \in \mathbb{R}_+$. If the distance between every pair of points x, y and z is greater than R then

$$|x-y|^2 \cdot |y-z|^2 \cdot |z-x|^2 v_R(x,y,z) = 2|B|^2 = 2B^\dagger B.$$

where we have defined the 2-blade $B := (x-y) \wedge (x-z)$.

Proof. We will start with the left-hand side and manipulate it using propositions from Section 3. The first step is to rewrite it in terms of the geometric product, using Proposition 3.12.

$$\begin{aligned}
\text{LHS} &= (x-y)*(x-z)|y-z|^2 + \text{cycl.} \\
&= \big((x-y)*(x-z)\big)\big((y-z)*(y-z)\big) + \text{cycl.} \\
&= \langle (x-y)(x-z)(y-z)(y-z) \\
&\qquad + (y-z)(y-x)(z-x)(z-x) \\
&\qquad + (z-x)(z-y)(x-y)(x-y)\rangle_0.
\end{aligned}$$

By Corollary 3.13, we can reorder the factors inside the scalar projection, in a certain way. Using this to factor out $(x-y)(x-z)$ from the left, we obtain

$$\text{LHS} = \langle (x-y)(x-z)\big[(y-z)(y-z) + (z-x)(y-z) - (z-y)(x-y)\big]\rangle_0.$$

In order to factor out another factor, $(y-z)$, we will use Propsition 3.11 to rewrite $(z-x)(y-z) = (y-z)(z-x) + 2(z-x) \wedge (y-z)$. This then gives

$$\begin{aligned}
\text{LHS} = \langle &(x-y)(x-z)(y-z)\overbrace{\big[(y-z)+(z-x)+(x-y)\big]}^{0} \\
&+ 2(x-y)(x-z)\cdot(z-x)\wedge(y-z)\rangle_0 \\
= \langle &(x-y)(x-z)\cdot(z-x)\wedge(y-z)\rangle_0.
\end{aligned}$$

If we decompose the geometric product in this expression, using Proposition 3.9, it becomes

$$\begin{aligned}\text{LHS} &= 2\langle\underbrace{(x-y)\wedge(x-z)}_{B}\cdot\underbrace{(z-x)\wedge(y-z)}_{-B}\rangle_0 + \underbrace{2\langle(x-y)*(x-z)\cdot(z-x)\wedge(y-z)\rangle_0}_{0} \\ &= 2|B|^2.\end{aligned}$$

In the last step, we used the fact that a scalar times a 2-blade is still a 2-blade, which is sent to zero under projection on the scalars. $\square$

Consider the triangle of which x, y, z are the vertices and denote its circumradius r. Identifying $|B|$ as twice the area of the triangle, Proposition 5.1 together with Proposition 2.1 gives us

$$v_R(x,y,z) = \frac{1}{2r^2}. \tag{9}$$

This results in the corollary

Corollary 5.2.

$$\frac{1}{2}r^{-2} = \sum_{\text{cyclic in } x,y,z} (x-y)^{-1} * (x-z)^{-1}.$$

Recall the two-particle repulsion in equation (5) in Section 4. The circumradius is a natural generalization of the distance between two particles to three particles. In this sense, the two- and three-particle interactions have an analogous behavior. Indeed, were you to take a guess about the behavior of the three-particle interaction based on the two-particle interaction, $\frac{1}{r^2}$ is an intuitive choice. Excluding the circumradius however, the next guess could be a total pairwise separation

$$\rho^2 := |x-y|^2 + |y-z|^2 + |z-x|^2.$$

The following result suggests a strong connection between r and ρ.

Proposition 5.3.

$$\frac{1}{2}\rho^2 = \sum_{\text{cyclic in } x,y,z} (x-y) * (x-z).$$

Proof. Rewriting this we have that

$$\begin{aligned}\rho^2 &= |x-y|^2 + |y-z|^2 + |z-x|^2 \\ &= \sum_{\text{cyclic in } x,y,z} (x-y)*(x-y) \\ &= \sum_{\text{cyclic in } x,y,z} \left(x^2 + y^2 - 2x*y\right) \\ &= 2x^2 + 2y^2 + 2z^2 - 2x*y - 2y*z - 2z*x \\ &= 2\left(\left[x^2 - x*y - x*z + y*z\right] + y^2 + z^2 - 2y*z\right) \\ &= 2\left((x-y)*(x-z) + \left[y^2 - y*x - y*z + x*z\right] + z^2 - y*z - x*z + x*y\right) \\ &= 2\left((x-y)*(x-z) + (y-x)*(y-z) + (z-x)*(z-y)\right).\end{aligned}$$

$\square$

Note the similarity if this result with Corollary 5.2. In light of this, it is natural to examine the scale invariant and dimensionless quantity $\rho^2 v_R$.

5.2 Inequalities involving v_R

The main result of this section is the following characterisation of v_R.

Theorem 5.4. For any $R \geq 0$ and $x, y, z \in \mathbb{R}^n$ we have that

$$0 \leq v_R(x, y, z) \leq \frac{C}{\rho^2},$$

where C is a constant.

We will divide the proofs into several parts.

Proposition 5.5. In the case when all sides of the triangle $\triangle xyz$ are greater than R, we have that

$$0 \leq \rho^2 v_R(x, y, z) \leq \frac{9}{2}.$$

Proof. By Proposition 5.1, $v_R = \frac{1}{2r^2}$. This means that we need to show

$$0 \leq \frac{\rho^2}{r^2} \leq 9.$$

By denoting the sides of $\triangle xyz$ by a, b, c we get

$$\rho^2 = a^2 + b^2 + c^2,$$

and by the law of sines (2.3) we get

$$\begin{aligned} \rho^2 &= d^2 \sin^2(\alpha) + d^2 \sin^2(\beta) + d^2 \sin^2(\gamma) \\ &= 4r^2 \left(\sin^2(\alpha) + \sin^2(\beta) + \sin^2(\gamma)\right) \\ \Rightarrow \frac{\rho^2}{r^2} &= 4 \left(\sin^2(\alpha) + \sin^2(\beta) + \sin^2(\gamma)\right). \end{aligned} \tag{10}$$

This gives rise to the optimization problem of finding the maximum of

$$f(\alpha, \beta, \gamma) := \sin^2(\alpha) + \sin^2(\beta) + \sin^2(\gamma),$$

under the constraint that

$$g(\alpha, \beta, \gamma) := \alpha + \beta + \gamma = \pi.$$

Using Lagrange multiplicators: $L(\alpha, \beta, \gamma) = f(\alpha, \beta, \gamma) - \lambda(g(\alpha, \beta, \gamma) - \pi)$,

$$\begin{aligned} & \nabla L = 0 \\ \Leftrightarrow \quad & 2\sin(\alpha)\cos(\alpha) = 2\sin(\beta)\cos(\beta) = 2\sin(\gamma)\cos(\gamma) \\ \Leftrightarrow \quad & \sin(2\alpha) = \sin(2\beta) = \sin(2\gamma). \end{aligned}$$

With the constraint ($g = \pi$) we obtain the following three solutions:

$$\alpha = \beta = \gamma = 0, \alpha = \beta = \frac{\pi}{2}, \gamma = 0 \text{ and } \alpha = \beta = \gamma = \frac{\pi}{3}. \tag{11}$$

Substituting these solutions into equation (10) gives

$$\frac{\rho^2}{r^2} = 0, \quad \frac{\rho^2}{r^2} = 8 \quad \text{and} \quad \frac{\rho^2}{r^2} = 9, \tag{12}$$

respectively. Finally

$$0 \leq \frac{\rho^2}{r^2} \leq 9, \tag{13}$$

with equality if and only if the triangle is equilateral. □

Proposition 5.6. In the case when all sides of $\triangle xyz$ are smaller than R, then

$$0 \leq \rho^2 v_R \leq \frac{9}{2}.$$

Proof. We have that

$$\begin{aligned}
v_R(x,y,z) &= \sum_{\text{cyclic in } x,y,z} (x-y)_R^{-1} * (x-z)_R^{-1} \\
&= \sum_{\text{cyclic in } x,y,z} \frac{(x-y)}{|x-y|_R^2} * \frac{(x-z)}{|y-z|_R^2} \\
&= \frac{1}{R_4} \underbrace{\left[\sum_{\text{cyclic in } x,y,z} (x-y) * (x-z) \right]}_{\frac{1}{2}\rho^2}
\end{aligned}$$

$$\implies \rho^2 v_R(x,y,z) = \frac{1}{2}\frac{\rho^4}{R^4} = \{\rho^2 = a^2 + b^2 + c^2 \leq 3R^2\} \leq \frac{9}{2}.$$

By the definition of ρ, we get $\rho^2 v_R \geq 0$. $\square$

We pause to illuminate an interesting point. In the proof of Proposition 5.6, we see that $v_R \propto \rho^2$ when the particle separation is small. Moreover, Proposition 5.1 tells us that $v_R \propto r^{-2}$ when the particle separation is large. Thus, v_R is, in a sense, an interpolation between ρ^2 and r^{-2} as the positions x, y, z move apart.

We will now treat the rest of the cases of Theorem 5.4.

Proposition 5.7. If one side of $\triangle xyz$ is greater than R and two are smaller than R, then

$$0 \leq \rho^2 v_R \leq 12.$$

Proof. Let, without loss of generality,

$$|x-y| \geq R \quad \text{and} \quad |y-z|, |z-x| \leq R,$$

We will do the proof in two steps. Firstly, we will show that

$$0 \leq |x-y|_R^2 \cdot |y-z|_R^2 \cdot |z-x|_R^2 \cdot v_R(x,y,z) \leq 2R^2|x-y|^2. \tag{14}$$

Then we will show that

$$2R^2|x-y|^2 \leq |x-y|_R^2 \cdot |y-z|_R^2 \cdot |z-x|_R^2 \cdot \frac{12}{\rho^2}. \tag{15}$$

Combining these inequalities gives the desired result.

The left-hand side of (14) can be rewritten and estimated as

$$\begin{aligned}
\text{LHS} &= |y-z|_R^2 \cdot (x-y) \cdot (x-z) + |z-x|_R^2 \cdot (y-x) \cdot (y-z) + |x-y|_R^2 \cdot (z-x) \cdot (z-y) \\
&= R^2 \cdot (x-y) \cdot (x-z) + R^2 \cdot \underbrace{(y-x) \cdot (y-z)}_{-(x-y)\cdot(y-z)} + |x-y|^2 \cdot (z-x) \cdot (z-y) \\
&= R^2 (x-y) \left[(x-z) - (y-z) \right] + |x-y|^2 \cdot (z-x) \cdot (z-y) \\
&= R^2 |x-y|^2 + |x-y|^2 \cdot (z-x) \cdot (z-y) \\
&= |x-y|^2 \left[R^2 + R^2 \right] \leq 2R^2 |x-y|^2.
\end{aligned}$$

Similarly we get a lower bound on the left-hand side of (14) by

$$\begin{aligned} \text{LHS} &= R^2|x-y|^2 + |x-y|^2 \cdot (z-x) \cdot (z-y) \\ &= |x-y|^2 \left[R^2 - R^2\right] \geq 0. \end{aligned}$$

For the right-hand side of expression (15) we have

$$\begin{aligned} \text{RHS} &= 12 \frac{|x-y|_R^2 \cdot |y-z|_R^2 \cdot |z-x|_R^2}{|x-y|^2 + |y-z|^2 + |z-x|^2} \\ &\geq 12 \frac{|x-y|^2 \cdot R^2 \cdot R^2}{|x-y|^2 + R^2 + R^2}, \end{aligned}$$

and using that $|x-y| \leq |y-z| + |z-x| \leq 2R$ in the denominator we arrive at

$$\begin{aligned} \text{RHS} &\geq 12 \frac{|x-y|^2 \cdot R^4}{4R^2 + 2R^2} \\ &\geq 2R^2|x-y|^2. \end{aligned}$$

Thus, we are done. □

Proposition 5.8. If one side of $\triangle xyz$ is smaller than R and two lengths are greater than R, then

$$0 \leq \rho^2 v_R \leq 24.$$

Proof. Let, without loss of generality,

$$|x-y| \leq R \quad \text{and} \quad |y-z|, |z-x| \geq R.$$

As in the last proof, we will prove two inequalities

$$0 \leq |x-y|_R^2 \cdot |y-z|_R^2 \cdot |z-x|_R^2 \cdot v_R(x,y,z) \leq 4R^2|x-z|^2, \tag{16}$$

and

$$4R^2|x-z|^2 \leq |x-y|_R^2 \cdot |y-z|_R^2 \cdot |z-x|_R^2 \cdot \frac{24}{\rho^2}. \tag{17}$$

Combining these then gives the desired result.

The left-hand side of (16) becomes

$$\begin{aligned} \text{LHS} &= |y-z|_R^2 \cdot (x-y) \cdot (x-z) + |z-x|_R^2 \cdot (y-x) \cdot (y-z) + |x-y|_R^2 \cdot (z-x) \cdot (z-y) \\ &= |y-z|^2 (x-y) * (x-z) + |z-x|^2 (y-x) * (y-z) + R^2 (z-x) * (z-y). \end{aligned}$$

The first two terms of this expression,

$$|y-z|^2 (x-y) * (x-z) + |z-x|^2 (y-x) * (y-z), \tag{18}$$

are almost the same as the expression analyzed in Proposition 5.1. With that result in mind, (18) becomes

$$\langle (x-y)(x-z)\big[(y-z)(y-z) + (z-x)(y-z)\big]\rangle_0 = -|x-y|^2 (x-z) * (y-z) + 2|B|^2.$$

Thus, the positivity of the left-hand side of (16) is obtained by

$$\begin{aligned}
\text{LHS} &= |y-z|^2(x-y)*(x-z) + |z-x|^2(y-x)*(y-z) + R^2(z-x)*(z-y) \\
&= -|x-y|^2(x-z)*(y-z) + 2|B|^2 + R^2(z-x)*(z-y) \\
&= (x-z)*(y-z)\big[-|x-y|^2 + R^2\big] + 2|B|^2 \geq 0.
\end{aligned}$$

To get the upper bound in (16), we will use

$$|B| = |(x-y)\wedge(x-z)| \leq |x-y||x-z|,$$

and the triangle inequality

$$|y-z| \leq |y-x| + |x-z| \leq R + |x-z| \leq 2|x-z|.$$

These estimates then imply, for the left-hand side of (16), that

$$\text{LHS} \leq \underbrace{(x-z)*(y-z)}_{\leq|x-z||y-z|\leq 2|x-z|^2}\underbrace{\big[-|x-y|^2+R^2\big]}_{\leq R^2} + 2|B|^2 \leq 4R^2|x-z|^2.$$

To show (17), we estimate the right-hand side as

$$\begin{aligned}
R.H.S. &\geq 24\frac{R^2|y-z|^2\cdot|z-x|^2}{R^2+|y-z|^2+|z-x|^2} \\
&\geq 24\frac{R^2|y-z|^2\cdot|z-x|^2}{R^2+|y-z|^2+(2|y-z|)^2} \\
&\geq 24\frac{R^2|y-z|^2\cdot|z-x|^2}{|y-z|^2+|y-z|^2+4|y-z|^2} \\
&\geq 4R^2\cdot|z-x|^2
\end{aligned}$$

□

This concludes our proof of Theorem 5.4.

6 Conclusions and Further Directions

The purpose of this thesis has been to examine the relationship between geometry and two-dimensional magnetically interacting particles using Clifford algebra.

The main results of this thesis are Proposition 5.1 and Theorem 5.4. The former provided us with a strong geometric interpretation of the three-particle interaction that emerged in Section 4. The latter says that the three-particle interaction v_R is positive and decreasing with distance, when the particles are separated enough. This implies that this potential is in fact repulsive, just as its two-particle counterpart. In the case where the particles are very close to each other, v_R is proportional to the sum of the pairwise distances squared (which we defined as ρ^2). The potential v_R is thus like an interpolation between ρ^2 and r^{-2} as the particles move apart.

In conclusion we have successfully used Clifford algebra to investigate connections between particle interactions and geometry. There is no doubt about the benifits of using Clifford algebra in mathematics and physics.

6.1 Further directions

We will now go through some open questions and ideas we have had when working on this thesis.

Firstly, we shall present a multivector X with interesting properties. We defined $B = (x-y)\wedge(x-z)$ which is easily seen to be invariant under cyclic permutation so that $B = (y-z)\wedge(y-x) = (z-x)\wedge(z-y)$. Furthermore, we found that $\frac{1}{2}\rho^2 = \sum_{\text{cycl.}}(x-y)*(x-z)$. If we define

$$X := \sum_{\text{cycl.}}(x-y)(x-z)$$

and apply Proposition 3.9 we obtain

$$\begin{aligned} X &= \sum_{\text{cycl.}}\left[(x-y)*(x-z) + (x-y)\wedge(x-z)\right] \\ X &= \frac{1}{2}\rho^2 + 3B. \end{aligned}$$

Then we get

$$XX^\dagger = \frac{1}{4}\rho^4 + 9BB^\dagger \in \mathbb{R},$$

since $B^\dagger = -B$. This is certainly an intrigueing connection when one considers for example Propositions 5.1 and 5.5. Is there a geometric interpretation of X? Can X be used to prove anything? There are things to explore here.

An obvious direction to go is that of generalisation to n-particle interactions. Given n particles, can we find an analogue of v_R that is related to a radius of some sphere associated to those particles? Clifford algebra could be an excellent tool here, especially a domain known as *conformal split*. Introduced in [17], it is a procedure where euclidean space $\mathbb{R}^n$ is extended to a higher dimensional space $\mathbb{R}^{n+1,1}$ with a mixed signature. Points in $\mathbb{R}^n$ are then related to points in a supspace of $\mathbb{R}^{n+1,1}$ called the *horosphere*. It turns out that spheres in $\mathbb{R}^n$ become hyperplanes in the horosphere.

Another potential point of interest could be $\sum_{\text{cycl.}} \rho_R^2 := |x-y|_R^2$ or something similar. There could be some such quantity which is to ρ what v_R is to r. Maybe that quantity has an interpolating behaviour similar to that of v_R but opposite, i.e. it is related to r for small triangles and to ρ for large triangles.

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