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1. Quaternionic analysis

The division ring of quaternions is denoted by \( \mathbb{H} \) and is equal to
\[
\mathbb{H} = \{ t + x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \mid t, x, y, z \in \mathbb{R} \}
\]
where \( \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \) and \( \mathbf{i} \mathbf{j} = \mathbf{k}, \mathbf{j} \mathbf{i} = \mathbf{k}, \mathbf{k} \mathbf{i} = \mathbf{j} \). Quaternions inherit the topological notions of continuity from Euclidean space since \( \mathbb{H} \cong \mathbb{R}^4 \) as \( \mathbb{R} \)-vector spaces. We can therefore begin our study of quaternionic analysis with the definition of a quaternionic derivative \([3]\).

**Definition 1.1.** A function \( f: U \to V \) where \( U, V \subseteq \mathbb{H} \) has a *left quaternionic derivative* at a point \( w_0 \in U \) if the limit
\[
\lim_{h \to 0} h^{-1}(f(w_0 + h) - f(w_0))
\]
exists. Similarly we say that \( f \) has a *right quaternionic derivative* at a point \( w_0 \in U \) if the limit
\[
\lim_{h \to 0} (f(w_0 + h) - f(w_0))h^{-1}
\]
exists.

The following examples show that this definition is in fact to restrictive to be of any practical use.

**Example 1.2.** Let us consider the function \( f: \mathbb{H} \to \mathbb{H} \) defined as \( f(w) = w^2 \). We then calculate that
\[
h^{-1}(f(w + h) - f(w)) = h^{-1}((w + h)^2 - w^2) = h^{-1}(wh + hw + h^2) = h^{-1}wh + w + h.
\]
It is now easy to see that the limit (1) does not exist for our function \( f \). We first choose \( h = h_0 \) where \( h_0 \in \mathbb{R} \). Limit (1) then becomes
\[
\lim_{h_0 \to 0} (h_0^{-1}wh_0 + w + h_0) = \lim_{h_0 \to 0} (2w + h_0) = 2w.
\]
We now choose \( h = h_0 \mathbf{i} \) where \( h_0 \in \mathbb{R} \). Limit (1) then becomes
\[
\lim_{h_0 \to 0} ((h_0\mathbf{i})^{-1}wh_0\mathbf{i} + w + h_0\mathbf{i}) = \lim_{h_0 \to 0} (i^{-1}wi + w + h_0\mathbf{i}) = i^{-1}wi + w.
\]
These two limits are of course not the same for a generic \( w \in \mathbb{H} \).

An analogous calculation shows that limit (2) also does not exist. Therefore \( f \) has neither a left nor a right quaternionic derivative.

**Example 1.3.** Let us consider the function \( f: \mathbb{H} \to \mathbb{H} \) defined as \( f(w) = a + bw \) where \( a, b \in \mathbb{H} \) fixed. We then calculate that
\[
h^{-1}(f(w + h) - f(w)) = h^{-1}(a + b(w + h) - a - bw) = h^{-1}bh.
\]
Thus limit (1) will not exits unless \( b \in Z(\mathbb{H}) = \mathbb{R} \) where \( Z(\mathbb{H}) \) denotes the centre of \( \mathbb{H} \). On the other hand, we calculate that
\[
(f(w + h) - f(w))h^{-1} = (a + b(w + h) - a - bw)h^{-1} = b.
\]
Thus limit (2) always exists. The function \( f \) thus has a right quaternionic derivative and also has a left quaternionic derivative in the special case when \( b \in \mathbb{R} \).
Example 1.4. In analogy to the previous example one can show that $f : \mathbb{H} \to \mathbb{H}$ defined as $f(w) = a + wb$ where $a, b \in \mathbb{H}$ fixed, always has a left quaternionic derivative and also has a right quaternionic derivative in the special case when $b \in \mathbb{R}$.

On has in fact the following theorem [4].

**Theorem 1.5.** Let $f : U \to \mathbb{H}$ where $U \subseteq \mathbb{H}$ is a connected open set. Then $f$ has a left quaternionic derivative at every point in $U$ if and only if $f$ has the form

$$f(w) = a + wb$$

for some $a, b \in \mathbb{H}$.

These examples show that quaternionic polynomials in general are not differentiable. Suppose then that we would like to find a definition of analyticity such that quaternionic polynomials would become analytic. To do so we first note that natural monomial functions of a quaternionic variable of degree $r$ are of the form

$$w \mapsto a_0 w a_1 \ldots a_{r-1} w a_r$$

where $a_0, a_1, \ldots, a_r \in \mathbb{H}$. The elementary monomials can be considered to be those with $a_0, a_1, \ldots, a_r \in \{1, i, j, k\}$. However, if one writes $w = t + xi + yj + zk$ we then have

$$t = \frac{1}{4}(w - i \, w \, i - j \, w \, j - \mathfrak{k} \, w \, \mathfrak{k})$$

$$x = \frac{1}{4}(w - i \, w \, i + j \, w \, j + \mathfrak{k} \, w \, \mathfrak{k})$$

$$y = \frac{1}{4}(w + i \, w \, i - j \, w \, j + \mathfrak{k} \, w \, \mathfrak{k})$$

$$z = \frac{1}{4}(w + i \, w \, i + j \, w \, j - \mathfrak{k} \, w \, \mathfrak{k})$$

Thus a definition which would make all quaternionic polynomials analytic would also make all polynomials in four variables analytic (in a quaternionic sense), something which we do not want.

In search of a better definition of a derivative we recall from elementary complex analysis the Cauchy-Riemann operators (also called the Wirtinger derivatives) that are defined as

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

where the $z = x + iy$ are the coordinates on $\mathbb{C}$. An analogous definition for $\mathbb{H}$ is the following [3].

**Definition 1.6.** A function $f : U \to V$ where $U, V \subseteq \mathbb{H}$ is called left regular on $U$ if

$$D_L f = \frac{\partial f}{\partial t} + \mathfrak{k} \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} + \frac{\partial f}{\partial \mathfrak{k}} = 0.$$  \hfill (3)

Similarly $f$ is called right regular on $U$ if

$$D_R f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial \mathfrak{k}} = 0.$$  \hfill (4)

A function $f$ is called simply regular if it is both left- and right-regular.

The operators $D_L$ and $D_R$ are sometimes called the Fueter operators and equation (3) (or (4)) is sometimes called the left (or right) Cauchy-Riemann-Fueter equation.
We continue with an example that shows that we indeed have non-trivial functions that are left (or right) regular contrary to definition 1.1 where even certain linear functions fail to have a left (or right) quaternionic derivative.

**Example 1.7.** Consider the function \( f : \mathbb{H} \to \mathbb{H} \) defined by \( f(w) = w \). Then \( f \) is neither left nor right regular since \((D_L f)(w) = -2 \) and \((D_R f)(w) = -2 \).

**Example 1.8.** Consider the function \( f : \mathbb{H} \to \mathbb{H} \) defined by
\[
 f(w) = \frac{\overline{w}}{|w|^4} = \frac{t - x^2 i - y^2 j - z^2 k}{(t^2 + x^2 + y^2 + z^2)^2}.
\]
Then \( f \) is both left and right regular on \( \mathbb{H} \) to show this we first calculate that
\[
\frac{\partial f}{\partial t}(w) = \frac{(t^2 + x^2 + y^2 + z^2) - 4t(t - x^2 i - y^2 j - z^2 k)}{(t^2 + x^2 + y^2 + z^2)^3},
\]
\[
\frac{\partial f}{\partial x}(w) = -i(t^2 + x^2 + y^2 + z^2) - 4x(t - x^2 i - y^2 j - z^2 k),
\]
\[
\frac{\partial f}{\partial y}(w) = -j(t^2 + x^2 + y^2 + z^2) - 4y(t - x^2 i - y^2 j - z^2 k),
\]
\[
\frac{\partial f}{\partial z}(w) = -k(t^2 + x^2 + y^2 + z^2) - 4z(t - x^2 i - y^2 j - z^2 k).
\]
It is now trivial to check that
\[
(D_L f)(w) = (D_R f)(w) = 0
\]
for all \( w \in \mathbb{H} \setminus \{0\} \).

**Definition 1.9.** The conjugate Fueter operators are defined as
\[
(D_L)^* = \frac{\partial}{\partial t} - i \frac{\partial}{\partial x} - j \frac{\partial}{\partial y} - k \frac{\partial}{\partial z}
\]
and
\[
(D_R)^* = \frac{\partial}{\partial t} - \frac{\partial}{\partial x} i - \frac{\partial}{\partial y} j - \frac{\partial}{\partial z} k.
\]

We recall now that the Cauchy-Riemann operators factorize the Laplacian on \( \mathbb{R}^2 \) in the sense that \( \partial \overline{\partial} = \overline{\partial} \partial = \frac{1}{i} \Delta \). Similarly the Fueter operators and their conjugates factorize the Laplacian on \( \mathbb{R}^4 \) in the sense that
\[
D_L \overline{D}_L = \overline{D}_L D_L = D_R \overline{D}_R = \overline{D}_R D_R = \Delta.
\]

One also has a quaternionic version of Cauchy’s theorem as follows [4].

**Theorem 1.10.** Suppose \( f \) is regular in an open set \( U \). Let \( w_0 \) be a point in \( U \), and let \( C \) be a rectifiable 3-chain which is homologous, in the singular homology of \( U \setminus \{w_0\} \), to a differentiable 3-chain whose image is \( \partial B \) for some ball \( B \subseteq U \). Then
\[
\frac{1}{2\pi^2} \int_C \frac{(w - w_0)^{-1}}{|w - w_0|^2} Dwf(w) = nf(w_0)
\]
where \( n \) is the wrapping number of \( C \) about \( w_0 \).

The details are available in [4]. On can then also generalize other theorems form complex analysis that depend only on Cauchy’s formula, such as the maximum principle, Morera’s theorem and Liouville’s theorem [4].
2. Dirac operators

Dirac operators are generalized Cauchy-Riemann operators. More precisely, we have the following construction [3].

Let \( Cl = Cl(V, q) \) be the Clifford algebra associated with a real non-degenerated quadratic vector space \((V, q)\). The space \( C^\infty(U, Cl) \) of smooth \( Cl \)-valued functions on an open set \( U \subseteq V \) is a \( Cl \)-module under pointwise multiplication. By identifying \( V \) with a subspace of \( Cl \) we can regard any \( X \in V \) as an element in \( Cl \) and hence a multiplier on \( C^\infty(U, Cl) \). On the other hand, each \( X \in V \) gives rise to a vector field \( \partial_X \) acting on \( C^\infty(U, Cl) \). The action of this vector field is usually defined as

\[
\partial_X f(v) = \left. \frac{d}{dt} \right|_{t=0} f(v + t X), \quad v \in U.
\]

It then holds that

\[
\partial_{\alpha X + \beta Y} = \alpha \partial_X + \beta \partial_Y, \quad \alpha, \beta \in \mathbb{R}, \ X, Y \in V
\]

and

\[
\partial_X (af + bg) = a \partial_X f + b \partial_X g, \quad a, b \in \mathbb{R}, \ f, g \in C^\infty(U, Cl), \ X \in V.
\]

Let now \( \{e_i\}_{i=1}^n \) be a normalized basis for \( V \) and denote by \( \partial_i \) the vector field corresponding to \( e_i \). We then have the following definition [3].

**Definition 2.1.** The Dirac operator \( D \) associated with the real non-degenerated quadratic vector space \((V, q)\) is the first order differential operator

\[
D = \sum_{i=1}^n q(e_i)e_i \partial_i
\]

acting on \( C^\infty(U, Cl) \). The coefficients of \( D \) are the generators of \( Cl(V, q) \) acting by pointwise multiplication on \( C^\infty(U, Cl) \). The Laplace operator \( \Delta \) is the second order constant-coefficient differential operator

\[
\Delta = \sum_{i=1}^n q(e_i)\partial_i^2
\]

acting on \( C^\infty(U, Cl) \).

It follows immediately from the definition of \( D \) that

\[
D^2 = \left( \sum_{i=1}^n q(e_i)e_i \partial_i \right)^2 = \sum_{i=1}^n q(e_i)^2 e_i^2 \partial_i^2 = \sum_{i=1}^n q(e_i)\partial_i^2 = \Delta.
\]

Although the definition of both \( D \) and \( \Delta \) depends on a choice of a basis of \( Cl \) the following proposition shows the definitions are in fact independent of the particular choice of a basis [3].

**Proposition 2.2.** Let \( \{e_i\}_{i=1}^n \) and \( \{f_i\}_{i=1}^n \) be two normalized bases of \( V \) such that \( q(e_i) = q(f_i) \) for all \( i = 1, 2, \ldots, n \) and let \( \partial_i \) be the vector field associated to \( e_i \) and \( \widetilde{\partial}_i \) the vector field associated to \( f_i \). Then

\[
\sum_{i=1}^n q(e_i)e_i \partial_i = \sum_{i=1}^n q(f_i)f_i \widetilde{\partial}_i, \quad \sum_{i=1}^n q(e_i)\partial_i^2 = \sum_{i=1}^n q(f_i)\widetilde{\partial}_i^2.
\]
Proof. Let \( A \in O(V,q) \) be such that \( f_i = Ae_i \) for all \( i \). Let \( A = \{a_{j,k}\}_{j,k=1}^n \). Then
\[
f_i = \sum_{j=1}^n a_{i,j}e_j, \quad \sum_{i=1}^n q(f_i)a_{i,j}a_{i,k} = q(e_j)\delta_{j,k} \quad \text{and} \quad \tilde{\partial}_i = \sum_{k=1}^n a_{i,k}\partial_k
\]
where the middle identity follows from the fact that \( A \in O(V,q) \). Thus
\[
\sum_{i=1}^n q(f_i)\tilde{\partial}_i = \sum_{j,k=1}^n \left( \sum_{i=1}^n q(f_i)a_{i,j}a_{i,k} \right) e_j\partial_k = \sum_{j=1}^n q(e_j)e_j\partial_j
\]
and
\[
\sum_{i=1}^n q(f_i)\tilde{\partial}_i^2 = \sum_{j,k=1}^n \left( \sum_{i=1}^n q(f_i)a_{i,j}a_{i,k} \right) \partial_j\partial_k = \sum_{j=1}^n q(e_j)e_j\partial_j^2.
\]
This completes the proof. \( \square \)

Let now \( \{e_\tilde{i}\}_\tilde{i} \) be a basis for \( Cl \) where the multi-index \( \tilde{i} \) satisfies \( 0 \leq |\tilde{i}| < n \) and \( \tilde{i} = (i_1, i_2, \ldots, i_k), \; k = |\tilde{i}| \). We then have the following proposition [3].

**Proposition 2.3.** If \( f = \sum_{\tilde{i}} f_\tilde{i}e_\tilde{i} \) is a solution of \( Df = 0 \) in \( C\infty(U,Cl) \) with each \( f_\tilde{i} \) real-valued then \( \triangle f_\tilde{i} = 0 \).

**Proof.** Using (9) we get that
\[
0 = D0 = D^2f = \nabla f = \sum_{\tilde{i}} (\triangle f_\tilde{i})e_\tilde{i},
\]
Since \( \{e_\tilde{i}\}_\tilde{i} \) be a basis for \( Cl \) it must hold that \( \triangle f_\tilde{i} = 0 \). \( \square \)

We now consider how the Dirac and Laplace operator act from the left or the right. Since the Laplace operator has scalar coefficients we have that
\[
\triangle f = f \nabla.
\]
The Dirac operator on the other hand has coefficients that are non-scalar elements of \( Cl \). Therefore we have to distinguish between left and right action. Suppose we have a function \( f \in C\infty(U,Cl) \). As in Proposition 2.3 we can then write
\[
f(x) = \sum_{\tilde{i}} f_\tilde{i}(x)e_\tilde{i}, \quad x \in U,
\]
where the “coordinate” functions are scalar valued. We then have
\[
Df = \sum_{j=1}^n \sum_{\tilde{i}} q(e_j)e_\tilde{i}\partial_j f_\tilde{i} \quad \text{and} \quad fD = \sum_{j=1}^n \sum_{\tilde{i}} q(e_j)e_\tilde{i}\partial_j f_\tilde{i}.
\]
We thus have the following definition [1].

**Definition 2.4.** A function \( f \in C\infty(U,Cl) \) is called **left-monogenic** if \( Df = 0 \) and **right-monogenic** if \( fD = 0 \). A function is called **monogenic** if it is both left- and right-monogenic.

Sometimes the terms used are left- and right- Clifford analytic [3].
We finish by returning to the case of quaternions.

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Example 2.5. In Example 1.8 we showed that the function \( f(w) = \frac{w}{|w|^4} \) is regular. By proposition 2.3 its coordinate functions are harmonic. We will now check this via a direct computation. Let

\[
f_1(t, x, y, z) = \frac{t}{(t^2 + x^2 + y^2 + z^2)^2}.
\]

We then have

\[
\begin{align*}
\frac{\partial^2 f_1}{\partial t^2}(t, x, y, z) &= \frac{\partial}{\partial t} \left( \frac{-3t^2 + x^2 + y^2 + z^2}{(t^2 + x^2 + y^2 + z^2)^3} \right) = \frac{-12t(-t^2 + y^2 + z^2)}{(t^2 + x^2 + y^2 + z^2)^4}, \\
\frac{\partial^2 f_1}{\partial x^2}(t, x, y, z) &= \frac{\partial}{\partial x} \left( \frac{4tx}{(t^2 + x^2 + y^2 + z^2)^3} \right) = \frac{-4t(t^2 - 5x^2 + y^2 + z^2)}{(t^2 + x^2 + y^2 + z^2)^4}, \\
\frac{\partial^2 f_1}{\partial y^2}(t, x, y, z) &= \frac{\partial}{\partial y} \left( \frac{-4ty}{(t^2 + x^2 + y^2 + z^2)^3} \right) = \frac{-4t(t^2 + x^2 + y^2 - 5z^2)}{(t^2 + x^2 + y^2 + z^2)^4}, \\
\frac{\partial^2 f_1}{\partial z^2}(t, x, y, z) &= \frac{\partial}{\partial z} \left( \frac{-4tz}{(t^2 + x^2 + y^2 + z^2)^3} \right) = \frac{-4t(t^2 + x^2 + y^2 + z^2 - 5z^2)}{(t^2 + x^2 + y^2 + z^2)^4}.
\end{align*}
\]

It is now trivial to see that we indeed have \( \Delta f_1 = 0 \). Similar calculations can be done for the other coordinate functions.

References