Anyons in Algebraic Quantum Field Theory (AQFT)

Jakob Yngvason

University of Vienna

NORDITA, Stockholm, March 12, 2019

Topics

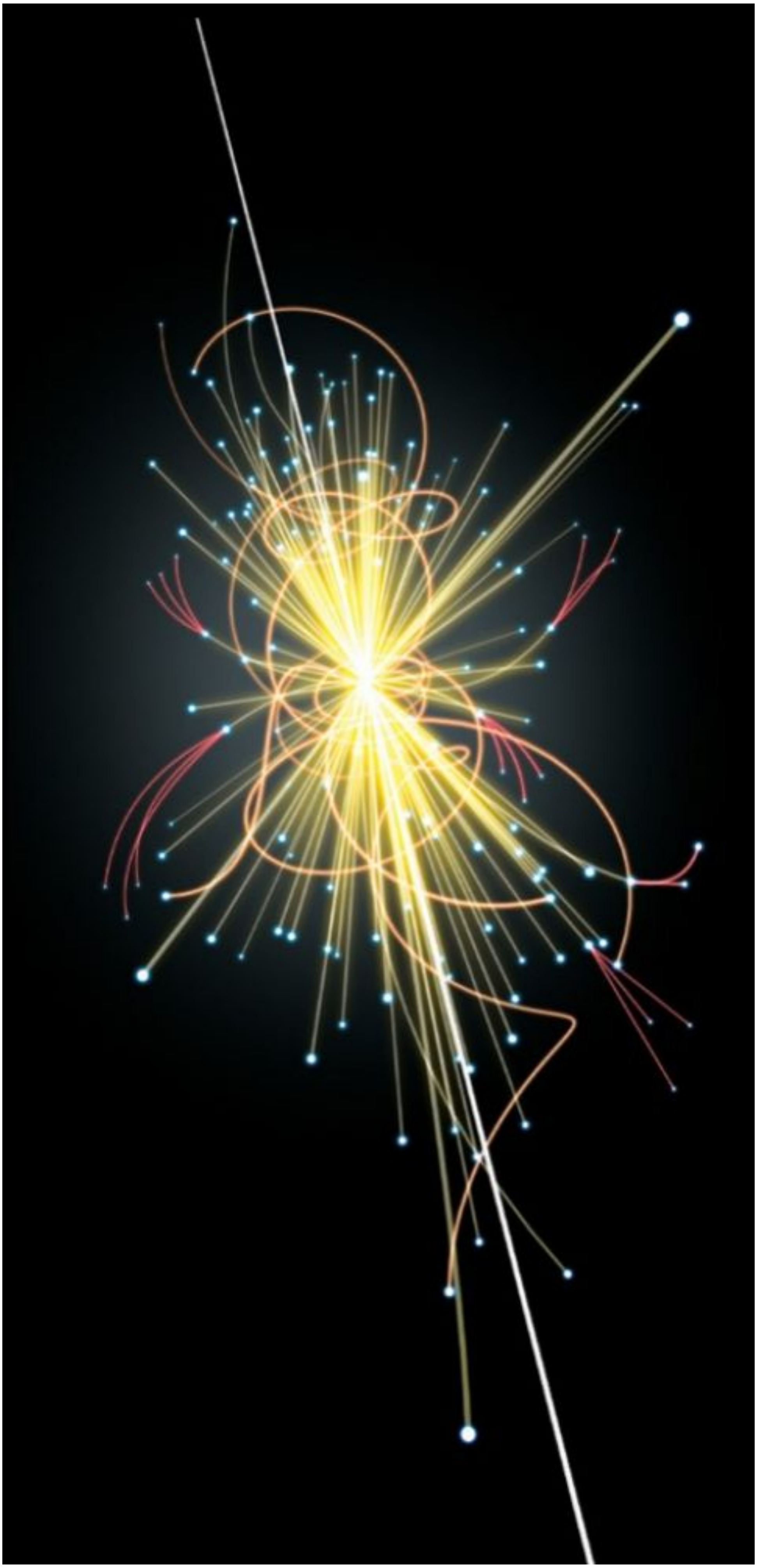
- The general framework of AQFT
- Superselection sectors
- Fields and statistics
- There are no “free” anyon fields
- Scattering theory
- Conclusions

The general framework of AQFT

AQFT, also called *Local Quantum Physics* (monograph of R. Haag, 1992), incorporates basic principles of quantum physics and the (special) theory of relativity in an elegant and mathematically rigorous framework. (R. Haag, D. Kastler, H. Araki, S. Doplicher, J. Roberts; H. Borchers; D. Buchholz; K. Fredenhagen,...)

Important aspect: No position operators for “particles”. Localization by means of **local observables**. Particles emerge **asymptotically** via scattering theory.

In the 80’s and 90’s the framework was adapted to include anyonic statistics in D=2+1 and 1+1 space-time dimensions. (K. Fredenhagen, H. Rehren, B. Schroer, R. Schrader, J. Mund,...; J. Fröhlich, P. Marchetti,...)



Assumptions

Basic object: Net

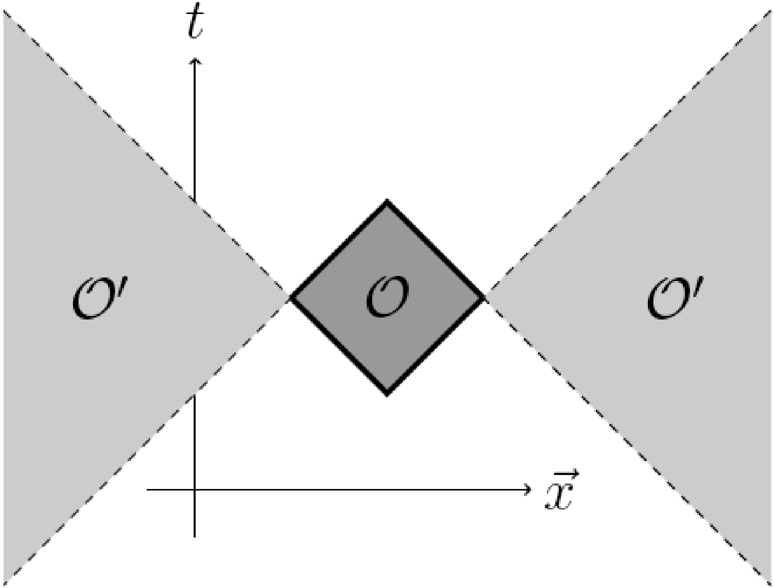
$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

of von Neumann algebras, $\mathcal{O} \subset \mathbb{R}^d$ ("algebras of local observables").

Subalgebras of a global C^*-algebra $\mathcal{A}$ ("quasi-local algebra").

Assumptions:

- ***Isotony:*** $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$
- ***Locality (Micro-causality):*** $\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$
- ***Covariance:*** Automorphisms α_g of $\mathcal{A}$ for $g \in \mathcal{P}_+^\uparrow$ with $\alpha_g \mathcal{A}(\mathcal{O}) = \mathcal{A}(g\mathcal{O})$



t
$\vec{x}$
$\mathcal{O}'$
$\mathcal{O}$
$\mathcal{O}'$

Assumptions (cont.)

- ***Vacuum representation:*** Irreducible representation π_0 of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_0$, and a unitary representation $U_0(x) = \exp(\mathrm{i} \sum_\mu P^\mu x_\mu)$ of the translation group $\mathbb{R}^d$ implementing α_x such that
  - (i) spec $\{P^\mu\} \subset \overline{V_+}$
  - (ii) There exists a translationally invariant $\Omega_0 \in \mathcal{H}_0$ (vacuum).
- ***Haag duality:*** $\pi_0(\mathcal{A}(\mathcal{O}))' = \pi_0(\mathcal{A}(\mathcal{O}'))$ for a suitable class of $\mathcal{O} \subset \mathbb{R}^d$.

Superselection sectors; spectrum condition

Besides the vacuum representation there are in general other unitarily **inequivalent** representations π **(superselection sectors)** of $\mathcal{A}$ of interest.

- **DHR (Doplicher-Haag-Roberts) criterion:** There is a bounded $\mathcal{O}$ such that π is unitarily equivalent to the vacuum representation π_0 when restricted to $\mathcal{A}(\mathcal{O}')$.
- **B (Borchers) criterion:** Space-time translations α_x are unitarily implemented on $\mathcal{H}_\pi$ and spec $\{P^\mu\} \subset \overline{V_+}$. **(Spectrum condition.)**

It can be shown that DHR implies B, but the converse may not hold in general.

Important: The spectrum condition implies im particular that $x \mapsto U_\pi(x)\psi$, $\psi \in \mathcal{H}_\pi$, can be extended to a holomorphic function on $\mathbb{R}^d + iV_+$. Crucial for many results!

Localization in spacelike cones

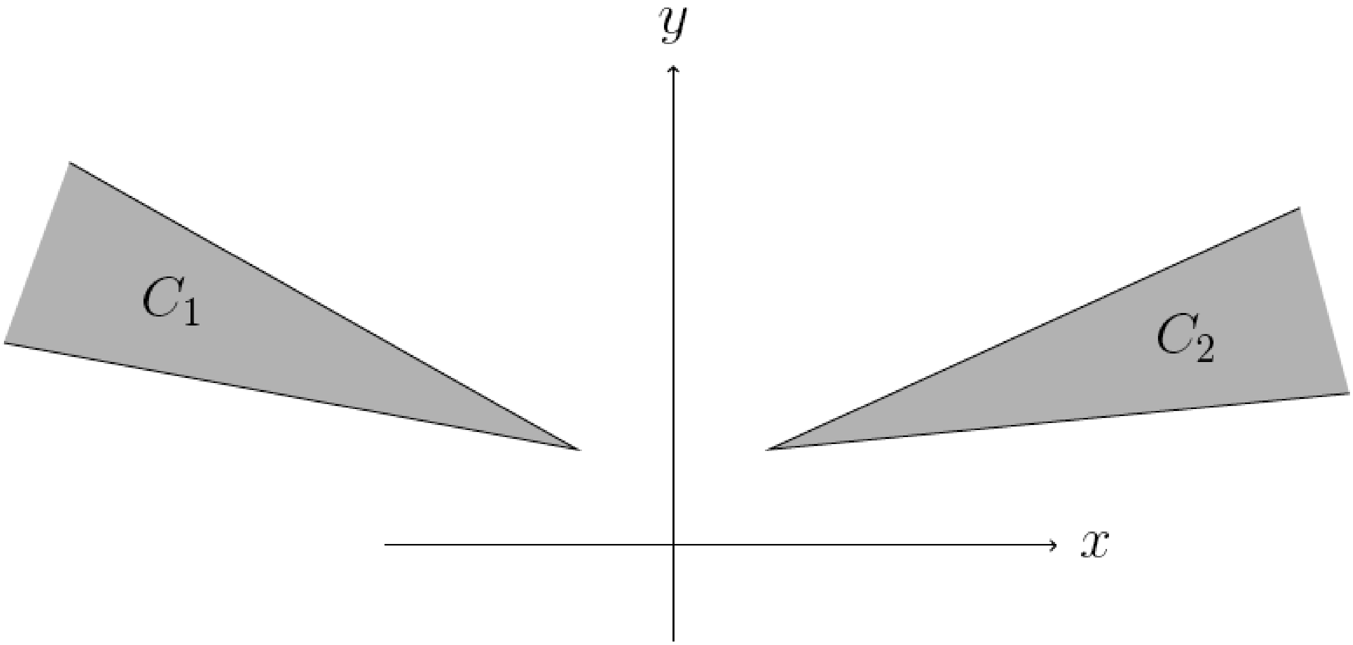
Special class of representations fulfilling B:

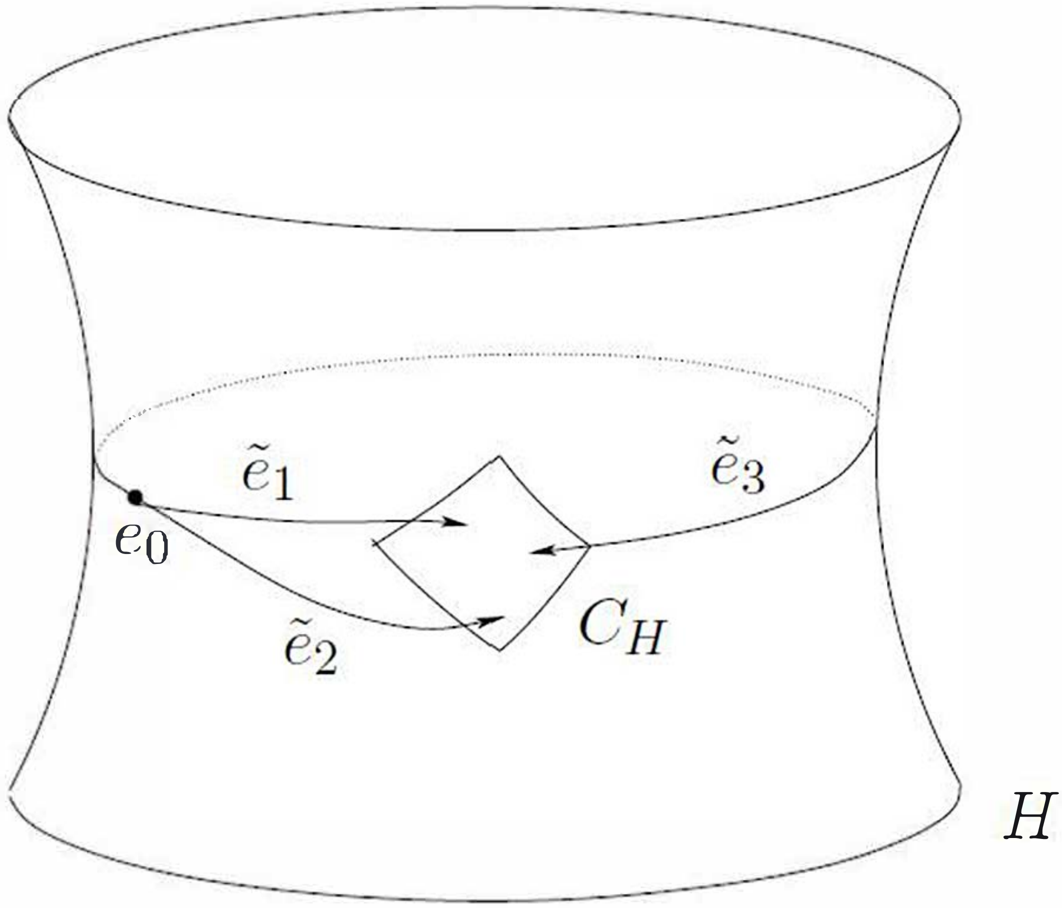
Massive particle representations: Translation group is unitarily implemented and the lower boundary of of spec $\{P^\mu\}$ is an isolated mass shell.

By a theorem of Buchholz and Fredenhagen (1982) it is always possible to localize massive particle representations in **spacelike cones**. These are regions of the form

$$\mathcal{C} = a + \bigcup_{\lambda>0} \lambda C^{\mathrm{H}}$$

with $a \in \mathbb{R}^d$, C^{H} a double cone in the spacelike hyperboloid $x \cdot x = -1$. Localization means here, by definition, that π is unitarily equivalent to π_0 when restricted to $\mathcal{A}(\mathcal{C}')$.


y
x
C1
C2



$\tilde{e}_1$
$\tilde{e}_3$
e_0
C_H
$\tilde{e}_2$
H

Charges; statistics

Parameters labelling (equivalence classes of) of superselection sectors are often called "charges". In the DHR situation (localization in bounded regions) one speaks of simple charges. If cone-localization is the tightest possible one speaks of gauge- or topological charges.

On the set Δ of (equivalence classes of) superselection sectors there is an intrinsic definition (based on locality, Haag duality, and covariance) of composition of charges: $(\pi, \sigma) \to \pi \times \sigma$.

This leads to the definition of statistic operators, which are unitary operators

$$\varepsilon_{\pi\sigma} : \mathcal{H}_{\pi\times\sigma} \to \mathcal{H}_{\sigma\times\pi}$$

The family of statistic operators satisfies the relations of the braid group. In $D \geq 3+1$ this leads to Bose- or Fermi representations of the permutation group as the only possibilities.

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.

Statistics (cont.)

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.

2. Let π be localized in some $\mathcal{C}$. There there is a unitary U such that $\pi(A) = UAU^{-1}$ for $A \in \mathcal{A}(\mathcal{C}')$.

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.

2. Let π be localized in some $\mathcal{C}$. There there is a unitary U such that $\pi(A) = UAU^{-1}$ for $A \in \mathcal{A}(\mathcal{C}')$.

3. Define a localized morphism on $\mathcal{A}$ by $\rho(A) := UAU^{-1}$ for all A.

Statistics (cont.)

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.

2. Let π be localized in some $\mathcal{C}$. There there is a unitary U such that $\pi(A) = UAU^{-1}$ for $A \in \mathcal{A}(\mathcal{C}')$.

3. Define a localized morphism on $\mathcal{A}$ by $\rho(A) := UAU^{-1}$ for all A.

4. If $\mathcal{C}_1 \supset \mathcal{C}$ and $A \in \mathcal{A}(\mathcal{C}_1)$, $B \in A(\mathcal{C}_1') \subset \mathcal{C}'$, then $B = UBU^{-1}$ and hence $[\rho(A), B] = U[A, B]U^{-1} = 0$. Thus $\rho(A) \in \mathcal{A}(\mathcal{C}_1')' \stackrel{!}{=} \mathcal{A}(\mathcal{C}_1)$ by Haag duality. Hence ρ is an endomorphism of $\mathcal{A}$.

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.

2. Let π be localized in some $\mathcal{C}$. There there is a unitary U such that $\pi(A) = UAU^{-1}$ for $A \in \mathcal{A}(\mathcal{C}')$.

3. Define a localized morphism on $\mathcal{A}$ by $\rho(A) := UAU^{-1}$ for all A.

4. If $\mathcal{C}_1 \supset \mathcal{C}$ and $A \in \mathcal{A}(\mathcal{C}_1)$, $B \in A(\mathcal{C}_1') \subset \mathcal{C}'$, then $B = UBU^{-1}$ and hence $[\rho(A), B] = U[A, B]U^{-1} = 0$. Thus $\rho(A) \in \mathcal{A}(\mathcal{C}_1')' \stackrel{!}{=} \mathcal{A}(\mathcal{C}_1)$ by Haag duality. Hence ρ is an endomorphism of $\mathcal{A}$.

5. If π_1 corresponds to ρ_1 and π_2 to ρ_2 then $\pi_1 \times \pi_2$ is, by definition, given by $\rho_1\rho_2$.

Statistics (cont.)

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.

2. Let π be localized in some $\mathcal{C}$. There there is a unitary U such that $\pi(A) = UAU^{-1}$ for $A \in \mathcal{A}(\mathcal{C}')$.

3. Define a localized morphism on $\mathcal{A}$ by $\rho(A) := UAU^{-1}$ for all A.

4. If $\mathcal{C}_1 \supset \mathcal{C}$ and $A \in \mathcal{A}(\mathcal{C}_1)$, $B \in A(\mathcal{C}_1') \subset \mathcal{C}'$, then $B = UBU^{-1}$ and hence $[\rho(A), B] = U[A, B]U^{-1} = 0$. Thus $\rho(A) \in \mathcal{A}(\mathcal{C}_1')' \stackrel{!}{=} \mathcal{A}(\mathcal{C}_1)$ by Haag duality. Hence ρ is an endomorphism of $\mathcal{A}$.

5. If π_1 corresponds to ρ_1 and π_2 to ρ_2 then $\pi_1 \times \pi_2$ is, by definition, given by $\rho_1\rho_2$.

6. If the ρ_i are localized in spacelike separated regions (achieved by means of "charge transporters" $U_i \in \mathcal{A}$) the statistics operator $\varepsilon_{12} = (U_2\rho_2(U_1))^*(U_1\rho_1(U_2))$ intertwines between $\rho_1\rho_2$ and $\rho_2\rho_1$.

Statistics (cont.)

Sketch of the composition of charges

1. Represent the algebra on the vacuum Hilbert space, i.e., identify A and $\pi_0(A)$.
2. Let π be localized in some $\mathcal{C}$. There there is a unitary U such that $\pi(A) = UAU^{-1}$ for $A \in \mathcal{A}(\mathcal{C}')$.
3. Define a localized morphism on $\mathcal{A}$ by $\rho(A) := UAU^{-1}$ for all A.
4. If $\mathcal{C}_1 \supset \mathcal{C}$ and $A \in \mathcal{A}(\mathcal{C}_1)$, $B \in A(\mathcal{C}_1') \subset \mathcal{C}'$, then $B = UBU^{-1}$ and hence $[\rho(A), B] = U[A, B]U^{-1} = 0$. Thus $\rho(A) \in \mathcal{A}(\mathcal{C}_1')' \stackrel{!}{=} \mathcal{A}(\mathcal{C}_1)$ by Haag duality. Hence ρ is an endomorphism of $\mathcal{A}$.
5. If π_1 corresponds to ρ_1 and π_2 to ρ_2 then $\pi_1 \times \pi_2$ is, by definition, given by $\rho_1\rho_2$.
6. If the ρ_i are localized in spacelike separated regions (achieved by means of "charge transporters" $U_i \in \mathcal{A}$) the statistics operator $\varepsilon_{12} = (U_2\rho_2(U_1))^*(U_1\rho_1(U_2))$ intertwines between $\rho_1\rho_2$ and $\rho_2\rho_1$.

Statistics (cont.)

For simple charges the Bose-Fermi alternative holds in any dimension $\geq 2+1$.

In $d = 2+1$ genuine representations (abelian or non-abelian) of the braid group are theoretically possible for cone-localized charges.

For charges with braid group statistics the term **"plektons"** (from Greek πλεκτος, "woven") is sometimes used.

Remarks:

1. In the simplest situations $\varepsilon_{\sigma\pi}$ is a phase factor (abelian anyons) with $\varepsilon_{\pi\pi} = e^{-2\pi \mathrm{i} s}$ and ± 1 corresponding to Bose or Fermi statistics.

2. "Statistics" is a property of superselection sectors, not primarily of "particles".

3. The statistics shows up in the commutation relations of charge generating fields and symmetries of scattering states.

Fields

From now on we shall be concerned with a local net in d=2+1 and its superselection sectors Δ.

The results on compositions of sectors and localization of charges can be stated in terms of field operators:

For every $\pi \in \Delta$ there is a linear space $\mathcal{F}_\pi$ of charge generating operators on the Hilbert space bundle $\mathcal{H} = \bigcup_{\sigma\in\Delta} \mathcal{H}_\sigma$ such that

$$\mathcal{F}_\pi : \mathcal{H}_\sigma \to \mathcal{H}_{\sigma\times\pi}.$$

The fields are localized in space-like cones, in the sense that for every $F \in \mathcal{F}_\pi$ there is a space-like cone $\mathcal{C}$ such that for all observables $A \in \mathcal{A}(\mathcal{C}')$

$$F\sigma(A) = (\sigma \times \pi)(A)F.$$

Paths of cones

In the case of anyonic statistics a refinement of the localization concept for fields is needed.

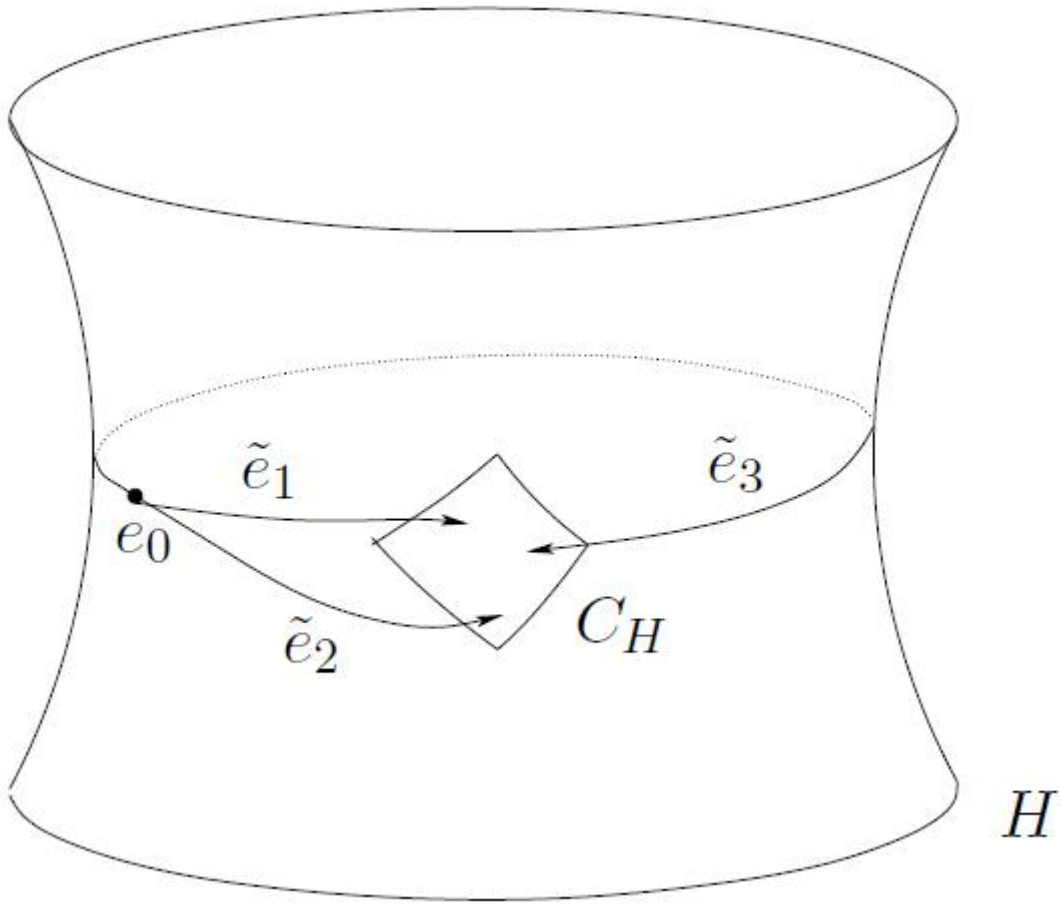
A spacelike cone $\mathcal{C} \subset \mathbb{R}^{3+1}$ corresponds by definition to a subset of $\mathbb{R}^3 \times H$:

$$\mathcal{C} \leftrightarrow \{a\} \times C^H$$

with a the apex and C^H a double cone in the hyperboloid H.

Consider a "lift" of $\mathcal{C}$ to a family of cones in the universal covering space of $\mathbb{R}^3 \times H$ which contains infinitely many copies ("sheets") of $\mathcal{C}$. Any particular sheet $\tilde{\mathcal{C}}$ (also called generalized cone) is fixed by a homotopy equivalence class $[\tilde{e}]$ of paths $\tilde{e}$ in H that start in some fixed point $e_0 \in H$ and end in C^H:

$$\tilde{\mathcal{C}} \leftrightarrow \mathcal{C} \times [\tilde{e}] \leftrightarrow \{a\} \times C^H \times [\tilde{e}].$$


$\tilde{e}_1$
$\tilde{e}_3$
e_0
$\tilde{e}_2$
C_H
H

Localization of fields; Reeh-Schlieder property

We say that $\tilde{\mathcal{C}}_1 \subset \tilde{\mathcal{C}}_2$ if $\mathcal{C}_1 \subset \mathcal{C}_2$ and the paths $\tilde{e}_1$ and $\tilde{e}_2$ differ by a path in C_2. The field operators $\mathcal{F}_{\pi_i}(\tilde{\mathcal{C}}_i)$ localized in $\tilde{\mathcal{C}}_i$, $i = 1, 2$, satisfy

$$\mathcal{F}_{\pi_1}(\tilde{\mathcal{C}}_1) \subset \mathcal{F}_{\pi_2}(\tilde{\mathcal{C}}_2) \text{ if } \tilde{\mathcal{C}}_1 \subset \tilde{\mathcal{C}}_2.$$

Moreover, the **Reeh-Schlieder property**

$$\mathcal{F}_\pi(\tilde{\mathcal{C}})\Omega_0 \text{ is dense in } \mathcal{H}_\pi$$

is required for all $\tilde{\mathcal{C}}$.
This implies in turn that the vacuum Ω_0 is a separating vector for localized operators, i.e., it suffices to test relations for such operators on the vacuum.

Commutation relations for fields

Consider now two sheets, $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$. The relative winding number $N = N(\mathcal{C}_1, \mathcal{C}_2)$ of the paths $\tilde{e}_1$ and $\tilde{e}_2$ is defined in a straightforward way. If C_1 is causally separated from C_2 and $F_i \in \mathcal{F}_{\pi_i}(\tilde{\mathcal{C}}_i)$ then the "twisted commutator" vanishes:

$$F_1 F_2 - R_{12}\, F_2 F_1 = 0$$

with

$$R_{12} = (\varepsilon_{\pi_1,\pi_2}\varepsilon_{\pi_2,\pi_1})^N \varepsilon_{\pi_1,\pi_2}$$

Simplest case:

One species of abelian anyons with spin s:

$$R_{12} = e^{-2\pi \mathrm{i} s(2N+1)}$$

Free fields with B or F statistics

In every space-time dimension there are infinitely many models with Bose- or Fermi statistics fulfilling all the assumptions. These are frequently (but not always!) defined in terms of fields which are operator valued distributions $\Phi_\alpha(x)$; observable algebras $\mathcal{A}(\mathcal{O})$ and field algebras $\mathcal{F}(\mathcal{O})$ are then generated by bounded functions of smeared field operators.

Simplest case: **Free fields** satisfying a linear equation (KG or Dirac equation) and realized on a Fock space. They have the property that their (anti-)commutators are c-numbers:

$$[\Phi_\alpha(x), \Phi_\beta(y)]_\mp = \langle\Omega_0|[\Phi_\alpha(x), \Phi_\beta(y)]_\mp|\Omega_0\rangle \mathbf{1}.$$

Free fields are often the staring point for the construction (perturbative or non-perturbative!) of models with interactions (nontrivial scattering).

Absence of free fields for anyons

In the case of anyonic statistics it might appear natural to generalize the free field concept by requiring all twisted commutators

$$F_1 F_2 - R_{12}\, F_2 F_1$$

of a generating set of F_i's to be c-numbers, provided at least the asymptotic directions of the localization cones are causally separated. This possibility is excluded for genuine anyons, however:

Theorem (J. Mund, 2002)

If the twisted commutators are c-numbers, then $R_{12} = \pm 1$.

Mund starts, in fact, from a different hypothesis, namely the existence of cone-localized operators that create only one-particle states when applied to the vacuum ("polarization-free generators"). He shows that this implies the c-number property and then that the latter implies $R_{12} = \pm 1$.

Braid group statistics implies scattering

A considerable strengthening of this result has been proved by Mund and J. Bros:

Theorem (J. Bros, J. Mund,, 2012)

Let π be a massive single particle representation with abelian anyon statistics. Then for any non-empty open sets of energy momenta U_1, U_2, V_1, V_2 admitted by energy-momentum conservation there is elastic two-particle scattering from $U_1 \times U_2$ to $V_1 \times V_2$ in the channel $\pi \times \pi \to \pi \times \pi$.

In the case of non-abelian anyons the same holds for the channel $\bar{\pi} \times \pi \to \bar{\pi} \times \pi$ where $\bar{\pi}$ is the conjugate sector.

The framework for the theorem is Haag-Ruelle scattering theory, generalized to accomodate anyons by Fredenhagen, Gaberdiel and Rüger in 1996.

Scattering in non-relativistic QM

Consider first non-relativistic potential scattering theory of n identical bosons with free Hamiltonian H_0 and interacting Hamiltonian $H = H_0 + V$ with V defined by a short range interaction potential (square integrability suffices). Then, for given 1-particle states $\psi_1, \dots, \psi_n$ there are state vectors, denoted

$$\psi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \psi_n \quad \text{and} \quad \psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n,$$

such that

$$\lim_{t \to \pm\infty} \| e^{-\mathrm{i}tH} (\psi_1 \overset{\text{ex}}{\times} \cdots \overset{\text{ex}}{\times} \psi_n) - e^{-\mathrm{i}tH_0} (\psi_1 \otimes_s \cdots \otimes_s \psi_n) \| = 0$$

i.e.,

$$\psi_1 \overset{\text{ex}}{\times} \cdots \overset{\text{ex}}{\times} \psi_n = \lim_{t \to \pm\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} (\psi_1 \otimes_s \cdots \otimes_s \psi_n).$$

In words: For large positive or negative times the scattering states look more and more like states of freely moving particles.

Scattering in RQFT

Challenges in RQFT:

- No splitting of the Hamiltonian in a "free" part and an interaction part.
- $\psi_1 \otimes_s \cdots \otimes_s \psi_n$ is in general not defined in the state space (except for free fields).

However, **1-particle states are well defined** (via the representation of the Poincaré group); they can be described by wave functions in momentum space with time evolution $\psi(\mathbf{p}) \to e^{-\mathrm{i}\omega(\mathbf{p})t}\psi(\mathbf{p})$ where $\omega(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}$.

A substitute for $e^{-\mathrm{i}tH} e^{-\mathrm{i}tH_0}(\psi_1 \otimes_s \cdots \otimes_s \psi_n)$ is obtained by means of almost local one-particle creators. These are obtained from field operators localized in generalized cones $\tilde{C}$ by cutting smoothly in momentum space around the mass shell.

Scattering in RQFT (cont.)

More precisely, such operators have the form

$$F(f_t) = \int d^3x \, f_t(x) F(x)$$

where $F(x) = \alpha_x F$ with a field operator F localized in some $\tilde{\mathcal{C}}$, and the Fourier transform of f_t has the form

$$\tilde{f}_t(p) = e^{\mathrm{i}(p^0 - \omega(\mathbf{p}))t} h(p)$$

with a smooth h whose support cuts the energy spectrum only on the mass shell. Then $\psi = F(f_t)\Omega_0$ is a one-particle state, independent of t. For suitable choices of h_i, $i = 1, \dots, n$ and F_i localized in $\tilde{\mathcal{C}}_i$ the state vectors

$$\Psi(t) = F_1(f_{1,t}) \cdots F_1(f_{n,t})\Omega_0$$

are a substitute for $e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0}(\psi_1 \otimes_s \cdots \otimes_s \psi_n)$.

Scattering in RQFT (cont.)

One now writes $\Psi(t) = \int_{-\infty}^{t} \frac{d}{dt'}\Psi(t')dt'$ and uses locality and time independence of $F_i(f_{i,t})\Omega_0 =: \psi_i$ to show that the derivative tends rapidly to zero for $t \to \pm\infty$. This implies that the limits

$$\lim_{t\to\pm\infty} F_1(f_{1,t}) \cdots F_n(f_{n,t})\Omega_0 =: (\psi_1, \tilde{\mathcal{C}}_1) \overset{\text{ex}}{\times} \cdots \overset{\text{ex}}{\times} (\psi_n, \tilde{\mathcal{C}}_n)$$

exist (by the Cauchy criterion).

The scattering states inherit the braid group symmetries from the fields generating them. They can be concretely realized as L_2-spaces of sections of hermitian vector bundles over products of the mass shells with the diagonal removed.

The theorem of Bros and Mund is proved by using LSZ reduction formulas to analyze scattering marix elements

$$\langle (\psi_1, \tilde{\mathcal{C}}_1) \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} (\psi_n, \tilde{\mathcal{C}}_n) | (\phi_1, \tilde{\mathcal{D}}_1) \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} (\phi_n, \tilde{\mathcal{D}}_n)\rangle.$$

Conclusions

The subtle interplay between quantum mechanics, localization in space-time (relativistic causality!) and spectrum condition in energy-momentum space (stability!), expressed mathematically in the framework of AQFT, leads to the concepts of

- superselection sectors
- statistics
- scattering

and fundamental properties like

- Bose-Fermi statistics as the only alternatives in $d \geq 3+1$
- possibility of braid group statistics in $d \leq 2+1$
- Non-trivial scattering as a consequence of braid group statistics

References

S. Doplicher, J.R. Roberts, *Why There is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics*, Commun. Math. Phys. **131**, 51–107 (1990)

K. Fredenhagen, M.R. Gaberdiel, S.M. Rüger, *Scattering States of Plektons (Particles with Braid Group Statistics) in 2 + 1 Dimensional Quantum Field Theory*, Commun. Math. Phys. **175**, 319–335 (1996)

J. Bros, J. Mund, *Braid Group Statistics Implies Scattering in Three-Dimensional Local Quantum Physics*, Commun. Math. Phys. **315**, 465–488 (2012)

M. Plaschke, *Local Ayonic Quantum Fields on the Circle Leading to Cone-Local Anyons in Two Dimensions*, Lett. Math. Phys. **105**, 1033–1055 (2015)