Emergent gauge fields and anyons in quantum impurity problems

Enderalp Yakaboylu*

Institute of Science and Technology Austria

*enderalp.yakaboylu@ist.ac.at

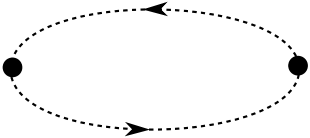
11-16 March 2019

Nordita, Stockholm

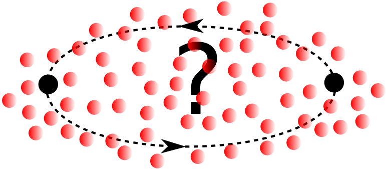
Motivation

- Consider two non-interacting bosons. Spin-statistics theorem says

$$|\psi_1\psi_2\rangle = |\psi_2\psi_1\rangle$$



- Would the statistics be the same when we immerse these two bosons in a 2D many-particle bath?



Overview

1. Quantum Impurity Problems and Emergent Gauge Fields
   - Polaron and angulon
   - Emergence of gauge fields
   - Non-Abelian magnetic monopole
2. Realization of Anyons
   - Spin-statistics theorem and gauge fields
   - Two-impurity problem
3. N-anyon Problem
   - CS from bath
   - CS from impurity: Spin-boson model
4. Symmetries of Quantum Impurities
   - Quantum groups
   - Renormalization of B
5. Conclusion

1. Quantum Impurity Problems and Emergent Gauge Fields
   - Polaron and angulon
   - Emergence of gauge fields
   - Non-Abelian magnetic monopole
2. Realization of Anyons
   - Spin-statistics theorem and gauge fields
   - Two-impurity problem
3. N-anyon Problem
   - CS from bath
   - CS from impurity: Spin-boson model
4. Symmetries of Quantum Impurities
   - Quantum groups
   - Renormalization of B
5. Conclusion

Polaron

- Consider an electron immersed in a lattice

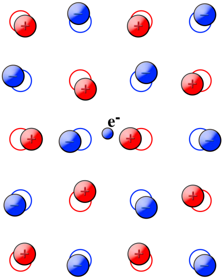


Figure: From Devreese

- Complicated many-body interaction

Polaron

- Consider an electron immersed in a lattice

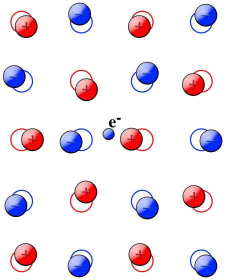


Figure: From Devreese

- Complicated many-body interaction
- Can be simplified within the quasiparticle picture

$$q, m \to q^*, m^*$$

- Polaron: electron dressed by lattice excitation
  Landau, Pekar, Fröhlich, Bogoliubov, Feynman, Holstein, Devreese, Lieb, ..

Fröhlich Hamiltonian : weakly interacting bath

- The Hamiltonian

$$\hat{H}_{\rm pol} = \frac{\boldsymbol{P}^2}{2m} + \sum_{\boldsymbol{k}} \omega(k) \hat{b}^\dagger_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}} + \sum_{\boldsymbol{k}} V(k) \left(e^{-i\boldsymbol{k}\cdot\hat{\boldsymbol{x}}} \hat{b}^\dagger_{\boldsymbol{k}} + e^{i\boldsymbol{k}\cdot\hat{\boldsymbol{x}}} \hat{b}_{\boldsymbol{k}} \right)$$

- Conservation of linear momentum

$$[\hat{\boldsymbol{\Pi}}, \hat{H}_{\rm pol}] = 0 \,, \quad \hat{\boldsymbol{\Pi}} = \hat{\boldsymbol{P}} + \sum_{\boldsymbol{k}} \boldsymbol{k} \hat{b}^\dagger_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}}$$

- Variational approach: $|\Psi_{\boldsymbol{p}}\rangle = \sqrt{Z} |\boldsymbol{p}\rangle |0\rangle + \sum_{\boldsymbol{k}} \beta(\boldsymbol{k}) |\boldsymbol{p}-\boldsymbol{k}\rangle \hat{b}^\dagger_{\boldsymbol{k}} |0\rangle$

$$E = \frac{\boldsymbol{p}^2}{2m} - \underbrace{\sum_{\boldsymbol{k}} \frac{V(k)^2}{(\boldsymbol{p}-\boldsymbol{k})^2/(2m) + \omega(k) - E - i0^+}}_{\text{self-energy - summation of one-phonon diagrams}}$$

- Pekar ansatz: $|\Psi\rangle = |\varphi\rangle |\xi\rangle$

$$\varepsilon[\varphi] = \int d^3x \, |\nabla\varphi|^2 - \alpha \int d^3x \, d^3y \, |\varphi(\boldsymbol{x})|^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} |\varphi(\boldsymbol{y})|^2$$

Angulon

- Angulon – a quantum rotor dressed by bosonic field excitations.
  R. Schmidt and M. Lemeshko, PRL 114, 203001 (2015)

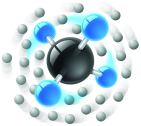


Figure: From Lemeshko

- The angulon Hamiltonian

$$\hat{H}_{\text{ang}} = B\hat{\boldsymbol{L}}^2 + \sum_{k\lambda\mu} \omega(k) \hat{b}^\dagger_{k\lambda\mu} \hat{b}_{k\lambda\mu} + \overbrace{\sum_{k\lambda\mu} U_\lambda(k) \left[Y^*_{\lambda\mu}(\hat{\theta},\hat{\phi}) \hat{b}^\dagger_{k\lambda\mu} + Y_{\lambda\mu}(\hat{\theta},\hat{\phi}) \hat{b}_{k\lambda\mu} \right]}^{\text{interaction}}$$

- Conservation of angular momentum

$$[\hat{\boldsymbol{J}}^2, \hat{H}_{\text{ang}}] = [\hat{J}_z, \hat{H}_{\text{ang}}] = 0\,, \quad \hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \overbrace{\sum_{k\lambda\mu\nu} \boldsymbol{\sigma}^\lambda_{\mu\nu} \hat{b}^\dagger_{k\lambda\mu} \hat{b}_{k\lambda\nu}}^{\hat{\boldsymbol{\Lambda}}}$$

- The variational state

$$|\psi_{LM}\rangle = \sqrt{Z}|LM\rangle|0\rangle + \sum_{k\lambda\mu jm} \beta_{\lambda j}(k) C^{LM}_{jm\lambda\mu} |jm\rangle \hat{b}^\dagger_{k\lambda\mu}|0\rangle$$

Emergence of gauge fields

- The general Hamiltonian:

$$\hat{H} = -\mu\,\nabla^2 + \hat{H}_{\text{mb}}(\boldsymbol{r};\varphi) \quad ; \quad \hat{H}\langle \boldsymbol{r}|\Psi_E\rangle = E\langle \boldsymbol{r}|\Psi_E\rangle$$

Emergence of gauge fields

- The general Hamiltonian:

$$\hat{H} = -\mu\,\nabla^2 + \hat{H}_{\text{mb}}(\boldsymbol{r};\varphi) \quad ; \quad \hat{H}\langle \boldsymbol{r}|\Psi_E\rangle = E\langle \boldsymbol{r}|\Psi_E\rangle$$

- Coordinate of the impurity, $\boldsymbol{r}$, is an external parameter in the many-body Hamiltonian: $\hat{H}_{\text{mb}}(\boldsymbol{r};\varphi)|\varphi_n(\boldsymbol{r})\rangle = \varepsilon_n(\boldsymbol{r})|\varphi_n(\boldsymbol{r})\rangle$.

- The state can be expanded as $\langle \boldsymbol{r}|\Psi_E\rangle = \sum_n \Phi_n^E(\boldsymbol{r})|\varphi_n(\boldsymbol{r})\rangle$

- The eigenvalue equation for the impurity:

$$\sum_n H^{\text{eff}}_{nm}\Phi^E_m(\boldsymbol{r}) = E\Phi^E_n(\boldsymbol{r})$$

with the effective impurity Hamiltonian

$$H^{\text{eff}}_{nm} = -\mu\sum_l \left[\delta_{nl}\nabla + \langle\varphi_n|\nabla|\varphi_l\rangle\right]\cdot\left[\delta_{lm}\nabla + \langle\varphi_l|\nabla|\varphi_m\rangle\right] + \varepsilon_n\delta_{nm}$$

Emergence of gauge fields - restriction of the basis vectors

The non-Abelian gauge field:

$$\boldsymbol{A}_{mn}(\boldsymbol{r}) = \langle \varphi_m(\boldsymbol{r}) | i\nabla | \varphi_n(\boldsymbol{r}) \rangle \tag{1}$$

- If $\boldsymbol{A}_{nn} \gg \boldsymbol{A}_{mn}$, we can neglect off-diagonal terms:

  $$\langle \boldsymbol{r} | \Psi^E \rangle \approx \Phi_n^E(\boldsymbol{r}) | \varphi_n(\boldsymbol{r}) \rangle \quad \rightarrow \quad \text{U(1) gauge field,}$$

  e.g., adiabatic approximation for a non-degenerate state.

- If

  $$\langle \boldsymbol{r} | \Psi^E \rangle \approx \sum_n^N \Phi_n^E(\boldsymbol{r}) | \varphi_n(\boldsymbol{r}) \rangle \quad \rightarrow \quad \text{U(N) gauge field,}$$

  e.g., adiabatic approximation for a degenerate state.

Emergence of gauge fields - restriction of the basis vectors

The non-Abelian gauge field:

$$\boldsymbol{A}_{mn}(\boldsymbol{r}) = \langle \varphi_m(\boldsymbol{r}) | i\nabla | \varphi_n(\boldsymbol{r}) \rangle \tag{1}$$

- If $\boldsymbol{A}_{nn} \gg \boldsymbol{A}_{mn}$, we can neglect off-diagonal terms:

  $$\langle \boldsymbol{r} | \Psi^E \rangle \approx \Phi_n^E(\boldsymbol{r}) | \varphi_n(\boldsymbol{r}) \rangle \quad \rightarrow \quad \text{U(1) gauge field,}$$

  e.g., adiabatic approximation for a non-degenerate state.

- If

  $$\langle \boldsymbol{r} | \Psi^E \rangle \approx \sum_n^N \Phi_n^E(\boldsymbol{r}) | \varphi_n(\boldsymbol{r}) \rangle \quad \rightarrow \quad \text{U(N) gauge field,}$$

  e.g., adiabatic approximation for a degenerate state.

- We truncate the number of basis vectors by a variational state.

Non-Abelian magnetic monopole

EY, A. Deuchert, and M. Lemeshko, PRL 119, 235301 (2017)

- Go back to the angulon Hamiltonian

$$\hat{H}_{\text{ang}} = B\hat{\boldsymbol{L}}^2 + \sum_{k\lambda\mu}\omega(k)\hat{b}^\dagger_{k\lambda\mu}\hat{b}_{k\lambda\mu} + \sum_{k\lambda\mu}U_\lambda(k)\left[Y^*_{\lambda\mu}(\hat{\theta},\hat{\phi})\hat{b}^\dagger_{k\lambda\mu} + Y_{\lambda\mu}(\hat{\theta},\hat{\phi})\hat{b}_{k\lambda\mu}\right]$$

- In the co-rotating frame

$$\hat{S} = e^{-i\hat{\phi}\otimes\hat{\Lambda}_z}\ e^{-i\hat{\theta}\otimes\hat{\Lambda}_y}\ e^{-i\hat{\phi}\otimes\hat{\Lambda}_z}$$

  the Hamiltonian

$$\hat{H}'_{\text{ang}} = \hat{S}^{-1}\hat{H}_{\text{ang}}\hat{S} = B(\hat{\boldsymbol{L}}' - \boldsymbol{\Lambda})^2 + \sum_{k\lambda\mu}\omega(k)\hat{b}^\dagger_{k\lambda\mu}\hat{b}_{k\lambda\mu} + \sum_{k\lambda}V_\lambda(k)\left[\hat{b}^\dagger_{k\lambda0} + \hat{b}_{k\lambda0}\right]$$

- Observe that

$$[\hat{\boldsymbol{L}}^2, \hat{H}'_{\text{ang}}] = [\hat{L}_z, \hat{H}'_{\text{ang}}] = 0 \quad \text{but} \quad [\hat{L}'_z, \hat{H}'_{\text{ang}}] \neq 0$$

- Variational state

$$|\Psi'_{LM}\rangle = g_0|\overbrace{L}^{\hat{\boldsymbol{L}}^2}\overbrace{M}^{\hat{L}_z}\overbrace{0}^{\hat{L}'_z}\rangle|0\rangle + \sum_{k\lambda n}\alpha_{\lambda n}(k)|LMn\rangle\hat{b}^\dagger_{k\lambda n}|0\rangle$$

Variational state

- Truncated state

$$\langle \Omega | \Psi_{LM} \rangle = \sum_{n=1}^{2\lambda_{\max}+1} \Phi_n^{LM}(\Omega) | \varphi_n(\Omega) \rangle$$

- The basis vectors

$$|\varphi_n(\Omega)\rangle = \hat{S}(\Omega) \left(\delta_{n0} g_0 |0\rangle + \sum_{k\lambda} \alpha_{\lambda n}(k) \hat{b}^\dagger_{k\lambda n} |0\rangle \right)$$

- The impurity wave function

$$\Phi_n^{LM}(\Omega) = \langle \Omega | LMn \rangle \quad : \text{spin weighted spherical harmonics}$$

Non-Abelian magnetic monopole

- $U_\lambda(k) = 0$ for $\lambda > 1$, then $U(3)$ gauge field:

$$A_\phi = \begin{pmatrix} -\cot\theta & \frac{-\kappa}{\sqrt{2}} & 0 \\ \frac{-\kappa^*}{\sqrt{2}} & 0 & \frac{-\kappa^*}{\sqrt{2}} \\ 0 & \frac{-\kappa}{\sqrt{2}} & \cot\theta \end{pmatrix}, \; A_\theta = \begin{pmatrix} 0 & \frac{i\kappa}{\sqrt{2}} & 0 \\ \frac{-i\kappa^*}{\sqrt{2}} & 0 & \frac{i\kappa^*}{\sqrt{2}} \\ 0 & \frac{-i\kappa}{\sqrt{2}} & 0 \end{pmatrix}$$

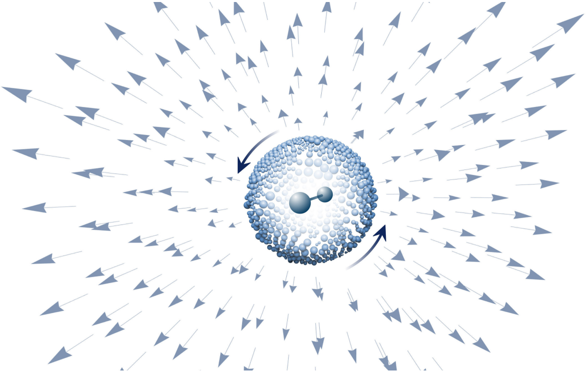
$\kappa \equiv \kappa(\omega, U_\lambda, B)$

- The curvature

$$F_{\phi\theta} = i[D_\phi, D_\theta] = \partial_\phi A_\theta - \partial_\theta A_\phi - i[A_\phi, A_\theta] = (1 - |\kappa|^2)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the strength of a $U(3)$ magnetic monopole with charge $g = 1 - |\kappa|^2$.

Non-Abelian magnetic monopole



Angulon: an impurity interacting with the field of a non-Abelian magnetic monopole.

Topology: Abelianization

- The Chern number

$$c = \frac{1}{2\pi} \oint d\Omega \, \mathrm{Tr}\, F = 0 \quad \text{trivial topology?}$$

Topology: Abelianization

- The Chern number

$$c = \frac{1}{2\pi} \oint d\Omega \; \mathrm{Tr}\, F = 0 \quad \text{trivial topology?}$$

- For $\kappa = 0$, the monopole gauge field becomes 'Abelianized,' i.e.

$$A_\phi = \cot\theta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\theta = 0 \quad \Rightarrow \quad \boldsymbol{A} = \boldsymbol{A}_- \oplus \boldsymbol{A}_0 \oplus \boldsymbol{A}_+$$

- $\boldsymbol{A}_\pm$ is the Dirac monopole field with the charge

$$g_\pm = \pm 1 = c_\pm \quad \rightarrow \quad \text{topological restriction}$$

Topology: Abelianization

- The Chern number

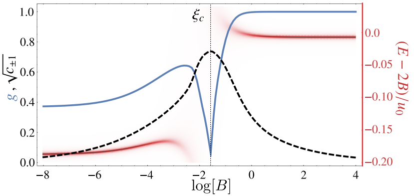
$$c = \frac{1}{2\pi} \oint d\Omega \ \mathrm{Tr}\, F = 0 \quad \text{trivial topology?}$$

- For $\kappa = 0$, the monopole gauge field becomes 'Abelianized,' i.e.

$$A_\phi = \cot\theta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\theta = 0 \quad \Rightarrow \quad \boldsymbol{A} = \boldsymbol{A}_- \oplus \boldsymbol{A}_0 \oplus \boldsymbol{A}_+$$

- $\boldsymbol{A}_\pm$ is the Dirac monopole field with the charge

$$g_\pm = \pm 1 = c_\pm \quad \to \quad \text{topological restriction}$$



The transition from a non-Abelian vector potential to an Abelian vector potential is a topological transition of the underlying vector bundle.

1. Quantum Impurity Problems and Emergent Gauge Fields
   - Polaron and angulon
   - Emergence of gauge fields
   - Non-Abelian magnetic monopole
2. Realization of Anyons
   - Spin-statistics theorem and gauge fields
   - Two-impurity problem
3. N-anyon Problem
   - CS from bath
   - CS from impurity: Spin-boson model
4. Symmetries of Quantum Impurities
   - Quantum groups
   - Renormalization of B
5. Conclusion

Spin-statistics theorem and gauge fields

- Spin-statistics theorem

$$|\psi_1\psi_2\rangle = (-1)^{2s}|\psi_2\psi_1\rangle \,.$$

- Two-body wave function in relative coordinates

$$\psi'(r,\varphi+\pi) = e^{i\xi}\,\psi'(r,\varphi)\,, \tag{2}$$

$$\begin{aligned} \xi = 0 \quad &\Rightarrow \quad \text{bosons} \\ \xi = \pi \quad &\Rightarrow \quad \text{fermions} \end{aligned}$$

Spin-statistics theorem and gauge fields

- Spin-statistics theorem

$$|\psi_1\psi_2\rangle = (-1)^{2s}|\psi_2\psi_1\rangle\,.$$

- Two-body wave function in relative coordinates

$$\psi'(r,\varphi+\pi) = e^{i\xi}\,\psi'(r,\varphi)\,, \tag{2}$$

$$\begin{aligned} \xi = 0 \quad &\Rightarrow \quad \text{bosons} \\ \xi = \pi \quad &\Rightarrow \quad \text{fermions} \end{aligned}$$

- **True only in 3+1** : Poincaré group $\rightarrow$ SO(3) $\rightarrow$ quantized spin

2D

- **In a 2-spatial dimensional world rotation is trivial:**

  Poincaré group $\to$ SO(2) $\to$ NO spin quantization $\to$ *any* statistics : $0 \leq \xi \leq \pi$
  J.M. Leinaas and J. Myrheim, Nuovo Cimento B37, 1 (1977)

2D

- **In a 2-spatial dimensional world rotation is trivial:**

  Poincaré group $\rightarrow$ SO(2) $\rightarrow$ NO spin quantization $\rightarrow$ *any* statistics : $0 \leq \xi \leq \pi$
  J.M. Leinaas and J. Myrheim, Nuovo Cimento B37, 1 (1977)

- Unusual boundary conditions

$$\psi'(r, \varphi + 2\pi) = e^{i2\xi}\, \psi'(r, \varphi) \neq \psi'(r, \varphi)$$

2D

- **In a 2-spatial dimensional world rotation is trivial:**

  Poincaré group $\rightarrow$ SO(2) $\rightarrow$ NO spin quantization $\rightarrow$ *any* statistics : $0 \leq \xi \leq \pi$
  J.M. Leinaas and J. Myrheim, Nuovo Cimento B37, 1 (1977)

- Unusual boundary conditions

$$\psi'(r, \varphi + 2\pi) = e^{i2\xi}\, \psi'(r, \varphi) \neq \psi'(r, \varphi)$$

- Single-valued wave function, $\psi(r, \varphi) = \exp[-2i\xi\varphi/(2\pi)]\psi'(r, \varphi)$, is governed by the Hamiltonian

$$e^{-2i\xi\varphi/(2\pi)}\, \hat{H}' \left\{ \frac{\partial}{\partial\varphi} \right\} e^{2i\xi\varphi/(2\pi)} = \hat{H} \left\{ \frac{\partial}{\partial\varphi} + i\frac{2\xi}{2\pi} \right\} \tag{3}$$

- $A = \dfrac{2\xi}{2\pi}$ is the statistical gauge field:
  anyon = boson/fermion interacting with the statistical gauge field.

Realization of anyon

- A magnetic field can substitute the role of the statistical gauge field.

Realization of anyon

- A magnetic field can substitute the role of the statistical gauge field.
- **Wilczek picture**: Flux-tube-charged-particle composites
- **Chern-Simons picture**: Charged particle coupled to Chern-Simons gauge field

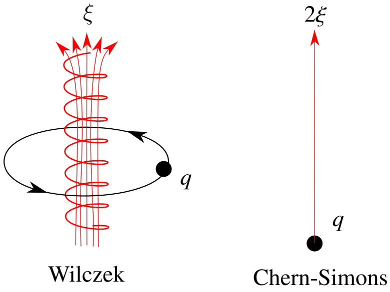


Figure: Realization of anyon

Two-impurity problem

EY and M. Lemeshko, Phys. Rev. B 98, 045402 (2018)

- How to realize anyons in a more realistic problem?

Two-impurity problem

EY and M. Lemeshko, Phys. Rev. B 98, 045402 (2018)

- How to realize anyons in a more realistic problem?

Emergent gauge field

Two-impurity problem

EY and M. Lemeshko, Phys. Rev. B 98, 045402 (2018)

- How to realize anyons in a more realistic problem?

Emergent gauge field

- Consider two non-interacting impurities immersed in a 2D bath

$$\hat{H}_{2\text{imp}} = \frac{1}{2m}\hat{\boldsymbol{P}}_1^2 + \frac{1}{2m}\hat{\boldsymbol{P}}_2^2 + \sum_{\boldsymbol{k}} \omega(k)\hat{b}^\dagger_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}} + \sum_{\boldsymbol{k}} V(k)\left[e^{-i\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_1} + e^{-i\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_2}\right]\hat{b}^\dagger_{\boldsymbol{k}} + \text{H.c.}$$

- In relative coordinates

$$\hat{H}_{\text{rel}} = \frac{1}{m}\hat{L}_z^2 + \sum_{k\mu}\tilde{\omega}(k)\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu} + \sum_{k\mu} Y_\mu(k)\left[e^{-i\mu\hat{\varphi}}\hat{b}^\dagger_{k\mu} + e^{i\mu\hat{\varphi}}\hat{b}_{k\mu}\right] + \hat{\Gamma}(\mathcal{O}(\hat{b}^2)) \quad (4)$$

- Observe that

$$[\hat{J}_z, \hat{H}_{\text{rel}}] = 0\,, \quad \hat{J}_z = \hat{L}_z + \hat{\Lambda}_z\,, \quad \hat{\Lambda}_z = \sum_{k\mu}\mu\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu}$$

Statistics of impurities

EY and M. Lemeshko, Phys. Rev. B 98, 045402 (2018)

- In the rotating frame, $\hat{S} = e^{-i\hat{\varphi}\otimes\hat{\Lambda}_z}$, the Hamiltonian

$$\hat{H}'_{\text{rel}} = \hat{S}^{-1}\hat{H}_{\text{rel}}\hat{S} = \frac{1}{m}(\hat{L}_z - \hat{\Lambda}_z)^2 + \sum_{k\mu}\tilde{\omega}(k)\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu} + \sum_{k\mu} Y_\mu(k)\left[\hat{b}^\dagger_{k\mu} + \hat{b}_{k\mu}\right] + \hat{\Gamma}$$

Statistics of impurities

EY and M. Lemeshko, Phys. Rev. B 98, 045402 (2018)

- In the rotating frame, $\hat{S} = e^{-i\hat{\varphi}\otimes\hat{\Lambda}_z}$, the Hamiltonian

$$\hat{H}'_{\text{rel}} = \hat{S}^{-1}\hat{H}_{\text{rel}}\hat{S} = \frac{1}{m}(\hat{L}_z - \hat{\Lambda}_z)^2 + \sum_{k\mu}\tilde{\omega}(k)\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu} + \sum_{k\mu} Y_\mu(k)\left[\hat{b}^\dagger_{k\mu} + \hat{b}_{k\mu}\right] + \hat{\Gamma}$$

- The angular part of the impurity decouples:

$$\langle\varphi|\Psi\rangle = \langle\varphi|M\rangle\hat{S}|\text{bos}_n\rangle \quad \to \quad \text{U(1) gauge field} \quad A = \langle\text{bos}_n|\hat{\Lambda}_z|\text{bos}_n\rangle\,, \qquad (5)$$

where $|\text{bos}_n\rangle$ is the eigenstate of

$$\hat{H}_{\text{bos}} = \frac{1}{m}(M - \hat{\Lambda}_z)^2 + \sum_{k\mu}\tilde{\omega}(k)\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu} + \sum_{k\mu} Y_\mu(k)\left[\hat{b}^\dagger_{k\mu} + \hat{b}_{k\mu}\right] + \hat{\Gamma}$$

Statistics of impurities

EY and M. Lemeshko, Phys. Rev. B 98, 045402 (2018)

- In the rotating frame, $\hat{S} = e^{-i\hat{\varphi}\otimes\hat{\Lambda}_z}$, the Hamiltonian

$$\hat{H}'_{\text{rel}} = \hat{S}^{-1}\hat{H}_{\text{rel}}\hat{S} = \frac{1}{m}(\hat{L}_z - \hat{\Lambda}_z)^2 + \sum_{k\mu}\tilde{\omega}(k)\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu} + \sum_{k\mu} Y_\mu(k)\left[\hat{b}^\dagger_{k\mu} + \hat{b}_{k\mu}\right] + \hat{\Gamma}$$

- The angular part of the impurity decouples:

$$\langle\varphi|\Psi\rangle = \langle\varphi|M\rangle\hat{S}|\text{bos}_n\rangle \quad \to \quad \text{U(1) gauge field} \quad A = \langle\text{bos}_n|\hat{\Lambda}_z|\text{bos}_n\rangle\,, \tag{5}$$

where $|\text{bos}_n\rangle$ is the eigenstate of

$$\hat{H}_{\text{bos}} = \frac{1}{m}(M - \hat{\Lambda}_z)^2 + \sum_{k\mu}\tilde{\omega}(k)\hat{b}^\dagger_{k\mu}\hat{b}_{k\mu} + \sum_{k\mu} Y_\mu(k)\left[\hat{b}^\dagger_{k\mu} + \hat{b}_{k\mu}\right] + \hat{\Gamma}$$

- By using $\partial\hat{H}_{\text{bos}}/\partial M = 2B(M - \hat{\Lambda}_z)$ and the Hellmann-Feynman theorem, one obtains

$$A = M - \frac{1}{2B}\frac{\partial E}{\partial M}\,. \tag{6}$$

Two-impurity problem

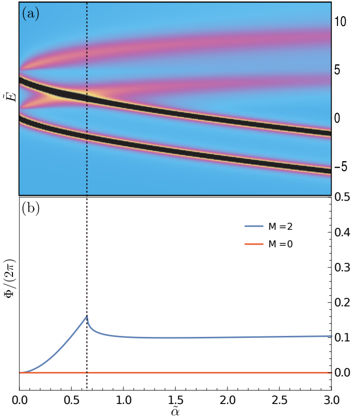
- The energy can be approximated by a variational energy

$$|\mathrm{var}\rangle = \sqrt{Z}|0\rangle + \sum_{k\mu} \beta_{k\mu} \hat{b}^{\dagger}_{k\mu}|0\rangle\,, \quad \delta\langle \mathrm{var}|\hat{H}_{\mathrm{bos}} - E|\mathrm{var}\rangle = 0 \tag{7}$$

Two-impurity problem

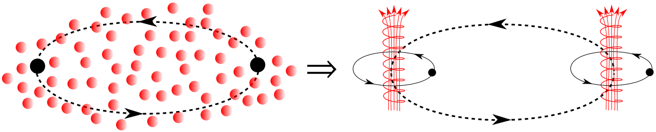
- The energy can be approximated by a variational energy

$$|\mathrm{var}\rangle = \sqrt{Z}|0\rangle + \sum_{k\mu} \beta_{k\mu} \hat{b}^{\dagger}_{k\mu}|0\rangle\,, \quad \delta\langle \mathrm{var}|\hat{H}_{\mathrm{bos}} - E|\mathrm{var}\rangle = 0 \tag{7}$$



Statistics of impurities

In the presence of a bath, each impurity turns into a tightly bound flux-tube-charged-particle composite.



exchange of impurities

1. Quantum Impurity Problems and Emergent Gauge Fields
   - Polaron and angulon
   - Emergence of gauge fields
   - Non-Abelian magnetic monopole
2. Realization of Anyons
   - Spin-statistics theorem and gauge fields
   - Two-impurity problem
3. N-anyon Problem
   - CS from bath
   - CS from impurity: Spin-boson model
4. Symmetries of Quantum Impurities
   - Quantum groups
   - Renormalization of B
5. Conclusion

Chern-Simons

- A particle coupled with the Chern-Simons field:

$$S = \frac{1}{2}\int dt \sum_{q=1}^{N} \dot{\boldsymbol{x}}_q^2 - \int d^3y\, A_\mu(y) j^\mu(y) + \frac{k}{2}\int d^3y\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \,,$$

  where k is the level parameter.

- The N-anyon Hamiltonian is given by

$$\hat{H}_{\text{N-anyon}} = \frac{1}{2}\sum_{q=1}^{N} \left[-i\nabla_q - \boldsymbol{A}_q(\boldsymbol{x}_q)\right]^2 \,,$$

  where the gauge field is

$$A_q^i(\boldsymbol{x}_q) = \alpha \sum_{p(\neq q)=1}^{N} \frac{\epsilon^{ij}\left(x_q^j - x_p^j\right)}{|\boldsymbol{x}_q - \boldsymbol{x}_p|^2} \,. \tag{8}$$

- $\alpha = 1/(2\pi k)$ which interpolates between 0 (boson) and 1 (fermion).
- In the Chern-Simons theory, the flux is given by $\Phi = 1/k \rightarrow \alpha = \Phi/(2\pi)$, whereas in the flux-tube-charged-particle composite picture, i.e., in the Maxwell theory, $\alpha = 2\xi/(2\pi)$ with ξ being the flux of the each composite.

Emergent Chern-Simons field

- Our aim is to define the statistics gauge field as an emergent gauge field:

$$\boldsymbol{A}_q(\boldsymbol{x}_q) = i\langle\psi_n(\boldsymbol{x}_q)|\nabla_q|\psi_n(\boldsymbol{x}_q)\rangle\,. \tag{9}$$

Emergent Chern-Simons field

- Our aim is to define the statistics gauge field as an emergent gauge field:

$$\boldsymbol{A}_q(\boldsymbol{x}_q) = i\langle\psi_n(\boldsymbol{x}_q)|\nabla_q|\psi_n(\boldsymbol{x}_q)\rangle\,. \tag{9}$$

- Consider a free N-boson system coupled to a *light* system

$$\hat{H}_{\text{tot}} = -\frac{1}{2}\sum_{q=1}^{N}\nabla_q^2 + \hat{H}_{\text{light}}(\boldsymbol{x}_q)\,. \tag{10}$$

The corresponding eigenvalue equation can be written as

$$\sum_m H^{\text{eff}}_{nm}\Phi^E_m(\boldsymbol{x}_q) = E\,\Phi^E_n(\boldsymbol{x}_q)\,,$$

where the effective Hamiltonian is

$$H^{\text{eff}}_{nm} = -\frac{1}{2}\sum_{q=1}^{N}\sum_{l}\left[\delta_{nl}\nabla_q + \langle\psi_n|\nabla_q|\psi_l\rangle\right]\cdot\left[\delta_{lm}\nabla_q + \langle\psi_l|\nabla_q|\psi_m\rangle\right] + \varepsilon_n\delta_{nm}\,,$$

and $|\psi_n(\boldsymbol{x}_q)\rangle$ is the eigenstate of the *light* Hamiltonian.

Emergent Chern-Simons field

- Assume that in the adiabatic limit $\langle\psi_n|\nabla_q|\psi_m\rangle \approx \delta_{nm}\langle\psi_n|\nabla_q|\psi_n\rangle$:

$$\frac{1}{2}\sum_{q=1}^{N}\left[-i\nabla_q - i\langle\psi_n|\nabla_q|\psi_n\rangle\right]^2 \Phi_n^E(\boldsymbol{x}_q) = \left(E - \tilde{\varepsilon}_n(\boldsymbol{x}_q)\right)\Phi_n^E(\boldsymbol{x}_q)\,,$$

- For an eigenstate in the form of

$$|\psi_n(\boldsymbol{x}_q)\rangle = \exp\left[-i\hat{\alpha}\Theta(\boldsymbol{x}_q)\right]|\tilde{\psi}_n\rangle\,,$$

  where

$$\Theta(\boldsymbol{x}_q) = \sum_{q>p}\arctan\left[\frac{y_q - y_p}{x_q - x_p}\right]\,,$$

  the emergent gauge field

$$i\langle\psi_n(\boldsymbol{x}_q)|\nabla_q|\psi_n(\boldsymbol{x}_q)\rangle = \alpha\sum_{p(\neq q)=1}^{N}\frac{\epsilon^{ij}\left(x_q^j - x_p^j\right)}{|\boldsymbol{x}_q - \boldsymbol{x}_p|^2}\,.$$

- The statistics parameter emerges as $\alpha = \langle\tilde{\psi}_n|\hat{\alpha}|\tilde{\psi}_n\rangle$ for some operator $\hat{\alpha}$ acting on the light system.

TWO APPROACHES

CS from bath

- Bath as a light system

CS from bath

- Bath as a light system
- N-impurity problem at the Fröhlich level

$$\hat{H}_{\text{tot}} = -\frac{1}{2}\sum_{q=1}^{N}\nabla_q^2 + \sum_{\mu}\omega_\mu\,\hat{b}_\mu^\dagger\hat{b}_\mu + \sum_{\mu}\lambda_\mu\left(e^{-i\beta_\mu\Theta(\boldsymbol{x}_q)}\hat{b}_\mu^\dagger + \text{H.c.}\right).$$

CS from bath

- Bath as a light system
- N-impurity problem at the Fröhlich level

$$\hat{H}_{\text{tot}} = -\frac{1}{2}\sum_{q=1}^{N}\nabla_q^2 + \sum_{\mu}\omega_\mu\,\hat{b}_\mu^\dagger\hat{b}_\mu + \sum_{\mu}\lambda_\mu\left(e^{-i\beta_\mu\Theta(\boldsymbol{x}_q)}\hat{b}_\mu^\dagger + \text{H.c.}\right).$$

- Adiabacity condition

$$\sum_{\nu}\beta_\nu\left(\frac{\lambda_\nu}{\omega_\nu}\right)^2 \gg \beta_\mu\frac{\lambda_\mu}{\omega_\mu} \quad \rightarrow \text{Emergent CS field}$$

Free N bosons become N anyons

CS from bath

- For instance, the $N_{\text{ph}} \to \infty$ solution of the following impurity Hamiltonian gives anyons

$$\hat{H}_{\text{tot}} = -\frac{1}{2}\sum_{q=1}^{N} \nabla_q^2 + \sum_{\mu=1}^{N_{\text{ph}}} \sqrt{N_{\text{ph}}}\, \hat{b}_\mu^\dagger \hat{b}_\mu + \sum_{\mu=1}^{N_{\text{ph}}} \sqrt{\alpha} \left(e^{-i\Theta(\boldsymbol{x}_q)} \hat{b}_\mu^\dagger + \text{H.c.} \right) \tag{11}$$

- Does it bring new insights to the N-anyon problem, such as the upper and lower bounds of the problem ?

CS from impurity

- Impurity as a light system

CS from impurity

- Impurity as a light system
- Quantum dissipation model

$$\hat{H}_{\text{tot}} = \frac{1}{2}\sum_{q=1}^{N}\left(-\nabla_q^2 + \boldsymbol{x}_q^2\right) + \mathit{\Delta}\left(\hat{J}_z\right)^n + h\left(e^{-i\Theta(\boldsymbol{x}_q)}\hat{J}_+ + \text{H.c.}\right)$$

- Spin-Boson model

$$\hat{H}_{\text{tot}} = \sum_{q=1,i=x,y}^{N}\left(\hat{a}^\dagger_{q,i}\hat{a}_{q,i} + 1\right) + \mathit{\Delta}\left(\hat{J}_z\right)^n + h\hat{J}_+\prod_{p>q}^{N}\frac{\hat{a}^\dagger_{q,y} - \hat{a}^\dagger_{p,y} + \hat{a}_{q,y} - \hat{a}_{p,y}}{\hat{a}^\dagger_{q,x} - \hat{a}^\dagger_{p,x} + \hat{a}_{q,x} - \hat{a}_{p,x}} + \text{H.c.} \tag{12}$$

- Bosonic bath turns into anyonic bath in the presence of a single impurity

1. Quantum Impurity Problems and Emergent Gauge Fields
   - Polaron and angulon
   - Emergence of gauge fields
   - Non-Abelian magnetic monopole
2. Realization of Anyons
   - Spin-statistics theorem and gauge fields
   - Two-impurity problem
3. N-anyon Problem
   - CS from bath
   - CS from impurity: Spin-boson model
4. Symmetries of Quantum Impurities
   - Quantum groups
   - Renormalization of B
5. Conclusion

Symmetries of quantum impurities

- Rigid rotor is given by SO(3)

$$H_{\mathrm{rotor}} = B \underbrace{\boldsymbol{J}^2}_{\text{Casimir operator}}$$

Symmetries of quantum impurities

- Rigid rotor is given by SO(3)

$$H_{\text{rotor}} = B \underbrace{\boldsymbol{J}^2}_{\text{Casimir operator}}$$

- Angulon: a rotor dressed by the field excitations

- $B \to B^*$: a deformed rotor

  in perturbative regime:

$$B \to B^* = B - \frac{1}{2}\sum_{j=0}^{2}\sum_{k\lambda j'}\binom{2}{j}\frac{(-1)^j V_\lambda(k)^2\left[C^{j'0}_{2-j0,\lambda 0}\right]^2}{Bj'(j'+1)+\omega(k)-B(2-j)(3-j)} + \mathcal{O}(U_\lambda(k)^4)$$

Symmetries of quantum impurities

- Rigid rotor is given by SO(3)

$$H_{\text{rotor}} = B \underbrace{\boldsymbol{J}^2}_{\text{Casimir operator}}$$

- Angulon: a rotor dressed by the field excitations
- $B \to B^*$: a deformed rotor

  in perturbative regime:

$$B \to B^* = B - \frac{1}{2} \sum_{j=0}^{2} \sum_{k\lambda j'} \binom{2}{j} \frac{(-1)^j V_\lambda(k)^2 \left[C_{2-j0,\lambda 0}^{j'0} \right]^2}{Bj'(j'+1) + \omega(k) - B(2-j)(3-j)} + \mathcal{O}(U_\lambda(k)^4)$$

Can we find a symmetry group for angulon?

Quantum groups

Conjuncture
EY, M. Shkolnikov, and M. Lemeshko, PRL 121, 255302 (2018)

let us deform the group $SO(3) \to SO_q(3)$

- deformation of the Lie algebra

$$[\hat{J}^q_z, \hat{J}^q_\pm] = \pm \hat{J}^q_\pm \quad [\hat{J}^q_+, \hat{J}^q_-] = [2\hat{J}^q_z]_q \,, \tag{13}$$

where the square bracket implies

$$[\hat{A}]_q = \frac{q^{\hat{A}} - q^{-\hat{A}}}{q - q^{-1}} \,, \tag{14}$$

with the deformation parameter q such that $\lim_{q\to 1}[\hat{A}]_q = \hat{A}$.

- The Hamiltonian

$$H_{\text{deformed rotor}} = B \underbrace{\left(\hat{J}^q_- \hat{J}^q_+ + [\hat{J}^q_z]_q [\hat{J}^q_z + 1]_q \right)}_{\text{Casimir of the quantum group}}$$

Renormalization of B

- Within the quantum group

$$B \to B^* = B\cos(3\tau) = B(1 - 9\tau^2/2 + \mathcal{O}(\tau^4))\,,$$

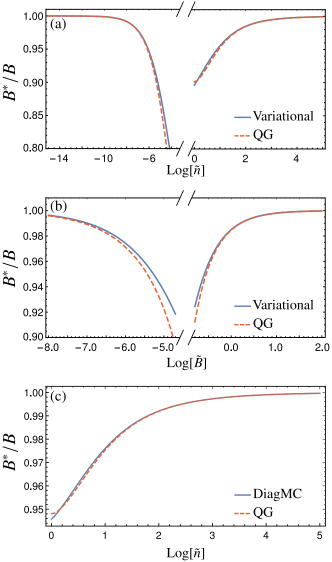
  where $q = \exp(i\tau)$.

- Match it with the perturbative result

$$\tau = \left(\frac{1}{9B}\sum_{k\lambda j'j}\binom{2}{j}\frac{(-1)^j V_\lambda(k)^2\left[C^{j'0}_{2-j0,\lambda 0}\right]^2}{Bj'(j'+1)+\omega(k)-B(2-j)(3-j)}\right)^{1/2}$$

$B^* = B\cos(3\tau)$ is valid in any coupling ???

Renormalization of B



1. Quantum Impurity Problems and Emergent Gauge Fields
   - Polaron and angulon
   - Emergence of gauge fields
   - Non-Abelian magnetic monopole
2. Realization of Anyons
   - Spin-statistics theorem and gauge fields
   - Two-impurity problem
3. N-anyon Problem
   - CS from bath
   - CS from impurity: Spin-boson model
4. Symmetries of Quantum Impurities
   - Quantum groups
   - Renormalization of B
5. Conclusion

Conclusion

Quantum impurity problems can be considered as charged particle/s coupled to a gauge field.

Conclusion

Quantum impurity problems can be considered as charged particle/s coupled to a gauge field.

In the angulon case, the emergent gauge field is a non-Abelian magnetic monopole

Conclusion

Quantum impurity problems can be considered as charged particle/s coupled to a gauge field.

In the angulon case, the emergent gauge field is a non-Abelian magnetic monopole

A 2D bath attaches a magnetic flux to impurities; fractional statistics

Conclusion

Quantum impurity problems can be considered as charged particle/s coupled to a gauge field.

In the angulon case, the emergent gauge field is a non-Abelian magnetic monopole

A 2D bath attaches a magnetic flux to impurities; fractional statistics

N-anyon setup is possible: either N bosonic impurities turn into N anyons, or bosonic bath into anyonic bath in the presence of an impurity

Conclusion

Quantum impurity problems can be considered as charged particle/s coupled to a gauge field.

In the angulon case, the emergent gauge field is a non-Abelian magnetic monopole

A 2D bath attaches a magnetic flux to impurities; fractional statistics

N-anyon setup is possible: either N bosonic impurities turn into N anyons, or bosonic bath into anyonic bath in the presence of an impurity

Non-Abelian anyons ???

THANK YOU FOR YOUR ATTENTION

European Commission

SEVENTH FRAMEWORK PROGRAMME

MARIE CURIE ACTIONS

The People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA Grant Agreement No. 291734.