Off-diagonal long-range order in 1D quantum systems

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Mathematical physics of anyons and topological states of matter
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Outline

- **Reminder on off-diagonal long-range order**

- **Off-diagonal long-range order for 1D quantum systems (including 1D anyons)**

- **Integrable Floquet Hamiltonian for a Lieb-Liniger 1D Bose gas**

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- Reminder on off-diagonal long-range order

- Off-diagonal long-range order for 1D quantum systems (including 1D anyons)

- Integrable Floquet Hamiltonian for a Lieb-Liniger 1D Bose gas

Off-diagonal long-range order (I)

One-body density matrix and its eigenvalues:

$$\rho_1(\vec{r},\overrightarrow{r'}) \equiv \langle \hat{\psi}^+(\vec{r})\,\hat{\psi}(\overrightarrow{r'})\rangle$$

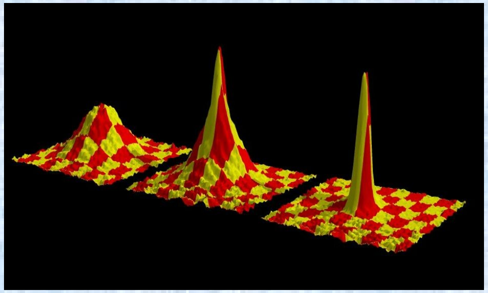
$$\int \rho_1(\vec{r},\overrightarrow{r'})\,\vartheta_i(\overrightarrow{r'})d\overrightarrow{r'} = \lambda_i\vartheta_i(\vec{r})$$

ODLRO: largest eigenvalue scale with N, while the others are O(1) [Penrose-Onsager, PR (1956) – Stringari-Pitaevskii, Bose-Einstein Condensation and Superfluidity (2016)]

Off-diagonal long-range order (II)

$$\lambda_0 \propto N^{\mathcal{C}}$$

$\mathcal{C}$ $\begin{cases} 0 & \text{Fermi-like} \\ 1 & \text{Bose-Einstein Condensation/ODLRO} \rightarrow \end{cases}$



For a 1D Lieb-Liniger Bose gas...

Lieb-Liniger Hamiltonian (I)

N interacting bosons in 1D:

$$H = -\frac{\hbar^2}{2\mathrm{m}} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\lambda \sum_{i<j} \delta\left(x_i - x_j\right)$$

One non-trivial coupling constant:

$$\gamma = \frac{2\mathrm{m}\lambda}{\hbar^2 n}$$

← density

Temperature typically in units of the degeneracy temperature:

$$k_B T_D = \frac{\hbar^2 n^2}{2\mathrm{m}}$$

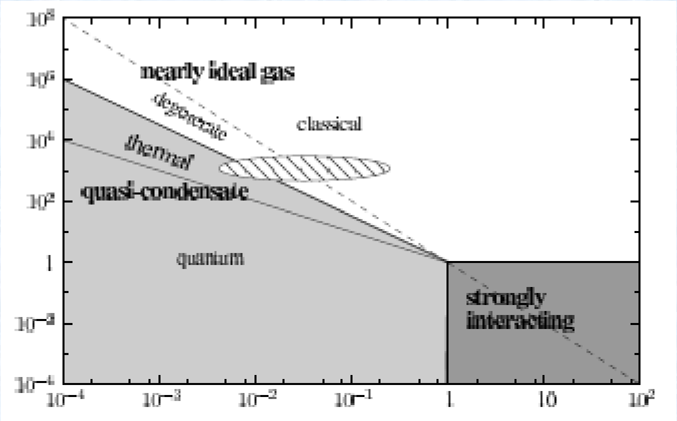
Coupling controllable from the 3D setup:

$$\lambda = \frac{\hbar^2 a_{3D}}{m a_\perp^2} \frac{1}{1 - C a_{3D}/a_\perp}$$

Lieb-Liniger Hamiltonian (II)

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\lambda \sum_{i<j} \delta\left(x_i - x_j\right)$$

τ



γ

[from I. Bouchole et al., 2009]

$$\gamma = \frac{2m\lambda}{\hbar^2 n} \qquad \tau = \frac{T}{T_D}$$

Large γ → Tonks-Girardeau limit

Lieb-Liniger Hamiltonian (III)

$$|\psi_N\rangle = \frac{1}{\sqrt{N!}}\int_0^L d^N z\, \chi_N(z_1, \dots, z_N)\psi_B^{+}(z_1)\dots\psi_B^{+}(z_N)|0\rangle$$

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2}{L}\sum_{k=1}^{N}\arctan\left(\frac{\lambda_j - \lambda_k}{c}\right), \qquad j = 1, \dots, N$$

$$\chi_N(z_1, \dots, z_N) = \mathcal{N}\det\left(e^{i\lambda_j z_m}\right)\prod_{n<l}[\lambda_l - \lambda_n - ic\, sign(z_l - z_n)]$$

Bethe ansatz solution

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Deviations from ODLRO

$$\frac{n(p=0)}{L} \propto N^{C}$$

$$n(p) = \frac{L}{2\pi}\int_0^L \rho(z)\, e^{ipz}\, dz$$

Large distance $\rho(z \gg 1)$ ⟹ Small momenta $n(p \approx 0)$

Luttinger Liquid prediction: $\rho(z \to \infty) \propto z^{-1/2K}$

$$K(\gamma) = \frac{v_F}{s(\gamma)}$$

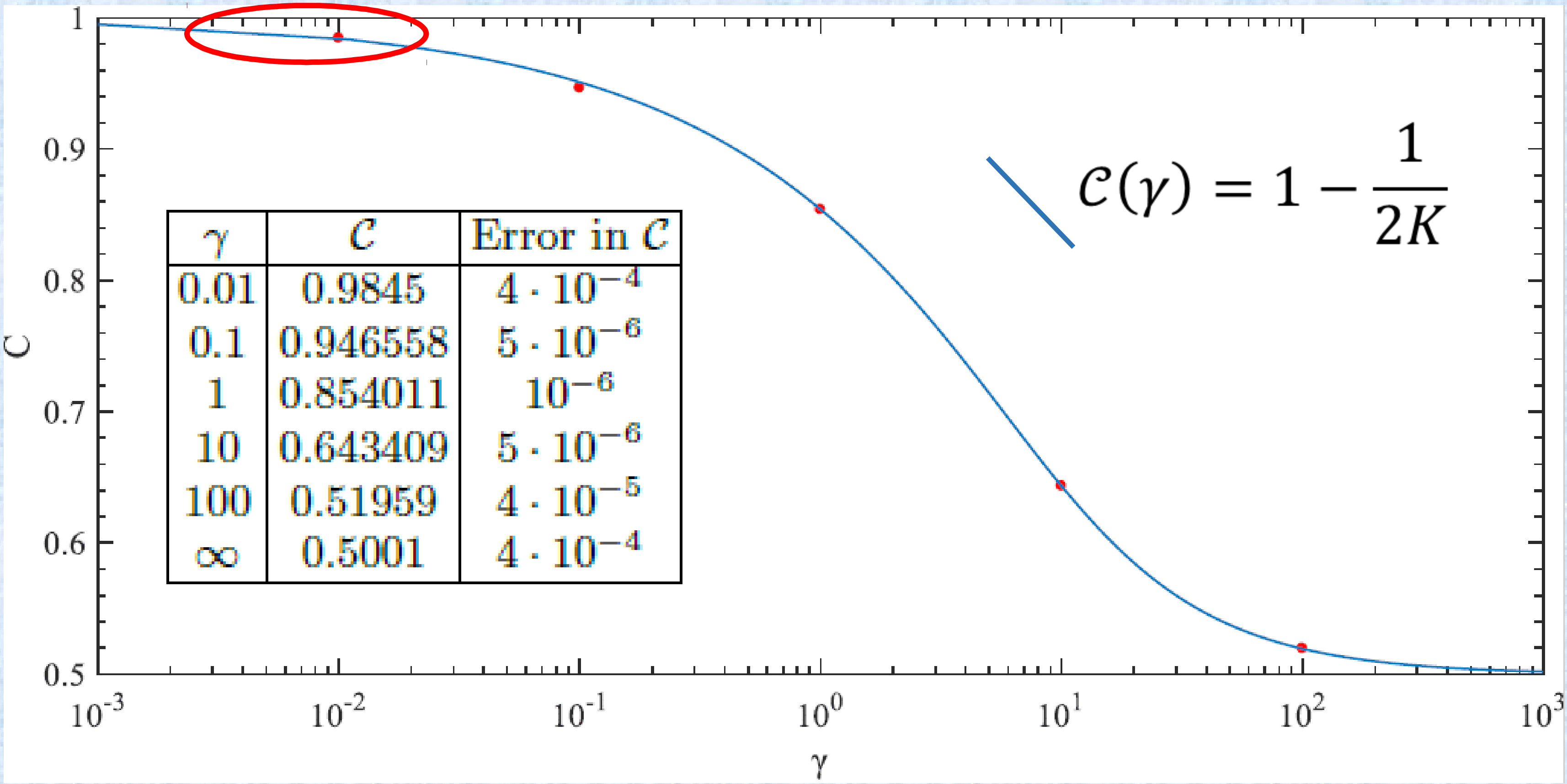
$$p_{min} \approx \frac{2\pi}{L} \propto N^{-1} \qquad n_{1D} = \frac{N}{L}$$

$$\frac{n(p \to 0)}{L} \propto \frac{1}{p^{1-1/2K}} \propto N^{1-1/2K}$$

$$C(\gamma) = 1 - \frac{1}{2K}$$

From γ=0 to infinite γ

For a 1D Lieb-Liniger Bose gas: if the largest eigenvalue of the one-body density matrix scale with N to the power C



γ	$\mathcal{C}$	Error in $\mathcal{C}$
0.01	0.9845	$4 \cdot 10^{-4}$
0.1	0.946558	$5 \cdot 10^{-6}$
1	0.854011	10^{-6}
10	0.643409	$5 \cdot 10^{-6}$
100	0.51959	$4 \cdot 10^{-5}$
∞	0.5001	$4 \cdot 10^{-4}$

$$\mathcal{C}(\gamma) = 1 - \frac{1}{2K}$$

[A. Colcelli, G. Mussardo and A. Trombettoni, Europhys. Lett. (2018)]

Hard-Core Anyons

$$\chi_N^\kappa(z_1,\dots,z_N) = \left[\prod_{1\le i<j\le N} A(z_j - z_i)\right]\chi_N^1(z_1,\dots,z_N) \qquad \kappa = \frac{m}{n} \in \mathbb{Q}$$

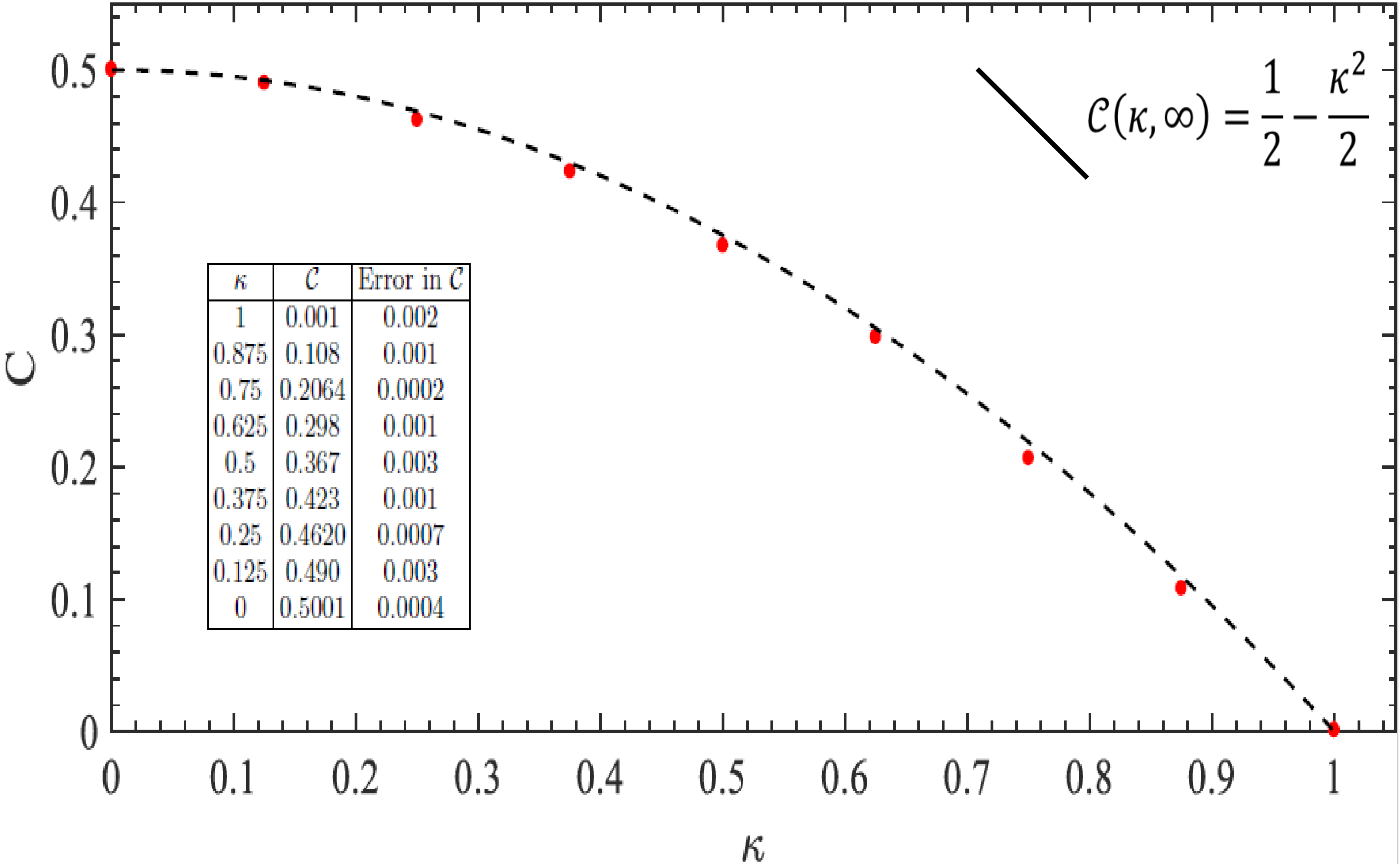
Anyon – Fermi mapping: $A(z_j - z_i) = [\theta(z_j - z_i) + \theta(z_i - z_j)e^{i\pi(1-\kappa)}]$ $\qquad \theta(0) = 0$

$\boxed{\kappa = 0}$ Boson – Fermi mapping: $A(z_j - z_i) = sign(z_j - z_i)$

$$\rho_A(t) = \det\left[\frac{2}{\pi}\int_0^{2\pi} d\tau\, e^{i(j-l)\tau} A(\tau - t)\sin\left(\frac{\tau - t}{2}\right)\sin\left(\frac{\tau}{2}\right)\right]_{j,l=1,\dots,N-1} \qquad t \equiv \frac{2\pi x}{L}$$

[R. Santachiara and P. Calabrese, JSTAT (2008)]

Hard-Core Anyons



κ	$\mathcal{C}$	Error in $\mathcal{C}$
1	0.001	0.002
0.875	0.108	0.001
0.75	0.2064	0.0002
0.625	0.298	0.001
0.5	0.367	0.003
0.375	0.423	0.001
0.25	0.4620	0.0007
0.125	0.490	0.003
0	0.5001	0.0004

Anyonic Lieb-Liniger Model

$$H = \int_0^L \{[\partial_z \psi_A^+(z)]\,[\partial_z \psi_A(z)] + c\,\psi_A^+(z)\psi_A^+(z)\psi_A(z)\psi_A(z)\}\,dz$$

$$\psi_A^+(z_1)\psi_A^+(z_2) = e^{i\pi\kappa\, sign(z_1 - z_2)}\psi_A^+(z_2)\psi_A^+(z_1)$$

$$\psi_A(z_1)\psi_A(z_2) = e^{i\pi\kappa\, sign(z_1 - z_2)}\psi_A(z_2)\psi_A(z_1)$$

$$\psi_A(z_1)\psi_A^+(z_2) = e^{-i\pi\kappa\, sign(z_1 - z_2)}\psi_A^+(z_2)\psi_A(z_1) + \delta(z_1 - z_2)$$

$$\kappa = \begin{cases} 0 & \text{Bosons} \\ 1 & \text{Fermions,} \end{cases}$$

$$\chi_N(z_1, \ldots, z_i, z_{i+1}, \ldots, z_N) = e^{-i\pi\kappa\, sign(z_i - z_{i+1})}\chi_N(z_1, \ldots, z_{i+1}, z_i, \ldots, z_N)$$

Anyonic Lieb-Liniger Model

Twisted BC: $\chi_N^\kappa(0, x_2, \dots) = e^{i\pi\kappa(N-1)}\chi_N^\kappa(L, x_2, \dots) \longleftrightarrow \chi_N^0(0, x_2, \dots) = \chi_N^0(L, x_2, \dots)$

Periodic BC

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2}{L}\sum_{k=1}^{N}\arctan\left(\frac{\lambda_j - \lambda_k}{c'}\right), \qquad j = 1, \dots, N \qquad c' = \frac{c}{\cos(\pi\kappa/2)} > 0$$

$$\chi_N^\kappa(z_1, \dots, z_N) = \mathcal{N}\exp\left(i\frac{\pi\kappa}{2}\sum_{j<k} sign(z_j - z_k)\right)\det\left(e^{i\lambda_j z_m}\right) \cdot$$

$$\cdot \prod_{n<l}[\lambda_l - \lambda_n - ic' sign(z_l - z_n)]$$

[O.I. Pâţu, V.E. Korepin, D.V. Averin, JPA (2007);
M.T. Batchelor, X.-W. Guan, N. Oelkers, PRL (2006)]

Predictions for the ODLRO scaling

$$n_A(p=0) \propto N^{\mathcal{C}(\kappa)}$$

$$n_A(p) = \frac{L}{2\pi}\int_0^L \rho_A(z)\, e^{ipz}\, dz$$

Large distance $\rho_A(z \gg 1)$ $\Longrightarrow$ "Small" momenta $n_A(p \approx -p_F\kappa)$

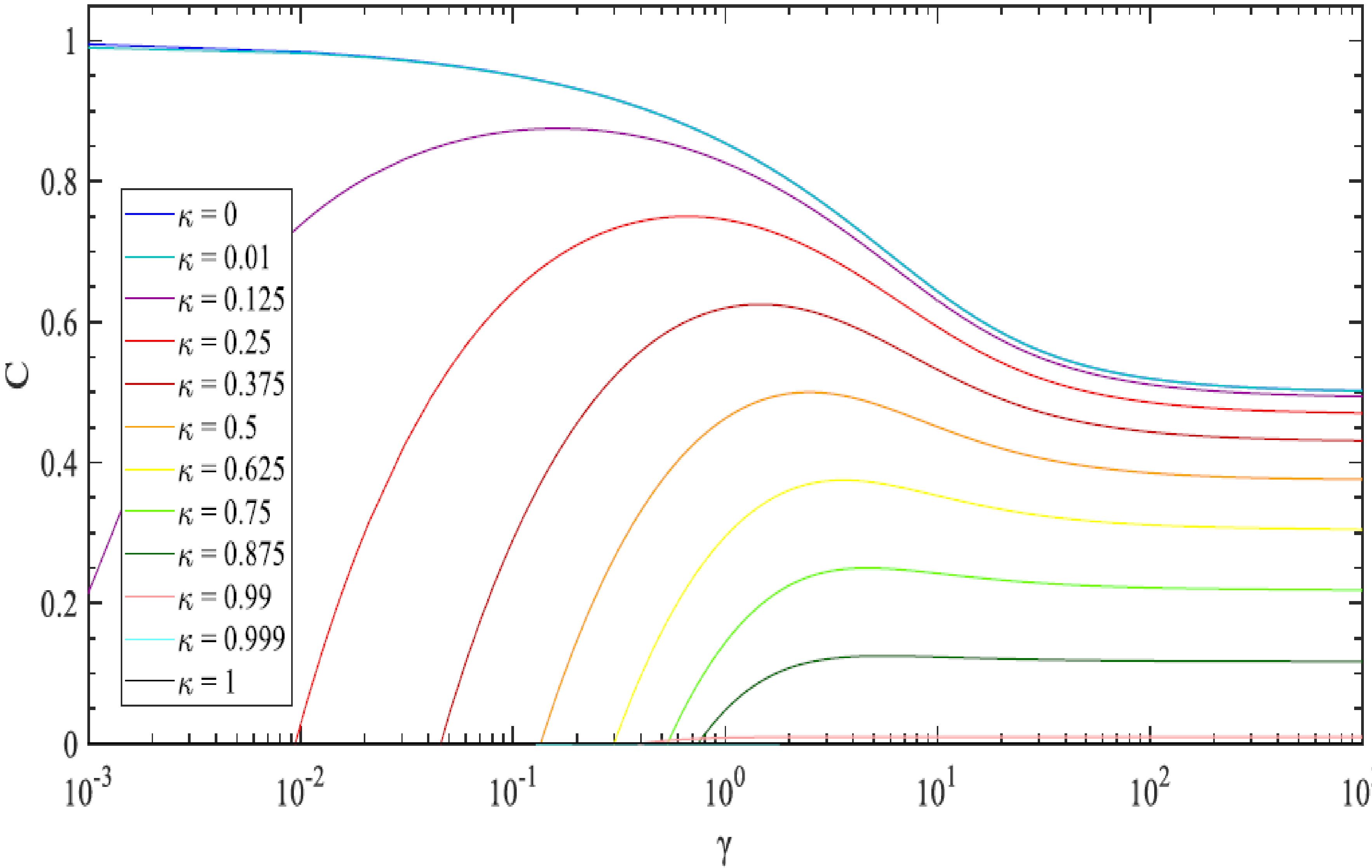
$\boxed{\kappa = 0}$ $\quad \mathcal{C}(\kappa = 0, \gamma) = 1 - \frac{1}{2K}$ $\quad K(\gamma) = \frac{v_F}{s(\gamma)}$

$$\frac{n_A(p)}{L} \propto \frac{1}{(p + p_F\kappa)^{1 - \frac{1}{2K} - \frac{K\kappa^2}{2}}} \Longrightarrow \boxed{\mathcal{C}(\kappa, \gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}}$$

Results for 1D Lieb-Liniger anyons →

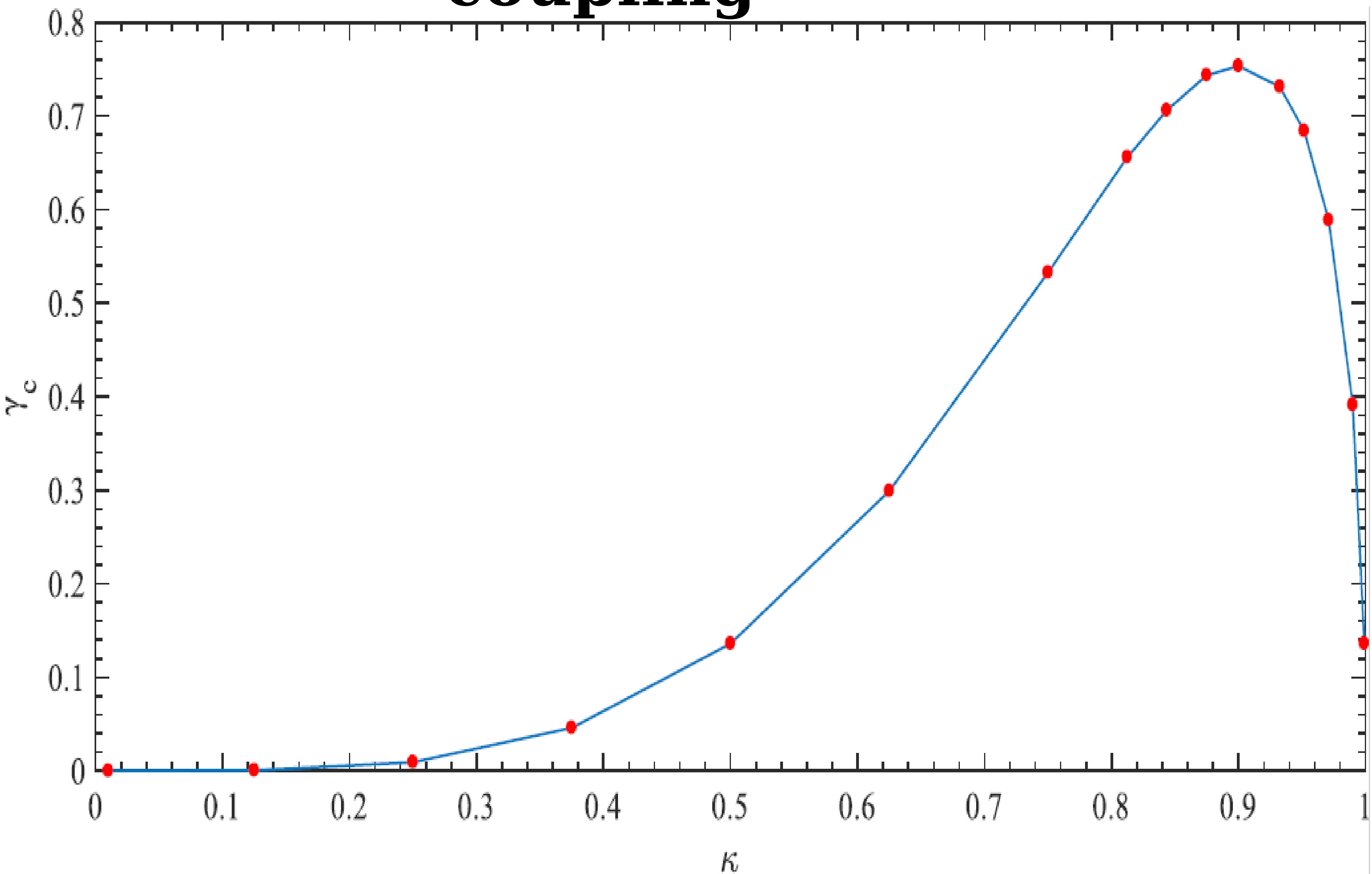
$$\mathcal{C}(\kappa,\gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}$$

Statistical parameter
1 → fermions
0 → bosons



“Critical” coupling

$\mathcal{C}(\kappa, \gamma_c) = 0$



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- Reminder on off-diagonal long-range order

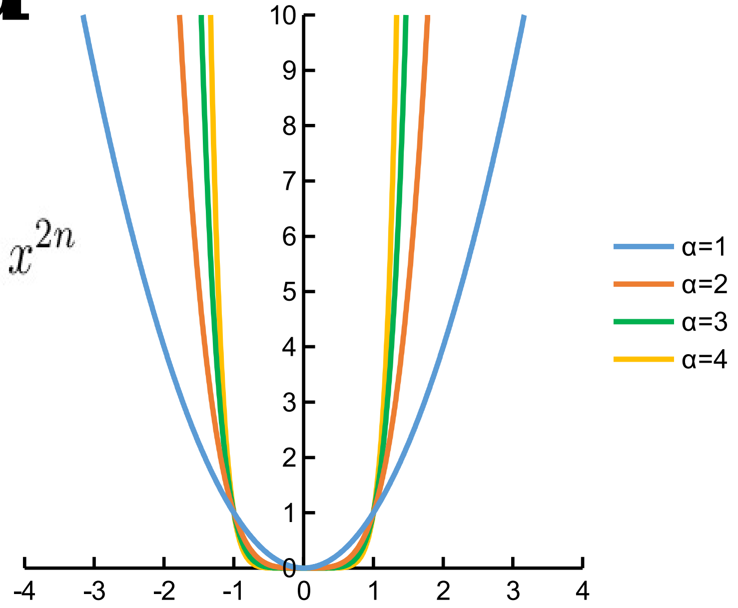
- Off-diagonal long-range order for 1D quantum systems → effect of a confining potential

- Integrable Floquet Hamiltonian for a Lieb-Liniger 1D Bose gas

Tonks-Girardeau gas

$$H = \sum_{i=1}^{N}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + V(x_i)\right]$$

$$V(x) = \Lambda x^{2n}$$



$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\phi_k(x) = \varepsilon_k \phi_k(x)$$

$$\psi|_{x_i=x_j} = 0\,, \quad \forall i,j = 1,\ldots,N \; with \; i \neq j$$

Bose – Fermi mapping: $\psi(x_1,\ldots,x_N) = \frac{1}{\sqrt{N!}}\det\left[\phi_k(x_l)\right]_{k,l=1,\ldots,N} \prod_{1\le i<j\le N} sgn(x_i - x_j)$

$$\rho(x,y) = \sum_{i,j=1}^{N}(-1)^{i+j}\,\phi_i(y)\,\phi_j^*(x)\det\left[\delta_{k,l} - 2\int_y^x dt\,\phi_l(t)\,\phi_k^*(t)\right]_{\substack{k,l=1,\ldots,N\\ k\neq j,\, l\neq i}}$$

Universality of the scaling of the ODLRO

$$\lambda_0 \sim \mathcal{B} N^{\mathcal{C}}$$

Semiclassical analysis:

$\hbar \to 0, \quad \text{with } m,\ V(x),\ \mu \text{ fixed} \qquad\qquad N\hbar = \text{const}$

$$\rho(x) = \frac{1}{\pi\hbar}\sqrt{2m\left[\mu - V(x)\right]}$$

$$\rho_{\text{cft}}(\tilde{x},\tilde{y}) = \sqrt{\frac{m}{2\hbar\widetilde{L}}}\frac{|C|^2\left|\sin\left(\frac{\pi\widetilde{x}}{\widetilde{L}}\right)\right|^{\frac{1}{4}}\left|\sin\left(\frac{\pi\widetilde{y}}{\widetilde{L}}\right)\right|^{\frac{1}{4}}}{\left|\sin\left(\frac{\pi}{\widetilde{L}}\frac{\widetilde{x}-\widetilde{y}}{2}\right)\right|^{\frac{1}{2}}\left|\sin\left(\frac{\pi}{\widetilde{L}}\frac{\widetilde{x}+\widetilde{y}}{2}\right)\right|^{\frac{1}{2}}}$$

$$\widetilde{x}(x) = \int_{x_1}^{x}\frac{du}{v(u)} \qquad v(x) = \sqrt{\frac{2}{m}\left[\mu - V(x)\right]}$$

[Y. Brun and J. Dubail, SciPost Physics (2017)]

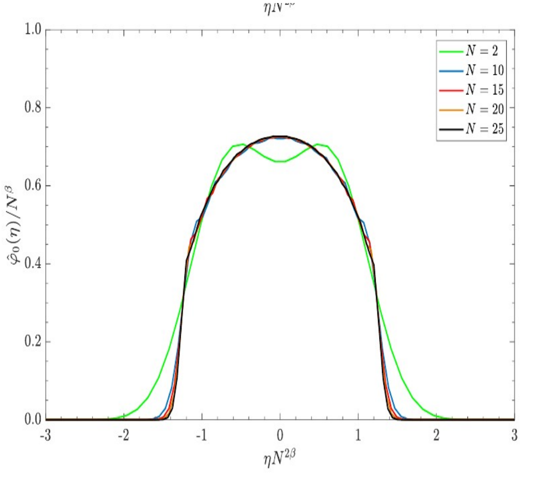
Universality of the scaling of the ODLRO

$\lambda_0 = O(\hbar^{-1/2}) \rightarrow \lambda_0 \sim \mathcal{B} N^{1/2}$ for *every* potential

But what about the coefficient $\mathcal{B}$?

$$\eta \equiv \frac{x}{\xi} \qquad \hat{\varphi}_0(\eta) \equiv \varphi_0(x)\sqrt{\xi} \qquad \int |\hat{\varphi}_0(\eta)|^2 \, d\eta = 1$$

$$\hat{\varphi}_0(\eta) \sim N^{\beta}$$



Universality of the scaling of the ODLRO

$$\tilde{\rho}(k) = \frac{1}{2\pi} \int dx \int dy \, \rho(x,y) e^{-i\,k\,(x-y)}$$

$$\frac{n_{\mathrm{peak}}}{\xi} \sim N^{\gamma}$$

$$\boxed{\gamma + 2\beta = \mathcal{C}}$$

Comparison of semiclassical limit with numerical results

$$\lambda_0 = \mathcal{A} + \mathcal{B}\, N^{\mathcal{C}} + \frac{\mathcal{D}}{N^{\mathcal{E}}}$$

n	$\mathcal{B}_{\rm fit}$	$\mathcal{B}$
1	1.4304(2)	1.430(4)
2	1.400(4)	1.392(4)
3	1.380(4)	1.378(3)
4	1.372(5)	1.368(2)
∞	1.31(1)	1.308(3)

n	$\mathcal{A}_{\rm fit}$	$\mathcal{B}_{\rm fit}$	$\mathcal{D}_{\rm fit}$	$\mathcal{E}_{\rm fit}$
1	$-0.554(2)$	1.4304(2)	0.122(1)	0.60(1)
2	$-0.55(4)$	1.400(4)	0.141(8)	0.79(6)
3	$-0.53(3)$	1.380(4)	0.16(2)	1.1(5)
4	$-0.56(3)$	1.372(5)	0.20(1)	0.9(3)
∞	$-0.6(1)$	1.31(1)	0.31(9)	0.3(1)

Fixing V and increasing the number (agreement with the CFT results for $\mathcal{B}$.)

Comparison with numerical results

n	$\mathcal{A}_{\text{fit}}$	$\mathcal{B}_{\text{fit}}$	$\mathcal{D}_{\text{fit}}$	$\mathcal{E}_{\text{fit}}$
1	$-0.56(3)$	$1.432(4)$	$0.13(3)$	$0.57(2)$
2	$-0.55(2)$	$1.407(4)$	$0.15(3)$	$1.0(2)$
3	$-0.56(2)$	$1.391(1)$	$0.18(2)$	$1.0(1)$
4	$-0.56(1)$	$1.38(2)$	$0.18(4)$	$0.8(1)$

Fixing instead the density at the center: same results!

Role of boundary conditions

$$\mathcal{B}_{\text{fit}} = 1.32(1)$$

Hard-wall with open BC: surprisingly different from the analytical results for periodic BC [1.54269..., see P.J. Forrester et al., PRA (2003)]

In other words: the thermodynamic limit "remembers" the boundary conditions
(for $\mathcal{B}$, but not for $\mathcal{C}$)

[A. Colcelli, J. Viti, G. Mussardo, and A. Trombettoni, Phys. Rev. A (2018)]

Comparison with numerical results

$$\int d\eta \propto \frac{1}{\xi} \propto \hbar^{-\frac{1}{n+1}} \qquad\qquad \beta = -\frac{1}{2n+2}$$

$$N^{-2\beta} \propto N^{\frac{1}{n+1}} \quad \rightarrow \quad \gamma = \frac{n+3}{2(n+1)}$$

n	$\mathcal{C}_{\text{fit}}$	$\mathcal{C}^{wkb}$	β_{fit}	γ_{fit}	β	γ
1	0.500(2)	0.496(8)	−0.25(1)	1.02(4)	$-\frac{1}{4}$	1
2	0.501(1)	0.54(3)	−0.16(1)	0.85(2)	$-\frac{1}{6}$	$\frac{5}{6}$
3	0.501(2)	0.54(7)	−0.12(2)	0.76(1)	$-\frac{1}{8}$	$\frac{3}{4}$
4	0.500(3)	0.54(9)	−0.10(1)	0.70(1)	$-\frac{1}{10}$	$\frac{7}{10}$
∞	0.500(1)		0.00(1)	0.502(2)	0	$\frac{1}{2}$

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Floquet engineering

Adding a time-periodic extermal potential → Floquet Hamiltonian

Typically the Floquet Hamiltonian is not integrable, even though the periodically driven system it is.

How to do it? For the Lieb-Liniger model → use a periodic tilting

Simple example: one-body case

$$i\hbar\frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2\,m}\frac{\partial^2 \chi}{\partial x^2} + x\,f(t)\,\chi(x,t)$$

function periodic in time with period T

One-body case (I)

$$i\hbar \frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \chi}{\partial x^2} + x\, f(t)\, \chi(x,t)$$

$$\chi(x,t) = e^{i\theta(x,t)}\, \eta(y(t),t)$$

$$y(t) \;=\; x \;-\; \xi(t)$$

$$i\hbar \frac{\partial \eta}{\partial t} = -\frac{\hbar^2}{2\,m}\frac{\partial^2 \eta}{\partial y^2}$$

$$\xi(t) = -\frac{1}{m}\int_0^t d\tau \int_0^\tau dt'\, f(t')$$

$$\theta(t) = -\frac{1}{2\,m\,\hbar}\int_0^t d\tau \left[\int_0^\tau dt'\, f(t')\right]^2$$

One-body case (II)

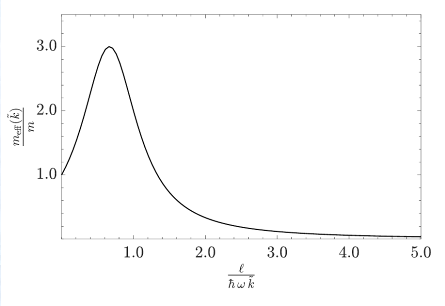
$$\chi(x, nT) = e^{-i\frac{nT}{\hbar}H_F}\,\chi(x, t=0)$$

$$H_F = \frac{\hat{p}^2}{2m} + \hat{p}\frac{\xi(nT)}{nT} - \hbar\frac{\theta(nT)}{nT}$$

$$H_F = \frac{\hat{p}^2}{2\,m_{\rm eff}(\hat{p})}$$

E.g.: $f(t) = \ell\,\sin(\omega t)$

$$\frac{m_{\rm eff}(\tilde{k})}{m} = \left[1 - \frac{2\,\ell}{\hbar\,\omega\,\tilde{k}} + \frac{3}{2}\left(\frac{\ell}{\hbar\,\omega\,\tilde{k}}\right)^2\right]^{-1}$$



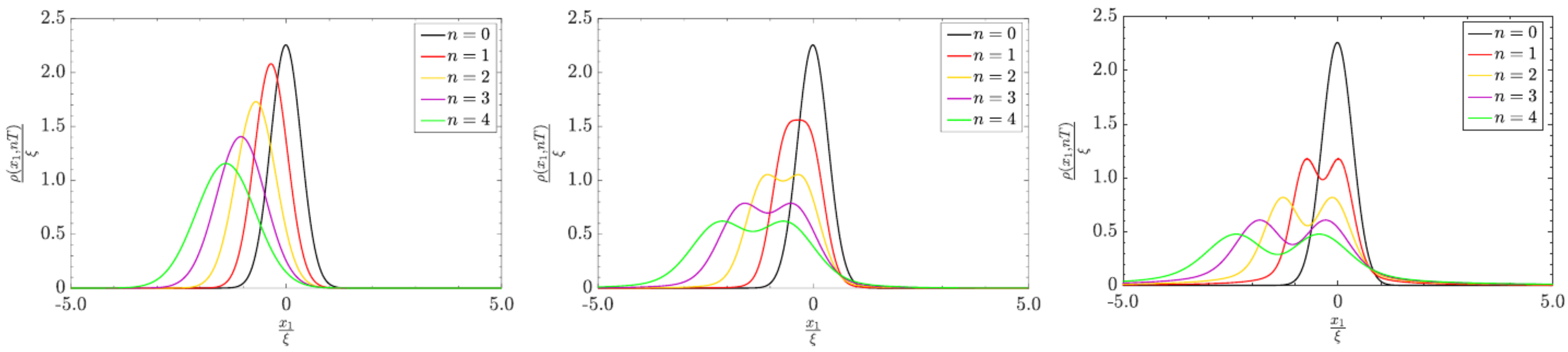
Two-body: the same happens

$$i\,\hbar\,\frac{\partial \chi}{\partial t} = \sum_{j=1}^{2}\Big(-\frac{\hbar^2}{2\,m}\frac{\partial^2}{\partial x_j^2} + V(x_j,t)\Big)\chi + V_{2b}(x_2 - x_1)\chi$$

$$y_j(t) = x_j - \xi(t).$$

$$i\hbar\frac{\partial \eta}{\partial t} = -\frac{\hbar^2}{2\,m}\left[\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right]\eta + V_{2b}(y_1 - y_2)\,\eta$$

$$H_F = \sum_{j=1}^{2}\left(\frac{\hat{p}_j^2}{2\,m} + \hat{p}_j\,\frac{\xi(nT)}{nT} - \frac{2\hbar}{nT}\theta(nT)\right) + V_{2b}(x_2 - x_1)$$



Two-body: the same happens also in 3D, but the Floquet Hamiltonian is not integrable…

For the 1D Bose gas with a periodic tilting:

$$\mathcal{L} = \frac{i\hbar}{2}\left[\psi^\dagger \frac{\partial \psi}{\partial t} - h.c.\right] - \frac{\hbar^2}{2m}\frac{\partial \psi^\dagger}{\partial x}\frac{\partial \psi}{\partial x} - \frac{\lambda}{2}\psi^\dagger\psi^\dagger\psi\psi - V(x,t)\psi^\dagger\psi$$

$$\psi(x,t) = e^{i\theta(x,t)}\varphi(y(t),t)$$

$$y = x - \xi(t)$$

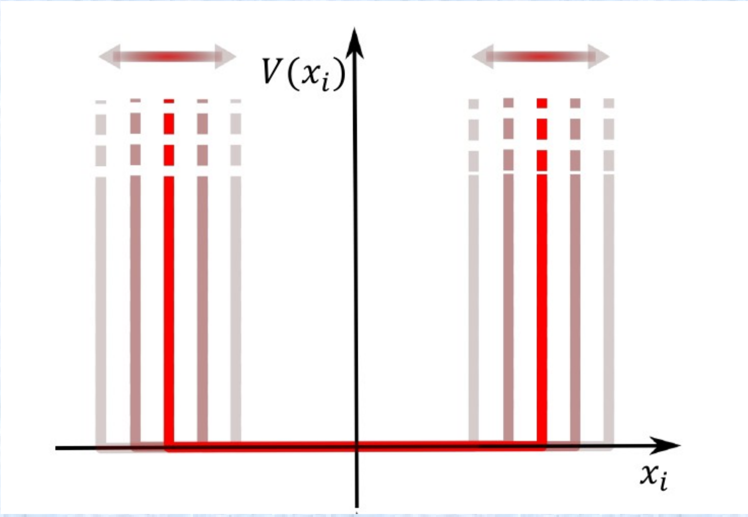
$$\mathcal{L} = \frac{i\hbar}{2}\left[\varphi^\dagger \frac{\partial \varphi}{\partial t} - h.c.\right] - \frac{\hbar^2}{2m}\frac{\partial \varphi^\dagger}{\partial y}\frac{\partial \varphi}{\partial y} - \frac{\lambda}{2}\varphi^\dagger\varphi^\dagger\varphi\varphi$$

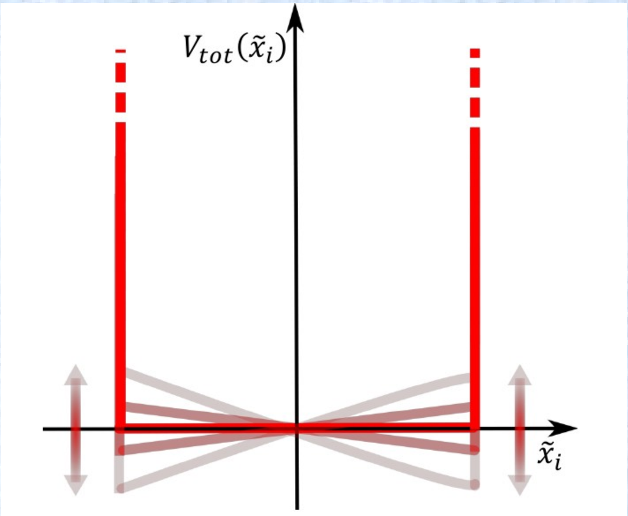
$$H_F = \sum_{j=1}^{N}\left(\frac{\hat{p}_j^2}{2\,m} + \frac{\xi(nT)}{nT}\hat{p}_j - \frac{\hbar\theta(nT)}{nT}\right) + \lambda\sum_{j<i}\delta(x_j - x_i)$$

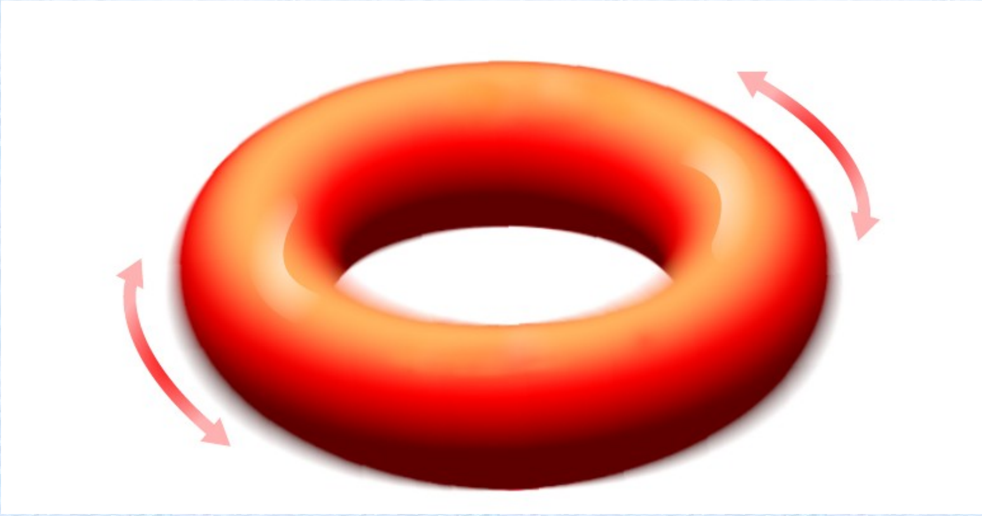
which is integrable! [A. Colcelli, G. Mussardo, G. Sierra, and A. Trombettoni, arXiv:1902.07809]

How to do it?

Shake back and forth an hard-wall or a ring:







Conclusions

→ Quantified deviations from ODLRO in 1D systems

→ Anyons interpolate between ODLRO of bosons and absence of it in fermions (non-monotonous behavior for small interactions)

→ Our results confirm the well know result that Gross-Pitaevskii works for experiments at small interactions in 1D

→ Universality of the scaling also in presence of external potential

→ An integrable Floquet Hamiltonian is found with a periodic tilting

→ Future work: 2D systems (and in particular 2D anyons)

Acknoledgements

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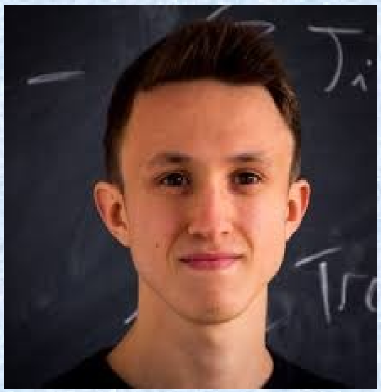
(SISSA)



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Thank you!