Vortex lattice solutions of the ZHK Chern-Simons equations

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Chern-Simons action

The (abelian) Chern-Simons (CS) action on $\mathbb{R}^3$ is [CS71]

$$S_{CS}(a) = -\frac{1}{2}\int_{\mathbb{R}^3} a \wedge \mathrm{d}a$$

where a is a 1-form.

It is one of the two gauge theories occurring in odd dimensional space-times, the other being the Maxwell action

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The CS action is gauge invariant on $\mathbb{R}^3$, and in general whenever boundary terms could be neglected.

Motivation - Condensed Matter Physics

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- We study a theory involving the Chern-Simons term, a constant external magnetic field and a double well potential, common in physics. This theory was first written down by Zhang, Hanson and Kivelson and is called the ZHK model [ZHK89].

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- We study a theory involving the Chern-Simons term, a constant external magnetic field and a double well potential, common in physics. This theory was first written down by Zhang, Hanson and Kivelson and is called the ZHK model [ZHK89].
- The ZHK model appears in the study of the fractional quantum hall effect in condensed matter physics [ZHK89].

The ZHK Chern-Simons action

The matter action we study, in the variables $(x_0, x_1, x_2) = (t, x_1, x_2)$, is

$$S_{mat}(\Psi, a, A^b) = \int_{\mathbb{R}^3} (i\bar{\Psi} D_0 \Psi - \frac{1}{2}\left|\nabla_{a+A^b}\Psi\right|^2 - \frac{g}{2}(|\Psi|^2 - 1)^2 \mathrm{d}t\mathrm{d}x$$

where $D_0\Psi = \partial_0\Psi + i(a_0 + A_0^b)\Psi$, $\nabla_{a+A^b}\Psi = \nabla\Psi + i(a + A^b)\Psi$ is the covariant derivative, $A^b = \frac{b}{2}(-x_2, x_1)$ satisfies curl $A^b = b > 0$ and $g > 0$.

We study the Euler-Lagrange equations of the ZHK action, which is

$$S_{matter}(\Psi, a, A^b) + S_{CS}(a)$$

The Zhang-Hanson-Kivelson equations

We define $\boldsymbol{A} = \boldsymbol{A}^b + \boldsymbol{a}$, then the Euler-Lagrange equations of the above action in terms of $\boldsymbol{A} = (A_0, A) = (A_0, A_1, A_2)$ are [ZHK89]

$$
\begin{aligned}
i\partial_t \Psi &= -\frac{1}{2}\Delta_A \Psi + A_0 \Psi + g(|\Psi|^2 - 1)\Psi \\
0 &= \operatorname{curl} A + |\Psi|^2 - b \qquad \text{(ZHK)} \\
* \, \partial_t A &= -curl^* A_0 + \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)
\end{aligned}
$$

where $-\Delta_A = \nabla_A^* \nabla_A$, $\operatorname{curl} A = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$, $curl^* A_0 = (\frac{\partial A_0}{\partial x_2}, -\frac{\partial A_0}{\partial x_1})$ is the adjoint of curl, and $*$ denotes the Hodge star.

The Ginzburg-Landau equations

$$\partial_t \Psi = \Delta_A \Psi - A_0 \Psi + \kappa^2 (1 - |\Psi|^2)\Psi$$
$$\partial_t A = -\operatorname{curl}^* \operatorname{curl} A - \nabla A_0 + \operatorname{Im}(\bar{\Psi} \nabla_A \Psi) \qquad \text{(GL)}$$

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These equations describe superconductors near phase transitions.

Gauge equivariance

For any function $\eta : C^{\infty}(\mathbb{R}^2) \to \mathbb{R}$, and any solution $(\Psi(x),\ \boldsymbol{A}(x))$ of the ZHK (GL) equations, the state $T_{\eta}^{\mathrm{gauge}}(\Psi(x),\ \boldsymbol{A}(x))$ defined by

$$T_{\eta}^{\mathrm{gauge}}(\Psi(x),\ \boldsymbol{A}(x)) = (e^{i\eta(x)}\Psi(x),\ \boldsymbol{A}(x) + \nabla\eta(x))$$

is also a solution of the ZHK (GL) equations.

Energy and ground state

$$E(\Psi, A) = \int_{\mathbb{R}^2} \frac{1}{2}(|\nabla_A \Psi|^2 + g(|\Psi|^4 - 2|\Psi|^2))\mathrm{d}x$$

The gauge invariant ground state $(\Psi, \boldsymbol{A}) = (0, (0, A^b))$ where curl $A^b = b$, is called the *normal state* u_0.

Brief History

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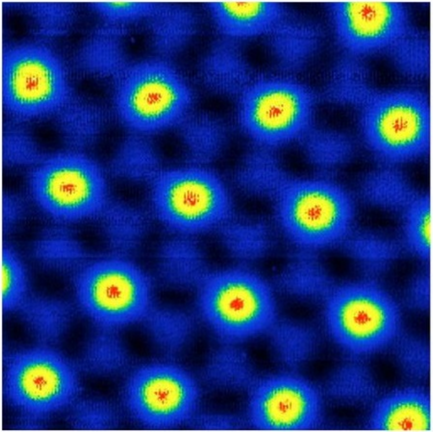
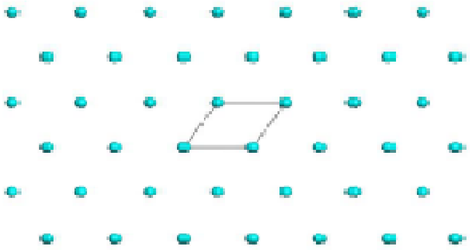
2011 Tzaneteas and Sigal rigorously proved the existence of Abrikosov lattice solutions of the GL equations.

Abrikosov lattice states in Superconductivity

Image of vortex lattice of a superconductor NbSe²



Sketch of a lattice $\mathcal{L}$.



Lattices and fundamental cells

Abrikosov lattice sate

We want time-independent solutions $(\Psi, (A_0, A))$ of the ZHK equations such that the quantities

$$\rho = |\Psi|^2 \qquad J = \mathrm{Im}(\bar{\Psi}\nabla_A \Psi)$$
$$B = \mathrm{curl}\, A \qquad A_0$$

are periodic with respect to a lattice $\mathcal{L}$. Such states $u = (\Psi, (A_0, A))$ are called *Abrikosov lattice states.*

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These states are more general than requiring Ψ and A to be $\mathcal{L}$-periodic.

Abrikosov lattice solutions

At $b = b_0 := 2g$ an Abrikosov lattice state bifurcates from the normal state, as the following theorem states.

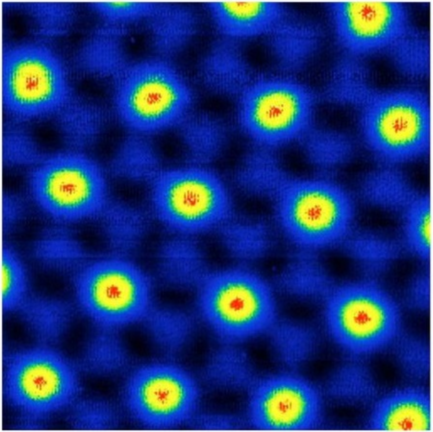
Theorem (Existence of a Bifurcation [RS18])

For any $g > 0$ and some b satisfying $0 < |2g - b| \ll 1$

1. *There exists an Abrikosov lattice state u_b, in a neighbourhood of the normal branch u_0, which solves the ZHK equations.*
2. *If $g < \frac{1}{2}$, the hexagonal lattice minimizes the average energy per lattice cell.*

Abrikosov lattice states in Superconductivity

Image of vortex lattice of a superconductor NbSe[2]



Abrikosov lattice solutions on Riemann surfaces

It turns out that $(\Psi, (A_0, A))$ is an Abrikosov lattice state iff Ψ lives on a line bundle over the $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathcal{L}}$, and A is a connection on it. This latter view point generalizes to arbitrary Riemann surfaces, and so does the bifurcation result.

Theorem (Existence of a Bifurcation [RS18])

Let $g > 0$ and suppose b satisfies $0 < |2g - b| \ll 1$. Then on a Riemann surface of genus h, as long as the first Chern number n of the line bundle satisfies $1 \leq n \leq h$, there exists an Abrikosov lattice state u_b, in a neighbourhood of the normal branch u_0.

Orbital stability of Abrikosov lattice solutions

Theorem ([Raj18])

For any solution $\Psi(t) \in C^1(\mathbb{R}^+, H^1(\mathbb{T}^2))$ of the ZHK equations, we can show that

$$\|\Psi_b - \Psi(0)\| < \delta \Rightarrow \left\|e^{-i\gamma(t)}\Psi_b - \Psi(t)\right\| < \epsilon \text{ for all } t$$

and some function $\gamma(t)$.

Key ideas behind Bifurcation theorem

- Write the time-independent ZHK equations as

$$F(b, u) = 0 \tag{3}$$

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- A bifurcation point b_0 occurs when $\mathrm{d}F_u(b_0, u_0)$ is not invertible.
- The map $\mathrm{d}F_u(b_0, u_0)$ is always non-invertible. However, this is fixed by working in the Coloumb gauge.
- The change in invertibility of $d_u F(b, u_0)$ is controlled by the following operator

$$-\frac{1}{2}\Delta_{A^b} - g$$

Its spectrum shall be studied using a Weitzenböck-type identity.

Weitzenböck-type identity

First we define

$$\partial_{A_b} = \partial - iA_c^b \qquad \partial_{A_b}^* = \bar{\partial} - i\bar{A}_c^b$$

$$A_c^b = \frac{1}{2}(A_1^b - iA_2^b)$$

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Then the Weitzenböck identity states that

$$-\frac{1}{2}\Delta_{A^b} = 2\partial_{A^b}\partial_{A_b}^* + \frac{\operatorname{curl} A^b}{2}$$

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Since $\partial_{A^b}\partial_{A_b}^* \geq 0$ and $\operatorname{curl} A^b = \frac{\partial A_1^b}{\partial x^2} - \frac{\partial A_2^b}{\partial x^1} = b$, we have

$$-\frac{1}{2}\Delta_{A^b} - g \geq \frac{b}{2} - g$$

□

Hamiltonian form of ZHK Equations

The ZHK equations can be written in Hamiltonian form using the energy functional

$$E(\Psi, A) = \int_{\mathbb{T}^2} \frac{1}{2}(|\nabla_A \Psi|^2 + g(|\Psi|^2 - 1)^2) + A_0(-\operatorname{curl} A + |\Psi|^2 + b)\mathrm{d}x$$

Then letting $J(\Psi, A) = (i\Psi, *A)$, the ZHK equations become

$$J\partial_t \begin{pmatrix} \Psi \\ A \end{pmatrix} = \nabla_{\Psi,A} E(\Psi, A_0, A)$$

$$0 = \nabla_{A_0} E(\Psi, A_0, A)$$

Stability analysis

If we let $H = \mathrm{Hess}\, E(u_b)$, then to leading order in b, we have

$$H = \begin{pmatrix} -\Delta_{A^{b_0}} - b_0 & 0 & 0 \\ 0 & 0 & -\operatorname{curl} \\ 0 & -\operatorname{curl}^* & 0 \end{pmatrix} \tag{4}$$

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The operator

$$M = \begin{pmatrix} 0 & -\operatorname{curl} \\ -\operatorname{curl}^* & 0 \end{pmatrix} \tag{5}$$

has an infinite number of negative eigenvalues, which makes stability impossible.

Modified equations

However we can use the second and third ZHK equations to solve for A and A_0 as a function of Ψ. Substituting these into the energy functional, we obtain

$$E(\Psi) = \int_{\mathbb{T}^2} \frac{1}{2}(\left|\nabla_{A(\Psi)}\Psi\right|^2 + g(|\Psi|^2 - 1)^2)\mathrm{d}x$$

whose Euler-Lagrange equation is

$$i\partial_t \Psi = -\frac{1}{2}\Delta_{A^b + A(|\Psi|)}\Psi + A_0(|\Psi|)\Psi + g(|\Psi|^2 - 1)\Psi$$

hereafter called the non-local ZHK equations.

The Hessian of the non-local energy to leading order in the bifurcation parameter is

$$-\Delta_{A^{b_0}} - b_0$$

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For the new equation, we can prove orbital stability.

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For any solution $\Psi(t) \in C^1(\mathbb{R}^+, H^1(\mathbb{T}^2))$ of the non-local ZHK equations, we can show that

$$\|\Psi_b - \Psi(0)\| < \delta \ \Rightarrow \ \left\|e^{-i\gamma(t)}\Psi_b - \Psi(t)\right\| < \epsilon \text{ for all } t$$

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Thank you for listening!

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