Non-Abelian Anyons

Statistical Repulsion and Topological Quantum Computation

Viktor Qvarfordt

Master's thesis in Mathematics

Supervised by
Douglas Lundholm

KTH Royal Institute of Technology
and Stockholm University

2017-05-23

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2. Identical quantum particles: Exchange symmetry

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3. Statistical repulsion

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4. Abstract anyon models

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5. Explicit model: Fibonacci anyons
6. Topological quantum computation with Fibonacci anyons

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Wave function, state of two identical quantum particles

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Hot topic, recent Nobel prize on Topological states of matter.

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Identifying particle configurations:

$$(\ldots, x_j, \ldots, x_k, \ldots) = (\ldots, x_k, \ldots, x_j, \ldots)$$

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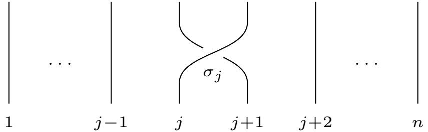
Exchange performs $\psi \mapsto U\psi$, s.t.

$$\|\psi\|^2 = \|U\psi\|^2.$$

Braid group B_n

Generators $\sigma_1, \ldots, \sigma_{n-1}$:



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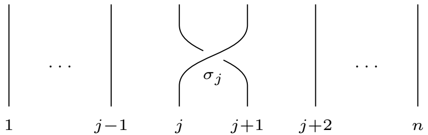

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$$\sigma_j \sigma_k = \sigma_k \sigma_j, \quad \text{if } |j-k| \geq 2,$$
$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$$
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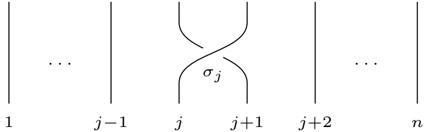

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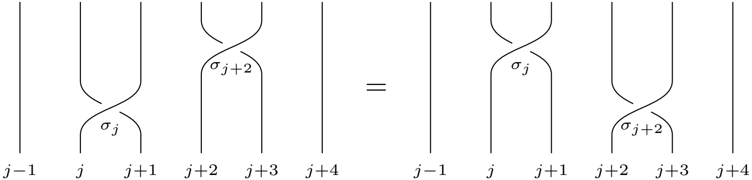

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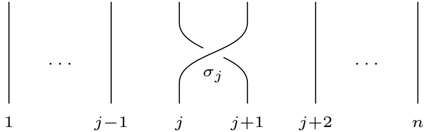

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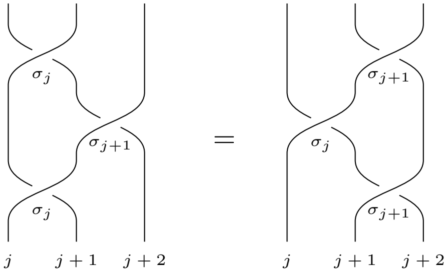

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Statistical repulsion: Introduction

Quantum state $\psi(x_1, x_2)$ of two fermions:

$$\psi(x_1, x_2) = -\psi(x_2, x_1)$$

Then $\psi(x, x) = 0$; an instance of the Pauli exclusion principle.

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Gives rise to an effective repulsion, statistical repulsion.
Kinetic energy operator: $\widehat{T} = -\nabla^2$.

Theorem (Many-body Hardy inequality for fermions)

$$\langle\psi|\widehat{T}|\psi\rangle = \int_{\mathbb{R}^{dn}} \sum_{j=1}^{n} |\nabla_j \psi|^2 dx \geq$$

$$\geq \frac{d^2}{n} \int_{\mathbb{R}^{dn}} \sum_{1 \leq j < k \leq n} \frac{|\psi|^2}{|x_j - x_k|^2} dx + \frac{1}{n} \int_{\mathbb{R}^{dn}} \left| \sum_{j=1}^{n} \nabla_j \psi \right|^2 dx.$$

Statistical repulsion: Towards anyons

Split the sum,

$$\sum_{j=1}^{n} |\nabla_j \psi|^2 = \frac{1}{n} \sum_{1 \leq j < k \leq n} |(\nabla_j - \nabla_k)\psi|^2 + \frac{1}{n} \left| \sum_{j=1}^{n} \nabla_j \psi \right|^2 .$$

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$$\int_{\mathbb{R}^{2n}} |(\nabla_j - \nabla_k)\psi|^2 \, dx = \int_{\mathbb{R}^{2(n-2)}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\nabla_{\mathsf{rel}} \psi|^2 \, 2 \, dx_{\mathsf{rel}} \, dx_{\mathsf{cm}} \, dx'.$$

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Polar coordinates $x_{\mathsf{rel}} = (r, \varphi)$,

$$\int_{\mathbb{R}^2} |\nabla_{\mathsf{rel}} \psi|^2 \, dx_{\mathsf{rel}} = \int_{r=0}^{\infty} \int_{\varphi=0}^{2\pi} \left(|\partial_r \psi|^2 + \frac{|\partial_\varphi \psi|^2}{r^2} \right) r \, d\varphi \, dr.$$

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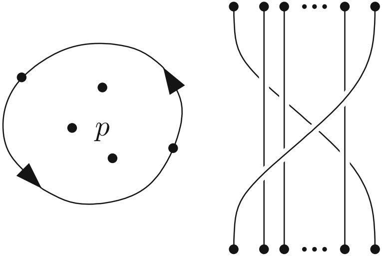

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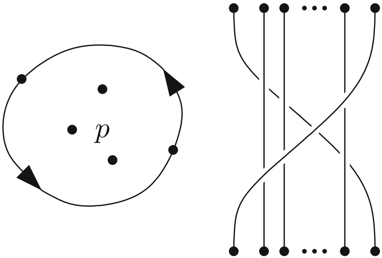

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Difficult to characterize U_p generally.

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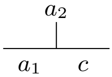

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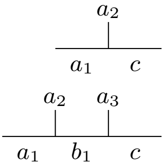

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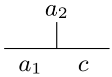

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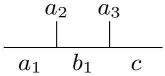

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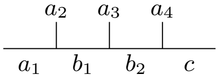

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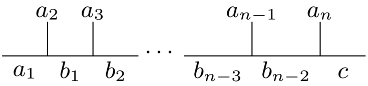
$$V^{c}_{a_1 \cdots a_n} \cong \bigoplus_{b_1, b_2, \ldots, b_{n-2}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes V^{b_3}_{b_2 a_4} \ldots \otimes V^{c}_{b_{n-2} a_n}$$

Standard basis for $V^{c}_{a_1 \cdots a_n}$:

$\Bigg\{$ 

$\Bigg|$ for all possible intermediate charges $b_1, b_2, \ldots, b_{n-2}$ $\Bigg\}$

Abstract anyon models: Fusion

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$$(a \times b) \times c = a \times (b \times c)$$

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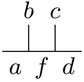

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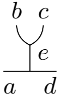

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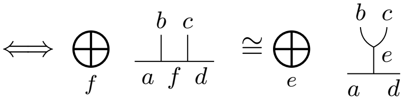
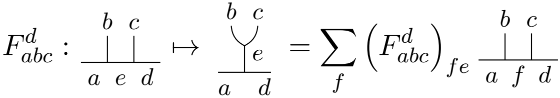
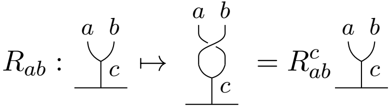


This isomorphism is given by the F operator:



Abstract anyon models: Braiding

The R operator: Isomorphism $R^c_{ab} : V^c_{ab} \to V^c_{ba}$:



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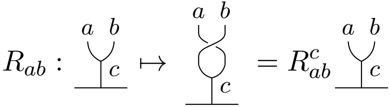

Define the B operator in terms of the F and R operator:

$$\left(B^d_{abc}\right)_{eg} = \sum_f \left(\left(F^{-1}\right)^d_{acb}\right)_{fe} R^f_{bc} \left(F^d_{abc}\right)_{gf} .$$

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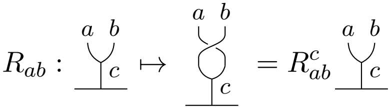

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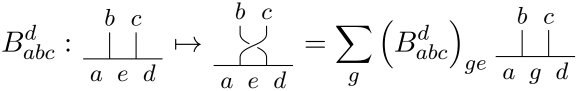

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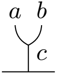
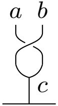
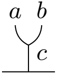
$R_{ab}:$ 

$\mapsto$ 

$= R^c_{ab}$ 

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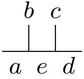
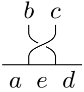
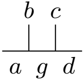
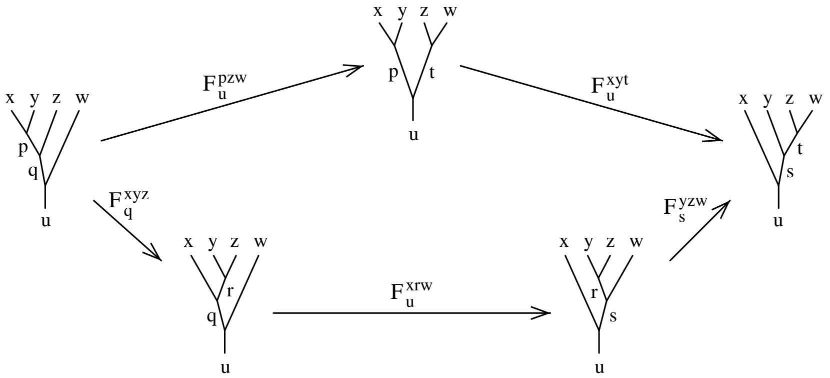
$B^d_{abc}:$ 

$\mapsto$ 

$= \sum_g \left(B^d_{abc}\right)_{ge}$ 

Symbolically: $B = F^{-1}RF$. Representation of B_n.

Abstract anyon models: Consistency: Pentagon equation



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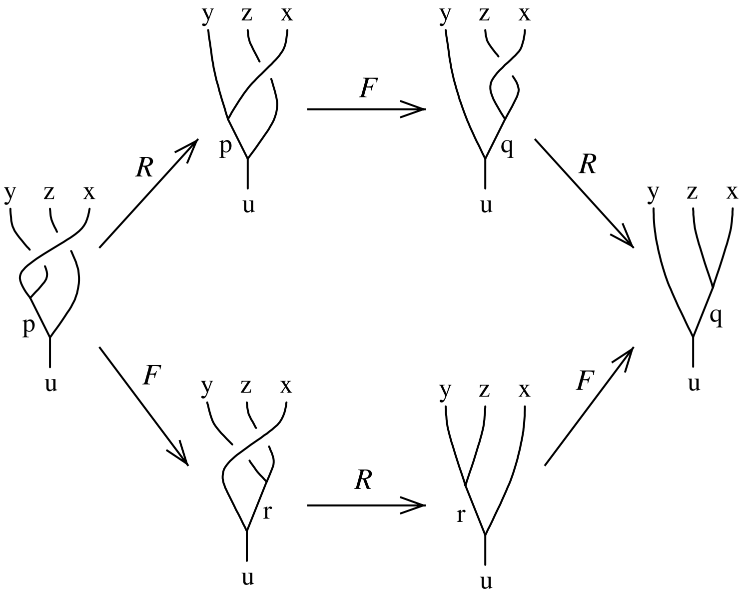

Abstract anyon models: Exchange operator U_p

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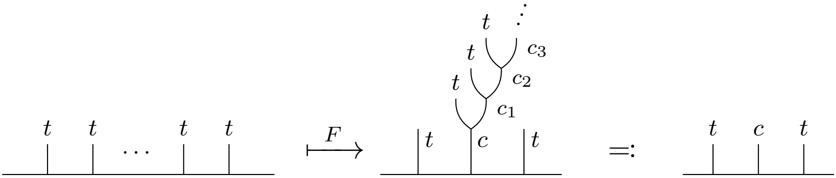
Non-trivial charge label t, fusion space $V_{t^{p+2}}$.



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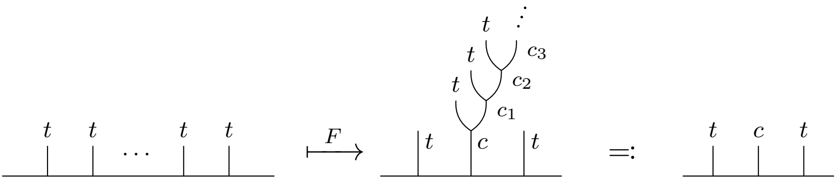

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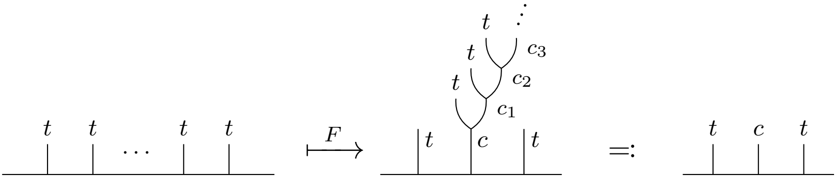

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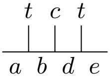
Theorem

The exchange operator is given by $U_p = \bigoplus_c U_{1,c}$ *where*

$U_{1,c}$ (

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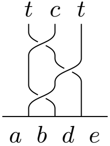
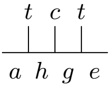


$$= \underbrace{\sum_{f,g,h} \left(B^d_{act}\right)_{fb} \left(B^e_{ftt}\right)_{gd} \left(B^g_{atc}\right)_{hf}}_{\rho(\sigma_1\sigma_2\sigma_1)}$$



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Fibonacci anyons

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Two charge labels:

- 1 (trivial, vacuum)
- τ (the Fibonacci anyon)

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\tau^n &= \mathrm{Fib}(n-1) \cdot 1 + \mathrm{Fib}(n) \cdot \tau
\end{aligned}
$$

n	0	1	2	3	4	5	6	7	$\cdots$
$\mathrm{Fib}(n)$	0	1	1	2	3	5	8	13	$\cdots$

Fibonacci anyons: Solving the model

Solving the Pentagon and Hexagon equations gives

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} (F^{\tau}_{\tau\tau\tau})_{11} & (F^{\tau}_{\tau\tau\tau})_{1\tau} \\ (F^{\tau}_{\tau\tau\tau})_{\tau 1} & (F^{\tau}_{\tau\tau\tau})_{\tau\tau} \end{pmatrix} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$$

$$R_{\tau\tau} = \begin{pmatrix} R^{1}_{\tau\tau} & 0 \\ 0 & R^{\tau}_{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix}$$

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$$\varphi = \lim_{n\to\infty} \frac{\mathrm{Fib}(n)}{\mathrm{Fib}(n-1)} = \frac{1+\sqrt{5}}{2}$$

is the golden ratio.

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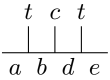
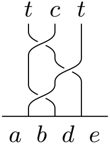
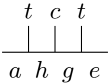
$U_{1,c}$ (

) = 

$= \rho(\sigma_1\sigma_2\sigma_1)$ 

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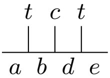
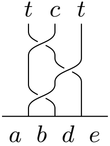
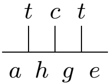
$U_{1,c}\left(\right.$ 

$\left.\right) =$ 

$= \rho(\sigma_1\sigma_2\sigma_1)$ 

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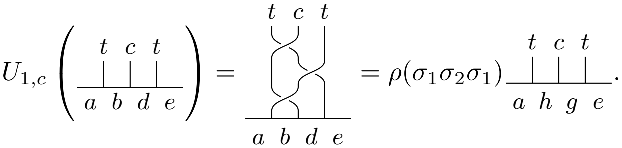

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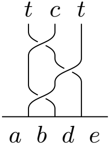

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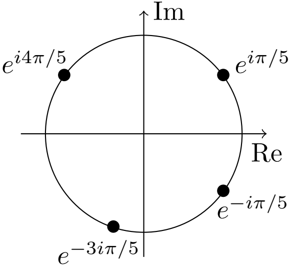

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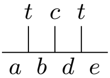
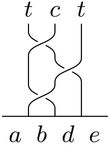
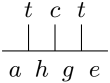
$U_{1,c}\Bigg($ 

$\Bigg) =$ 

$= \rho(\sigma_1\sigma_2\sigma_1)$ 

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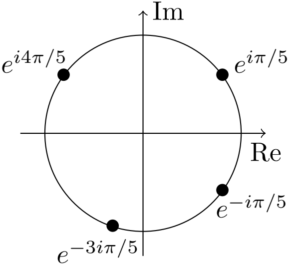

Statistical repulsion: $\lambda_{0,0}^2 = (3/5)^2, \quad \lambda_{0,p}^2 = (1/5)^2$ *for* $p \geq 1$.

Topological quantum computation with Fibonacci anyons

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Define the computational basis as

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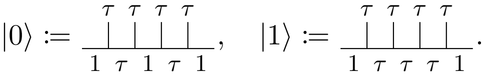
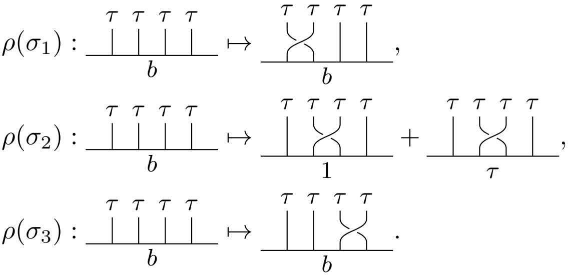


Braid group generators



Topological quantum computation with Fibonacci anyons

Available manipulations on the Fibonacci qubit

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and their inverses.

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Quantum dimension:

$$d_\tau = \lim_{n\to\infty} \frac{\dim\left(V^1_{\tau^{n+1}}\right)}{\dim\left(V^1_{\tau^n}\right)} = \lim_{n\to\infty} \frac{\mathrm{Fib}(n)}{\mathrm{Fib}(n-1)} = \varphi$$

compare to distinguishable particles with $h = \mathbb{C}^k$

$$\mathcal{H}_n = h^{\otimes n}, \quad d = \frac{\dim \mathcal{H}_{n+1}}{\dim \mathcal{H}_n} = \dim h = k \in \mathbb{N}.$$

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- `viktorq.se/nonabelions.pdf`