Edge universality in interacting $2d$ topological insulators

Marcello Porta

Joint with: G. Antinucci (UZH) and V. Mastropietro (Milan)

Summary

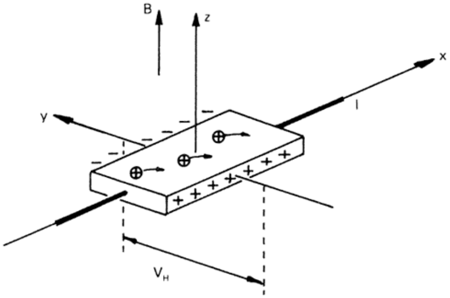
- Introduction: edge transport in noninteracting quantum Hall systems and time-reversal invariant systems. Bulk-edge duality.
- Many-body quantum systems. Results:
  - Edge transport coefficients for quantum Hall and TRI systems.
  - Interacting bulk-edge correspondence, Haldane relations.
- Sketch of the proof: Renormalization group and Ward identities.
- Conclusions.

Introduction: noninteracting systems

Integer quantum Hall effect

- Bulk topological order in condensed matter systems is deeply related to the emergence of gapless edge modes.

- Example. Integer quantum Hall effect [von Klitzing *et al.* '80]
  $2d$ insulators exposed to strong magnetic field and in-plane electric field.

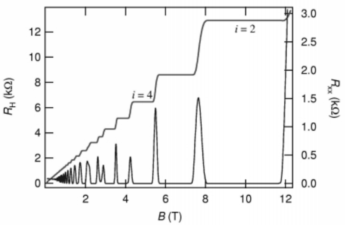


Integer quantum Hall effect

- Bulk topological order in condensed matter systems is deeply related to the emergence of gapless edge modes.

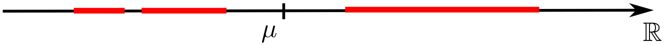
- Example. Integer quantum Hall effect [von Klitzing *et al.* '80]
  $2d$ insulators exposed to strong magnetic field and in-plane electric field.
  Linear response: $J = \sigma E + o(E)$ with $\sigma =$ conductivity matrix:

$$\sigma = \begin{pmatrix} 0 & \frac{n}{2\pi} \\ -\frac{n}{2\pi} & 0 \end{pmatrix}\ , \qquad n \in \mathbb{Z}.$$



Integer quantum Hall effect: theory

- Noninteracting fermions. H = 1-particle Hamiltonian, on $\ell^2(\mathbb{Z}^2; \mathbb{C}^M)$.
  Suppose that $\sigma(H)$ is gapped, μ = Fermi level $\in \text{gap}(H)$.



Integer quantum Hall effect: theory

- Noninteracting fermions. H = 1-particle Hamiltonian, on $\ell^2(\mathbb{Z}^2; \mathbb{C}^M)$. Suppose that $\sigma(H)$ is gapped, μ = Fermi level $\in$ gap(H).
- For simplicity, $H(x;y) \equiv H(x-y)$. Bloch decomp.: $H = \int_{\mathbb{T}^2}^{\oplus} dk\, \hat{H}(k)$ Let $\hat{P}_\mu(k) = \chi(\hat{H}(k) \leq \mu)$ = Fermi projector. Thouless *et al.* '82:

$$\boxed{\sigma_{12} = i \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2} \, \mathrm{Tr}_{\mathbb{C}^M} \hat{P}_\mu(k)[\partial_{k_1} \hat{P}_\mu(k), \partial_{k_2} \hat{P}_\mu(k)] \in \frac{1}{2\pi}\mathbb{Z}}$$

σ_{12} = Chern number of Bloch bundle:

$$\mathcal{E}_B = \{(k,u) \in \mathbb{T}^2 \times \mathbb{C}^M \mid u \in \mathrm{Ran}\hat{P}_\mu(k)\}$$

Integer quantum Hall effect: theory

- Noninteracting fermions. H = 1-particle Hamiltonian, on $\ell^2(\mathbb{Z}^2;\mathbb{C}^M)$. Suppose that $\sigma(H)$ is gapped, μ = Fermi level $\in$ gap(H).
- For simplicity, $H(x;y) \equiv H(x-y)$. Bloch decomp.: $H = \int_{\mathbb{T}^2}^{\oplus} dk\, \hat{H}(k)$
  Let $\hat{P}_\mu(k) = \chi(\hat{H}(k) \leq \mu)$ = Fermi projector. Thouless *et al.* '82:

$$\boxed{\sigma_{12} = i \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2}\, \mathrm{Tr}_{\mathbb{C}^M} \hat{P}_\mu(k)[\partial_{k_1}\hat{P}_\mu(k), \partial_{k_2}\hat{P}_\mu(k)] \in \frac{1}{2\pi}\mathbb{Z}}$$

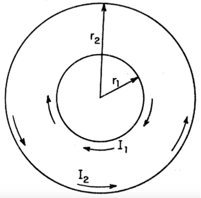
  σ_{12} = Chern number of Bloch bundle:

$$\mathcal{E}_B = \{(k,u) \in \mathbb{T}^2 \times \mathbb{C}^M \mid u \in \mathrm{Ran}\hat{P}_\mu(k)\}$$

- IQHE for general (disordered) systems:
  - Bellissard *et al.* '94. σ_{12} = Noncommutative Chern number.
  - Avron-Seiler-Simon '94. σ_{12} = index of a pair of projections.
  - Aizenman-Graf '98. Mobility gap regime.

Edge states in quantum Hall systems

- Halperin '82. Hall phases must come with robust edge currents.



- Field theoretic approach. For a weak, slowly varying vector potential A,

$$
\begin{aligned}
\frac{\mathcal{Z}(A)}{\mathcal{Z}(0)} &= e^{i\sigma_{12}\int_\Omega A\wedge dA + \text{irr.}} && \text{(gap assumption)} \\
&= e^{i\sigma_{12}\int_\Omega (A+d\alpha)\wedge d(A+d\alpha) + \text{irr.}} && \text{(gauge inv.)} \\
&= \frac{\mathcal{Z}(A)}{\mathcal{Z}(0)} e^{i\sigma_{12}\int_{\partial\Omega} d\alpha\wedge A + \text{irr.}} && \text{(Stokes)}
\end{aligned}
$$

$\sigma_{12} \neq 0 \Rightarrow$ The gap assumption cannot be true! [Fröhlich '90+].

Edge states in quantum Hall systems: more precise

- Let H be a lattice Schrödinger operator on the cylinder:

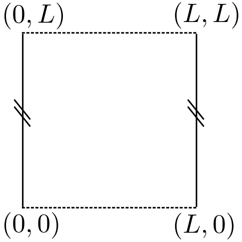


Figure: Dotted lines: Dirichlet boundary conditions. Identify vertical sides.

Edge states in quantum Hall systems: more precise

- Let H be a lattice Schrödinger operator on the cylinder:
- Let H_p the counterpart of H with periodic b.c.. Hyp.: H_p is gapped.

Edge states in quantum Hall systems: more precise

- Let H be a lattice Schrödinger operator on the cylinder:
- Let H_p the counterpart of H with periodic b.c.. Hyp.: H_p is gapped. $\sigma(H)$ might differ from $\sigma(H_p)$ by the presence of edge states.

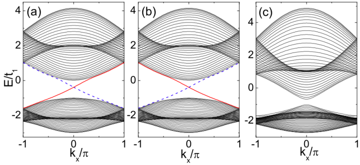


Figure: $H = \int_{\mathbb{T}^1}^{\oplus} dk_1\, \hat{H}(k_1)$, $\hat{H}(k_1) = 1d$ Hamiltonian. Spectrum of $\hat{H}(k_1)$.

- Red curve: eigenvalue branch $\varepsilon(k_1)$, with eigenstates (edge modes)

$$\varphi_x(k_1) = e^{ik_1 x_1} \xi_{x_2}(k_1)\,, \qquad \text{with } \xi_{x_2}(k_1) \sim e^{-cx_2}\,.$$

The bulk-edge correspondence

- Bulk-edge duality: relation between σ_{12} of H_p and the edge states of H.

$$\boxed{\sigma_{12} = \sum_e \frac{\omega_e}{2\pi}}$$

with $\omega_e = \pm 1$ (chirality of the edge state.)

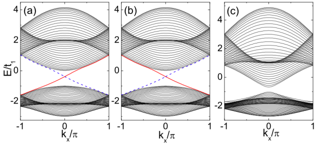


Figure: $(a): \ \sigma_{12} = \frac{1}{2\pi}$, $(b): \sigma_{12} = -\frac{1}{2\pi}$, $(c): \sigma_{12} = 0.$

The bulk-edge correspondence

- Bulk-edge duality: relation between σ_{12} of H_p and the edge states of H.

$$\boxed{\sigma_{12} = \sum_e \frac{\omega_e}{2\pi}}$$

with $\omega_e = \pm 1$ (chirality of the edge state.)

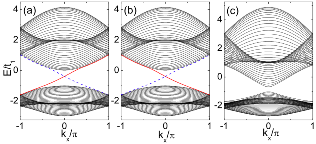


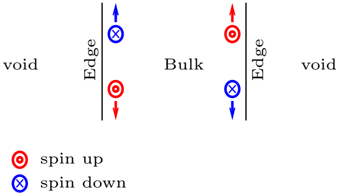
Figure: $(a): \sigma_{12} = \frac{1}{2\pi}$, $(b): \sigma_{12} = -\frac{1}{2\pi}$, $(c): \sigma_{12} = 0$.

- Rigorous results for noninteracting systems:
  - Hatsugai, '93: Translation invariant systems.
  - Schulz-Baldes *et al.* '00: Disordered systems (with bulk gap).
  - Graf et al. '02: Mobility gap regime.

Time-reversal invariant systems

- Quantum Hall systems are an example of topological insulators.
  Necessary condition for $\sigma_{12} \neq 0$: breaking of TRS (magnetic field).

- Unbroken TRS: charge transport is trivial but spin transport is possible.



- Spin Hall effect: Murakami-Nagaosa-Zhang '03, ... (Fröhlich et al. '93.) Model: Kane-Mele '05. Discovery: Bernevig-Hughes-Zhang '06 (theory), König *et al.* '07.

Edge spin transport

- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1\, \hat{H}(k_1)$.
  TRS: $\hat{H}(k_1) = \Theta^{-1}\hat{H}(-k_1)\Theta$, with $\Theta^2 = -1$, Θ antiunitary.

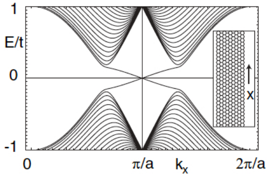
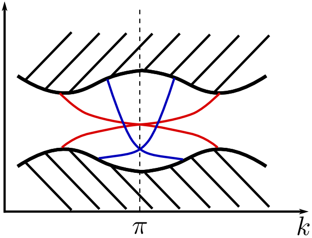


Figure: $\sigma(\hat{H}(k_1))$ for the Kane-Mele model. $\sigma(\hat{H}(k_1)) = \sigma(\hat{H}(-k_1))$.

- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of H: parity of # of pairs of edge modes at μ.

Edge spin transport

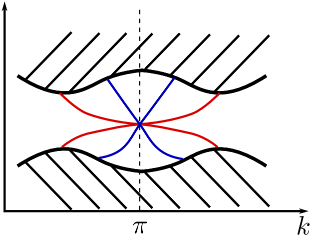
- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1\, \hat{H}(k_1)$.
  TRS: $\hat{H}(k_1) = \Theta^{-1}\hat{H}(-k_1)\Theta$, with $\Theta^2 = -1$, Θ antiunitary.



- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of H: parity of # of pairs of edge modes at μ.

Edge spin transport

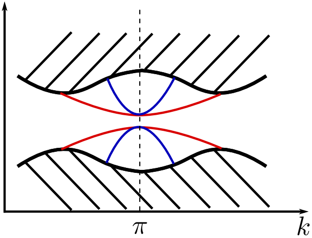
- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1\, \hat{H}(k_1)$.
  TRS: $\hat{H}(k_1) = \Theta^{-1}\hat{H}(-k_1)\Theta$, with $\Theta^2 = -1$, Θ antiunitary.



- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of H: parity of # of pairs of edge modes at μ.

Edge spin transport

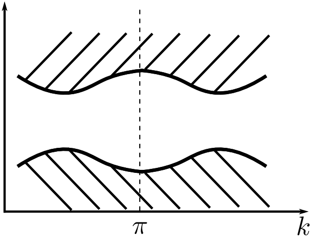
- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1\, \hat{H}(k_1)$.
  TRS: $\hat{H}(k_1) = \Theta^{-1}\hat{H}(-k_1)\Theta$, with $\Theta^2 = -1$, Θ antiunitary.



- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of H: parity of # of pairs of edge modes at μ.

Edge spin transport

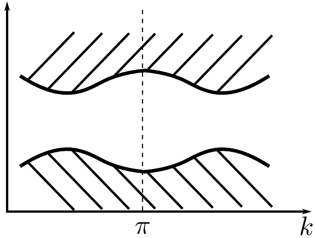
- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1\, \hat{H}(k_1)$.
  TRS: $\hat{H}(k_1) = \Theta^{-1}\hat{H}(-k_1)\Theta$, with $\Theta^2 = -1$, Θ antiunitary.



- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of H: parity of # of pairs of edge modes at μ.

Edge spin transport

- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{\mathbb{T}^1}^{\oplus} dk_1 \, \hat{H}(k_1)$.
  TRS: $\hat{H}(k_1) = \Theta^{-1}\hat{H}(-k_1)\Theta$, with $\Theta^2 = -1$, Θ antiunitary.



- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of H: parity of # of pairs of edge modes at μ.

- Bulk $\mathbb{Z}_2$ classif. is also possible (no direct connection with transport).

- Graf-P. '13: bulk-edge duality for TRI systems.

Many-body quantum systems

Many-body quantum systems

- Interacting many-body Fermi system on $\Lambda_L \subset \mathbb{Z}^2$.
- Fock space Hamiltonian: $\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{V}$ with

$$\mathcal{H}_0 = \sum_{x,y} \sum_{\sigma,\sigma'} a^+_{x,\sigma} H(x,y) a^-_{y,\sigma} \ , \quad \mathcal{V} = \sum_{x,y} \sum_{\sigma,\sigma'} v(x-y) a^+_{x,\sigma} a^+_{y,\sigma'} a^-_{y,\sigma'} a^-_{x,\sigma}$$

with $\{a^+_{x,\sigma}, a^-_{y,\sigma'}\} = \delta_{x,y}\delta_{\sigma,\sigma'}$, $\{a^+_{x,\sigma}, a^+_{y,\sigma'}\} = \{a^-_{x,\sigma}, a^-_{y,\sigma'}\} = 0$,

and H, v finite ranged.

- Finite volume, finite temperature Gibbs state:

$$\langle \cdot \rangle_{\beta,L} = \frac{\mathrm{Tr} \cdot e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathcal{Z}_{\beta,L}} \ , \qquad \mathcal{Z}_{\beta,L} = \mathrm{Tr}\, e^{-\beta(\mathcal{H}-\mu\mathcal{N})} \ , \qquad \beta = 1/T$$

with $\mu =$ chemical potential and:

$$\mathcal{N} = \sum_x \sum_\sigma a^+_{x,\sigma} a^-_{x,\sigma} \equiv \sum_x n_x \ .$$

Interacting bulk transport

- Periodic boundary conditions. Many-body Kubo formula:

$$\sigma_{ij} = \lim_{\eta \to 0^+} \lim_{\beta, L \to \infty} \frac{i}{\eta L^2} \Big(\int_{-\infty}^{0} dt\, e^{\eta t} \, \langle \big[\mathcal{J}_i(t), \mathcal{J}_j \big] \rangle_{\beta,L} - \langle \big[\mathcal{J}_i, \mathcal{X}_j \big] \rangle_{\beta,L} \Big)$$

$\mathcal{X} = \sum_x x n_x$, $\mathcal{J} = i[\mathcal{H}\,, \mathcal{X}] =$ current operator and $\mathcal{J}_i(t) = e^{i\mathcal{H}t} \mathcal{J}_i e^{-i\mathcal{H}t}$.

- Hastings-Michalakis '15. Quantization of σ_{12}. (Quasi-adiabatic methods)
  Hyp. The ground state of $\mathcal{H}$ is gapped.

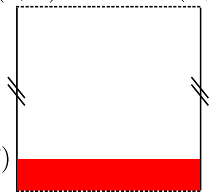
- Giuliani-Mastropietro-P. '16. Universality of σ_{ij}. (Ward identities)
  Hyp. Fast enough algebraic decay of corrs. E.g.: $\mathcal{H}_0$ gapped, λ small; graphene-like models (+Jauslin '16: critical Haldane model, RG).

- Bachmann-de Roeck-Fraas '17. Validity of Kubo formula.
  Hyp. Gapped (time-dependent) Hamiltonians.

Interacting edge transport

- Edge transport. Localize observables at distance $\leq \ell$ from $x_2 = 0$.

(Cylindric boundary conditions & transl. inv. in direction x_1.)



- Interesting quantities: charge density n_x and current density $\vec{j}_x$,

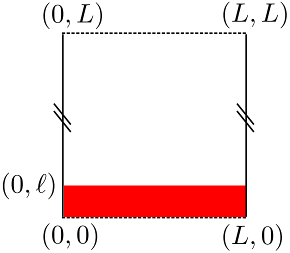
$$n_x = a_x^+ a_x^- \ , \qquad \partial_t n_x(t) + \mathrm{div}_x \vec{j}_x(t) = 0 \ .$$

Let: $\hat{n}_{p_1} = \sum_{x=(x_1,x_2)} e^{ip_1 x_1} n_x$ and $\hat{n}^{\ell}_{p_1} = \sum_{x_1} e^{ip_1 x_1} \sum_{x_2 \leq \ell} n_x$.

Interacting edge transport

- Edge transport. Localize observables at distance $\leq \ell$ from $x_2 = 0$.

(Cylindric boundary conditions & transl. inv. in direction x_1.)



- Interesting quantities: charge density n_x and current density $\vec{j}_x$,

$$n_x = a_x^+ a_x^- \ , \qquad \partial_t n_x(t) + \mathrm{div}_x \vec{j}_x(t) = 0 \ .$$

Let: $\hat{n}_{p_1} = \sum_{x=(x_1,x_2)} e^{ip_1 x_1} n_x$ and $\hat{n}^{\ell}_{p_1} = \sum_{x_1} e^{ip_1 x_1} \sum_{x_2 \leq \ell} n_x$.

- Spin transport (current well defined if $[\mathcal{H}, \mathcal{S}_3] = 0$, with $\mathcal{S}_3 = \sum_\sigma \sigma n_{x,\sigma}$):

$$n_x \to n_{x,\uparrow} - n_{x,\downarrow} \ , \qquad j_{1,x} \to j_{1,x,\uparrow} - j_{1,x,\downarrow}$$

Edge transport coefficients

- Edge charge susceptibility:

$$\kappa^{\ell}(\eta, p_1) \quad := \quad - \lim_{\beta, L \to \infty} \frac{i}{L} \int_{-\infty}^{0} dt\, e^{t\eta} \, \langle [\hat{n}_{p_1}(t) \, , \hat{n}^{\ell}_{-p_1}] \rangle_{\beta, L}$$

(response of the edge density to a density perturbation)

- Edge charge conductance:

$$G^{\ell}(\eta, p_1) \quad := \quad - \lim_{\beta, L \to \infty} \frac{i}{L} \int_{-\infty}^{0} dt\, e^{t\eta} \, \langle [\hat{n}_{p_1}(t) \, , \hat{j}^{\ell}_{1,-p_1}] \rangle_{\beta, L}$$

(response of the edge current to a density perturbation)

- Edge Drude weight:

$$D^{\ell}(\eta, p_1) \quad := \quad \lim_{\beta, L \to \infty} \frac{i}{L} \int_{-\infty}^{0} dt\, e^{t\eta} \, \langle [\hat{j}_{1,p_1}(t) \, , \hat{j}^{\ell}_{1,-p_1}] \rangle_{\beta, L}$$

(response of the edge current to an electric field)

Effective description of the edge modes

- Effective $1d$ theory for a single edge mode: chiral Luttinger model.

$$\mathcal{H}_{\chi L} = \sum_{\sigma=\uparrow\downarrow} \int dk\, v_{\mathrm{e}} k\, \hat{a}^{+}_{k,\sigma} \hat{a}^{-}_{k,\sigma} + \lambda \int dp dk dk'\, \hat{a}^{+}_{k+p,\uparrow} \hat{a}^{+}_{k'-p,\downarrow} \hat{a}^{-}_{k,\downarrow} \hat{a}^{-}_{k',\uparrow}$$

- Wen '90. Theory of interacting Hall edge currents based on χL.
  Advantage: χL exactly solvable by bosonization [Mattis-Lieb '65.]

- Effective $1d$ theory for TRI systems: helical Luttinger model.

$$\mathcal{H}_{HL} = \sum_{\sigma=\uparrow\downarrow} \int dk\, \sigma v_{\mathrm{e}} k\, \hat{a}^{+}_{k,\sigma} \hat{a}^{-}_{k,\sigma} + \lambda \int dp dk dk'\, \hat{a}^{+}_{k+p,\uparrow} \hat{a}^{+}_{k'-p,\downarrow} \hat{a}^{-}_{k,\downarrow} \hat{a}^{-}_{k',\uparrow}$$

Effective description of the edge modes

- Effective $1d$ theory for a single edge mode: chiral Luttinger model.

$$\mathcal{H}_{\chi L} = \sum_{\sigma=\uparrow\downarrow} \int dk\, v_{\mathrm{e}} k\, \hat{a}^{+}_{k,\sigma} \hat{a}^{-}_{k,\sigma} + \lambda \int dp dk dk'\, \hat{a}^{+}_{k+p,\uparrow} \hat{a}^{+}_{k'-p,\downarrow} \hat{a}^{-}_{k,\downarrow} \hat{a}^{-}_{k',\uparrow}$$

- Wen '90. Theory of interacting Hall edge currents based on χL.
  Advantage: χL exactly solvable by bosonization [Mattis-Lieb '65.]

- Effective $1d$ theory for TRI systems: helical Luttinger model.

$$\mathcal{H}_{HL} = \sum_{\sigma=\uparrow\downarrow} \int dk\, \sigma v_{\mathrm{e}} k\, \hat{a}^{+}_{k,\sigma} \hat{a}^{-}_{k,\sigma} + \lambda \int dp dk dk'\, \hat{a}^{+}_{k+p,\uparrow} \hat{a}^{+}_{k'-p,\downarrow} \hat{a}^{-}_{k,\downarrow} \hat{a}^{-}_{k',\uparrow}$$

- Remark. Integrability is nongeneric: broken by, e.g., nonlinearities of the dispersion relation or by the bulk degrees of freedom.

Interacting edge transport: single mode edge currents

Theorem (Antinucci-Mastropietro-P., Comm. Math. Phys. '18)

Suppose that H has one edge mode per edge. Then, $\exists\ \lambda_0 > 0$ s.t. for $|\lambda| < \lambda_0$ the Gibbs state is analytic in λ. Moreover, the edge transport coefficients are:

$$\begin{aligned} \kappa^\ell(\eta, p_1) &= \frac{1}{\pi|v|}\frac{vp_1}{-i\eta + vp_1} + R^\ell_\kappa(\eta, p_1) \\ G^\ell(\eta, p_1) &= \frac{\omega}{\pi}\frac{vp_1}{-i\eta + vp_1} + R^\ell_G(\eta, p_1)\,, \qquad (\omega = sgn(v)) \\ D^\ell(\eta, p_1) &= \frac{|v|}{\pi}\frac{-i\eta}{-i\eta + vp_1} + R^\ell_D(\eta, p_1) \end{aligned}$$

$v \equiv v(\lambda) =$ *dressed Fermi velocity,* $\lim_{\ell\to\infty}\lim_{\eta,p_1\to 0} R^\ell_\sharp(\eta, p_1) = 0.$

- The results agrees with the predictions based on bosonization:

  "edge states $\simeq$ noninteracting $1d$ Bose gas".

- Proof based on renormalization group methods and on a rigorous comparison with χL [Benfatto-Falco-Mastropietro '10+].

Bulk-edge correspondence and Haldane relations

- Interacting bulk-edge duality:

$$G = \lim_{\ell\to\infty} \lim_{p_1,\eta\to 0^+} G^\ell(\eta, p_1) = \frac{\omega}{\pi} \underbrace{=}_{\text{bulk-edge corresp.}} \sigma_{12}(\lambda = 0)$$

The bulk-edge duality follows from bulk universality: $\sigma_{12}(0) = \sigma_{12}(\lambda)$.

- In contrast, the Drude weight and the susceptibility are nonuniversal:

$$\kappa = \lim_{\ell\to\infty} \lim_{p_1,\eta\to 0^+} \kappa^\ell(\eta, p_1) = \frac{1}{\pi|v|}$$

$$D = \lim_{\ell\to\infty} \lim_{\eta,p_1\to 0^+} D^\ell(\eta, p_1) = \frac{|v|}{\pi}\,. \qquad (v \equiv v(\lambda))$$

Nevertheless, they satisfy the Haldane relation:

$$\frac{D}{\kappa} = v^2$$

first predicted to hold for $1d$ systems by [Haldane '80].

Interacting edge transport: TRI systems

Theorem (Mastropietro-P., Phys. Rev. B '17)

Suppose that $\mathcal{H}$ is TRS, and that H has one pair of edge states per edge. Also, suppose that $[\mathcal{H}, \mathcal{S}_3] = 0$. Then, $\exists\, \lambda_0 > 0$ s.t. for $|\lambda| < \lambda_0$:

$$G^s = \frac{\omega}{\pi}\,, \qquad \omega = sgn(v)\,.$$

Moreover the charge and spin edge Drude weights and susceptibilities are:

$$\kappa^c = \frac{K}{\pi v}\,, \qquad D^c = \frac{vK}{\pi}\,, \qquad \kappa^s = \frac{1}{\pi v K}\,, \qquad D^s = \frac{v}{\pi K}$$

with $K \equiv K(\lambda) = 1 + O(\lambda) \neq 1$, $v \equiv v(\lambda) = v_\uparrow + O(\lambda)$. Finally, the 2-point function decays with anomalous exponent $\eta = (K + K^{-1} - 2)/2$.

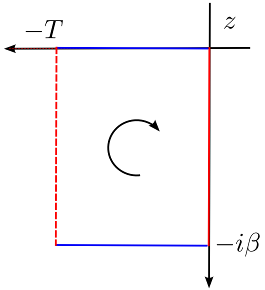
Remark. In the single edge mode case, $K = 1$ (no anomalous exponents.)

Sketch of the proof
(one edge mode)

Wick rotation

- Analytic continuation to imaginary times. Let $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}_+$. We have:

$$\int_{-\infty}^{0} dt\, e^{t\eta_\beta} \, \langle [A(t)\,,B]\rangle_{\beta,L} = i\int_0^\beta dt\, e^{-it\eta_\beta}\langle A(-it)B\rangle_{\beta,L}$$



- Errors (dotted red) estimated via bounds on Euclidean correlations:

$$|\langle A(T-it)B\rangle_{\beta,L}| \leq \langle A(-it)A(-it)^*\rangle_{\beta,L}^{1/2}\langle B^*B\rangle_{\beta,L}^{1/2}$$

Perturbation theory

- Transport coefficients can be expressed via imaginary time correlations:

$$G^{\ell}(\eta, p_1) = \lim_{\beta, L\to\infty} \int_{-\beta/2}^{\beta/2} e^{-i\eta t} \frac{1}{L} \langle \mathbf{T}\, \hat{n}_{p_1}(-it)\,;\hat{j}^{\ell}_{-p_1}\rangle_{\beta,L}\,.$$

- Let $A_t \equiv A(-it)$. Perturbative expansion of Euclidean correlations:

$$\langle \mathbf{T}\, A_t\,;B\rangle_{\beta,L} = \sum_{n\geq 0} \frac{\lambda^n}{n!} \int_{[0,\beta)^n} dt_1 \ldots dt_n \langle \mathbf{T}\, A_t\,;B\,;\mathcal{V}_{t_1}\,;\cdots\,;\mathcal{V}_{t_n}\rangle_{\beta,L}\big|_{\lambda=0}$$

$\rightsquigarrow$ Expansion in terms of Feynman diagrams. Propagator ($\beta, L\to\infty$):

$$g(t_1,x;t_2,y) = \langle \mathbf{T}\, a^-_{(t_1,x)} a^+_{(t_2,y)}\rangle\big|_{\lambda=0} =$$
$$\theta(t_1-t_2)e^{(t_2-t_1)(H-\mu)}P^{\perp}_{\mu}(H) - \theta(t_2-t_1)e^{(t_2-t_1)(H-\mu)}P_{\mu}(H)$$

- Problems. 1) $(2n)!$ diagrams; 2) gapless modes: slow space-time decay.

Grassmann integral formulation

$$\frac{\mathrm{Tr}\, e^{-\beta\mathcal{H}}}{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0}} = \int \mu(d\psi) e^{V(\psi)}$$

- $\psi^{\pm}_{\mathbf{x}}$ = Grassmann field, $V(\psi) = \text{“}\lambda\psi^4\text{”}$, $\mu(d\psi) = e^{-(\psi^+, g^{-1}\psi^-)} D\psi$
- $\psi = \psi_{\mathrm{e}} + \psi_{\mathrm{b}}$, where ψ_{b} has gapped covariance $g_{\mathrm{b}} \equiv g\chi(|H - \mu| > \delta)$.

$$\int \mu(d\psi) e^{V(\psi)} = \int \mu_{\mathrm{e}}(d\psi_{\mathrm{e}}) \mu_{\mathrm{b}}(d\psi_{\mathrm{b}}) e^{V(\psi_{\mathrm{e}} + \psi_{\mathrm{b}})} \equiv e^{F^{(\mathrm{b})}_{\beta,L}} \int \mu_e(d\psi_e) e^{V^{(\mathrm{e})}(\psi_{\mathrm{e}})}$$

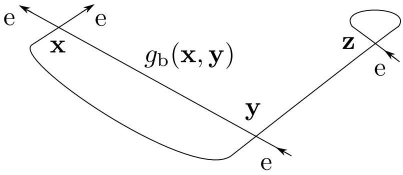


Figure: $V^{(\mathrm{e})}(\psi_{\mathrm{e}}) = \sum_P W^{(\mathrm{e})}_P \psi_{\mathrm{e}}(P)$. Contractions ≡ bulk props.
$O(\lambda^3)$ contribution, proportional to $\psi^+_{\mathbf{x}}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{y}}\psi^-_{\mathbf{z}}$ ($\mathbf{x} = (t, x)$).

Grassmann integral formulation

$$\frac{\mathrm{Tr}\, e^{-\beta\mathcal{H}}}{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0}} = \int \mu(d\psi) e^{V(\psi)}$$

- $\psi^{\pm}_{\mathbf{x}} =$ Grassmann field, $\quad V(\psi) =$ "$\lambda\psi^4$", $\quad \mu(d\psi) = e^{-(\psi^+, g^{-1}\psi^-)} D\psi$
- $\psi = \psi_{\mathrm{e}} + \psi_{\mathrm{b}}$, where ψ_{b} has gapped covariance $g_{\mathrm{b}} \equiv g\chi(|H - \mu| > \delta)$.

$$\int \mu(d\psi) e^{V(\psi)} = \int \mu_{\mathrm{e}}(d\psi_{\mathrm{e}})\mu_{\mathrm{b}}(d\psi_{\mathrm{b}}) e^{V(\psi_{\mathrm{e}}+\psi_{\mathrm{b}})} \equiv e^{F^{(\mathrm{b})}_{\beta,L}} \int \mu_e(d\psi_e) e^{V^{(\mathrm{e})}(\psi_{\mathrm{e}})}$$

Brydges-Battle-Federbush formula. Solution of $2n!$ problem.

$$\langle \psi^{P_1}_{\mathrm{b}}; \dots; \psi^{P_n}_{\mathrm{b}} \rangle_{\mu_{\mathrm{b}}} = \sum_{T \in \mathbf{T}} \alpha_T \prod_{\ell \in T} g_{\mathrm{b}}(\ell) \int \nu_T(d\mathbf{t}) \det G^T_{\mathrm{b}}(\mathbf{t})$$

$T =$ spanning tree of $\{P_i\}$ $\qquad \#\{T\} \le Cn!$

$\|\det G^T_{\mathrm{b}}(\mathbf{t})\|_\infty \le C^{(\#\text{ loop lines})}$ $\qquad |g_{\mathrm{b}}(\ell)| \le (C/\delta) e^{-c\delta\|\ell\|}$

Effective $1d$ model: RG analysis

$$\int \mu_e(d\psi_e) e^{V^{(\mathrm{e})}(\psi_\mathrm{e})} = \int \mu_{1\mathrm{d}}(d\varphi) e^{V^{(\mathrm{e})}(\varphi * \xi^{(\mathrm{e})})} \equiv \int \mu_{1\mathrm{d}}(d\varphi) e^{V^{(1\mathrm{d})}(\varphi)}$$

- $(\widehat{\varphi * \xi^{(\mathrm{e})}})(\omega, k_1, x_2) = \hat{\xi}^e_{x_2}(k_1)\hat{\varphi}_{(\omega,k_1)}$ with $\xi^{(\mathrm{e})}$ = edge state and:

$$\langle \hat{\varphi}^-_{(\omega,k_1)} \hat{\varphi}^+_{(\omega,k_1)} \rangle = \frac{\chi_\mathrm{e}(k_1)}{-i\omega + \varepsilon(k_1) - \mu} \simeq \frac{\chi_\mathrm{e}(k_1)}{-i\omega + v(k_1 - k_F)} \quad \chi L \text{ model}$$

- Multiscale evaluation of the Grassmann integral:

$\varphi = \sum_{h=h_\beta}^0 \varphi^{(h)}$ with $\hat{\varphi}^{(h)}_{(\omega,k_1)}$ supported for $|\omega|^2 + |k_1 - k_F|^2 \sim 2^{2h}$:

$$\int \mu_{1\mathrm{d}}(d\varphi) e^{V^{(1\mathrm{d})}(\varphi)} = \int \mu_{h_\beta}(d\varphi^{(h_\beta)}) \cdots \mu_h(d\varphi^{(h)}) e^{V^{(h)}(\varphi^{(h_\beta)} + \ldots + \varphi^{(h)})}$$

- Goal: control the map $(\mu_h, V^{(h)}) \to (\mu_{h-1}, V^{(h-1)})$.

The flow of the beta function

- Scaling of the covariance: $g^{(\le h)}(\cdot) \simeq 2^h g^{(\le 0)}(2^h \cdot)$. Rescale:

$$\varphi^{(\le h)}_{\cdot} =: 2^{\frac{h}{2}} \phi^{(\le 0)}_{2^h \cdot}$$

where $\phi^{(\le 0)}$ lives on scale $O(1)$. Power counting:

$$\int dt dx_1 \, \varphi^{(\le h)}_{(t,x_1)}(P) = 2^{h(\frac{|P|}{2}-2)} \int ds dy_1 \, \phi^{(\le 0)}_{(s,y_1)}(P)$$

$|P| = 2$: relevant. $|P| = 4$: marginal. $|P| \ge 6$: irrelevant.

$$\begin{aligned} V^{(h)}(\varphi^{(\le h)}) &= \int d^2\underline{x}\, \lambda_h \varphi^{(\le h)+}_{\underline{x},\uparrow} \varphi^{(\le h)-}_{\underline{x},\uparrow} \varphi^{(\le h)+}_{\underline{x},\downarrow} \varphi^{(\le h)-}_{\underline{x},\downarrow} + \text{irrelevant terms} \\ \lambda_h &= \lambda_{h+1} + \beta_{h+1}(\lambda_{h+1}, \ldots, \lambda_0)\,, \qquad \lambda_0 \equiv \lambda\,. \end{aligned}$$

The flow of the beta function

- Scaling of the covariance: $g^{(\leq h)}(\cdot) \simeq 2^h g^{(\leq 0)}(2^h \cdot)$. Rescale:

$$\varphi_{\cdot}^{(\leq h)} =: 2^{\frac{h}{2}} \phi_{2^h \cdot}^{(\leq 0)}$$

where $\phi^{(\leq 0)}$ lives on scale $O(1)$. Power counting:

$$\int dt dx_1 \, \varphi_{(t,x_1)}^{(\leq h)}(P) = 2^{h(\frac{|P|}{2}-2)} \int ds dy_1 \, \phi_{(s,y_1)}^{(\leq 0)}(P)$$

$|P| = 2$: relevant. $|P| = 4$: marginal. $|P| \geq 6$: irrelevant.

$$\begin{aligned} V^{(h)}(\varphi^{(\leq h)}) &= \int d^2\underline{x} \, \lambda_h \varphi_{\underline{x},\uparrow}^{(\leq h)+} \varphi_{\underline{x},\uparrow}^{(\leq h)-} \varphi_{\underline{x},\downarrow}^{(\leq h)+} \varphi_{\underline{x},\downarrow}^{(\leq h)-} + \text{irrelevant terms} \\ \lambda_h &= \lambda_{h+1} + \beta_{h+1}(\lambda_{h+1}, \ldots, \lambda_0) \, , \qquad \lambda_0 \equiv \lambda \, . \end{aligned}$$

- Crucial info: $\beta_{h+1} = \beta_{h+1}^{\chi L} + \delta\beta_{h+1}$, with [Falco-Mastropietro '08]

$$\beta_{h+1}^{\chi L} = 0 \, , \qquad |\delta\beta_{h+1}| \leq C 2^{\frac{h}{100}} \max_{k \geq h} |\lambda_k|^2 \, .$$

Summable iteration! Analytic, non-Gaussian fixed point $(\mu_{-\infty}, V^{(-\infty)})$.

Comparison with the effective $1d$ theory

- RG allows to express the lattice correlations via the χL:

$$\langle \mathbf{T}\, j_{\mu,(t,x)}\,; j_{\nu,y}\rangle = Z_\mu(x_2) Z_\nu(y_2) \langle \mathbf{T}\, n_{(t,x_1)}\,; n_{y_1}\rangle_{\chi L} + \text{“small errors”}$$

where $|Z_\mu(x_2)| \leq C e^{-c|x_2|}$ (from the decay of edge modes), and:

$$(FT\langle \mathbf{T}\, n_{(t,x_1)}\,; n_{y_1}\rangle_{\chi L})(\omega, p_1) = -\frac{1}{2\pi v}\frac{1}{Z^2(1-\tau)}\frac{-i\omega - v p_1}{-i\omega + \tilde{v} p_1}\ ,$$

$\tilde{v} = \left(\frac{1-\tau}{1+\tau}\right) v, \quad \tau = \frac{\lambda}{2\pi v} =$ anomaly, $\quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2).$

Comparison with the effective $1d$ theory

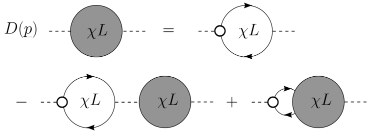
- RG allows to express the lattice correlations via the χL:

$$\langle \mathbf{T}\, j_{\mu,(t,x)}\,;j_{\nu,y}\rangle = Z_\mu(x_2)Z_\nu(y_2)\langle \mathbf{T}\, n_{(t,x_1)}\,;n_{y_1}\rangle_{\chi L} + \text{“small errors”}$$

where $|Z_\mu(x_2)| \le Ce^{-c|x_2|}$ (from the decay of edge modes), and:

$$(FT\langle \mathbf{T}\, n_{(t,x_1)}\,;n_{y_1}\rangle_{\chi L})(\omega,p_1) = -\frac{1}{2\pi v}\frac{1}{Z^2(1-\tau)}\frac{-i\omega - vp_1}{-i\omega+\tilde{v}p_1}\,,$$

$\tilde{v} = \left(\frac{1-\tau}{1+\tau}\right)v, \quad \tau = \frac{\lambda}{2\pi v} =$ anomaly, $\quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2).$



(empty bubble) $= \frac{\tau}{\lambda}(i\omega + vp_1)$

(i) $D(\underline{p}) = -i\omega + vp_1$; (ii) the circle localizes the lines on the UV cutoff scale; (iii) the last term vanishes as the UV cutoff is removed.

Comparison with the effective $1d$ theory

- RG allows to express the lattice correlations via the χL:

$$\langle \mathbf{T}\, j_{\mu,(t,x)}\,; j_{\nu,y}\rangle = Z_\mu(x_2) Z_\nu(y_2) \langle \mathbf{T}\, n_{(t,x_1)}\,; n_{y_1}\rangle_{\chi L} + \text{“small errors”}$$

where $|Z_\mu(x_2)| \leq C e^{-c|x_2|}$ (from the decay of edge modes), and:

$$(FT\langle \mathbf{T}\, n_{(t,x_1)}\,; n_{y_1}\rangle_{\chi L})(\omega, p_1) = -\frac{1}{2\pi v}\frac{1}{Z^2(1-\tau)}\frac{-i\omega - vp_1}{-i\omega + \tilde{v}p_1}\ ,$$

$\tilde{v} = \left(\frac{1-\tau}{1+\tau}\right) v, \quad \tau = \frac{\lambda}{2\pi v} =$ anomaly, $\quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2).$

- To prove universality, need to find a cancellation between Z, τ, v and the vertex renormalization:

$$Z_\mu = \sum_{x_2} Z_\mu(x_2)$$

The cancellation follows from Ward identities: consequences of the continuity equation $\partial_\mu j_{\mu,\mathbf{x}} = 0$ for the correlations.

Ward identities

- Both $\langle\cdot\rangle_{\beta,L}$ and $\langle\cdot\rangle_{\chi L}$ satisfy vertex WIs: $(\mathbf{x} = (t, x_1, x_2) = (\underline{x}, x_2))$

$$
\begin{aligned}
\partial_\mu \langle \mathbf{T} j_{\mu,\mathbf{z}}\,; a^-_{\mathbf{y}} a^+_{\mathbf{x}}\rangle_{\beta,L} &= \big[\delta_{\mathbf{x},\mathbf{z}} \langle \mathbf{T} a^-_{\mathbf{y}} a^+_{\mathbf{x}}\rangle_{\beta,L} - \delta_{\mathbf{y},\mathbf{z}} \langle \mathbf{T} a^-_{\mathbf{y}} a^+_{\mathbf{x}}\rangle_{\beta,L}\big] \\
(\partial_0 + v\partial_1)\langle \mathbf{T} n_{\underline{z}}\,; \varphi^-_{\underline{y}} \varphi^+_{\underline{x}}\rangle_{\chi L} &= \frac{1}{Z(1-\tau)}\big[\delta_{\underline{x},\underline{z}} \langle \mathbf{T} \varphi^-_{\underline{y}} \varphi^+_{\underline{x}}\rangle_{\chi L} - \delta_{\underline{y},\underline{z}} \langle \mathbf{T} \varphi^-_{\underline{y}} \varphi^+_{\underline{x}}\rangle_{\chi L}\big]
\end{aligned}
$$

Ward identities

- Both $\langle\cdot\rangle_{\beta,L}$ and $\langle\cdot\rangle_{\chi L}$ satisfy vertex WIs: $(\mathbf{x} = (t, x_1, x_2) = (\underline{x}, x_2))$

$$\begin{aligned} \partial_\mu \langle \mathbf{T} j_{\mu,\mathbf{z}}\,; a^-_{\mathbf{y}} a^+_{\mathbf{x}}\rangle_{\beta,L} &= \big[\delta_{\mathbf{x},\mathbf{z}}\langle \mathbf{T} a^-_{\mathbf{y}} a^+_{\mathbf{x}}\rangle_{\beta,L} - \delta_{\mathbf{y},\mathbf{z}}\langle \mathbf{T} a^-_{\mathbf{y}} a^+_{\mathbf{x}}\rangle_{\beta,L}\big] \\ (\partial_0 + v\partial_1)\langle \mathbf{T} n_{\underline{z}}\,; \varphi^-_{\underline{y}} \varphi^+_{\underline{x}}\rangle_{\chi L} &= \frac{1}{Z(1-\tau)}\big[\delta_{\underline{x},\underline{z}}\langle \mathbf{T}\varphi^-_{\underline{y}} \varphi^+_{\underline{x}}\rangle_{\chi L} - \delta_{\underline{y},\underline{z}}\langle \mathbf{T}\varphi^-_{\underline{y}} \varphi^+_{\underline{x}}\rangle_{\chi L}\big] \end{aligned}$$

- For large space-time distances:

$$\begin{aligned} \big\langle \mathbf{T}\, j_{\mu,(t_1,x)}\,; a^-_{(t_2,y)} a^+_{z}\big\rangle_{\beta,L} &\simeq Z_\mu(x_2)\xi_{y_2}\overline{\xi}_{z_2}\big\langle \mathbf{T}\, n_{(t_1,x_1)}\,; \varphi^-_{(t_2,y_1)}\varphi^+_{z_1}\big\rangle_{\chi L} \\ \langle \mathbf{T}\, a^-_{(t,x)}\,; a^+_{y}\rangle_{\beta,L} &\simeq \xi_{x_2}\overline{\xi}_{y_2}\langle \mathbf{T}\,\varphi^-_{(t,x_1)}\,; \varphi^+_{y_1}\rangle_{\chi L} \qquad (*) \end{aligned}$$

- Plugging $(*)$ in the WIs, we get relations between Z_μ, Z, τ and v:

$$Z_0 = Z(1-\tau)\,, \qquad Z_1 = Zv(1-\tau)\,.$$

These identities imply the universality of the edge conductance.

Conclusions

- Today: Edge transport coefficients for $2d$ topological insulators with:

  (i) single-mode edge currents, or

  (ii) one pair of counterpropagating edge modes.

  Consequences: bulk-edge duality, Haldane relation.

- Based on RG, and on Ward identities for relativistic & lattice model.

- Open problems:

  (i) Multi-edge states topological insulators? (edge states scattering?)

  (ii) Validity of edge linear response theory? (already for $\lambda = 0$!)

  (iii) Disorder?

Thank you!