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The Calogero Model: Physics, Mathematics and Recent Results

March 11, 2019

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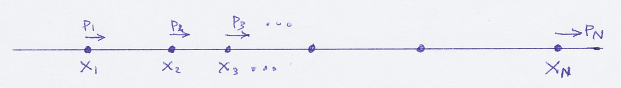
Case in point: Long-range one-dimensional billiards
(aka Calogero particles) and their fluids, spins, ghosts and avatars

Bill Sutherland, Francesco Calogero and Michel Gaudin
Winners of the 2019 Dannie Heineman Prize for Mathematical Physics

How it all started...

A system of nonrelativistic identical particles on the line ($m = 1$)



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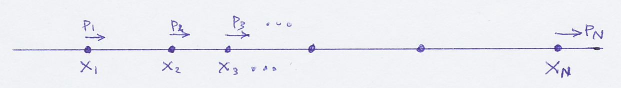

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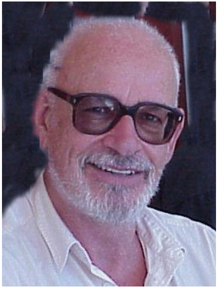
Francesco Calogero, 1969:



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  solved 3 particle problem Quantum Mechanically
- Solved N particle problem
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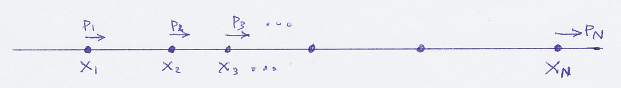

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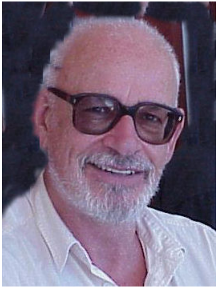

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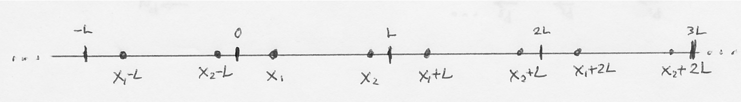
1989-1997 Secretary General of Pugwash
Conferences on Science and World Affairs

Peace Nobel Prize 1995

We can also 'confine' the particles in a box:

Either add an external harmonic trap: $\sum_i \frac{1}{2}\omega^2 x_i^2$ (Calogero 1971)

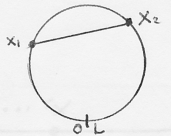
...or put the particles on a periodic space of length L



Each particle interacts with the images of all other particles, so the potential becomes

$$V(x) = \sum_{n=-\infty}^{\infty} \frac{g}{(x+nL)^2} = \frac{g}{\left(\frac{L}{\pi}\sin\frac{\pi x}{L}\right)^2}$$ (Sutherland 1972)

Same as interacting through the chord length on a circle.



By analytically continuing $L = iL'$ we can also consider a model with hyperbolic interactions

$$V(x) = \sum_{n=-\infty}^{\infty} \frac{g}{(x+inL')^2} = \frac{g}{\left(\frac{L'}{\pi}\sinh\frac{\pi x}{L'}\right)^2}$$

Unconfined model, interaction falls off exponentially with distance

Finally, we can periodize the hyperbolic model in $x \to x + L$. This gives the elliptic model with a Weierstrass elliptic function potential

$$V(x) = g\,\mathscr{P}(x; L, iL')$$

Little is known about the elliptic system's classical or quantum solution (Langmann) – it remains an interesting challenging topic.

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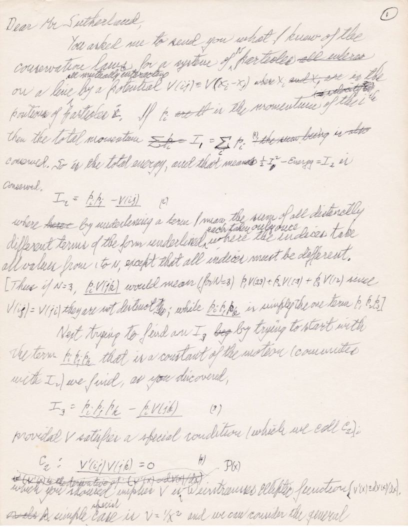

(1)

Dear Mr Sutherland,

You asked me to send you what I know of the conservation laws for a system of N particles all mutually interacting on a line by a potential $V(ij) = V(x_i - x_j)$ where x_i is the position of particle i. If p_i is the momentum of the i-th then the total momentum $I_1 = \sum_i p_i$ (1) is conserved. So is the total energy, and that means $\frac{1}{2}I_1^2 - \text{Energy} = I_2$ is conserved.

$$I_2 = \underline{p_i p_j} - \underline{V(ij)} \qquad (2)$$

where by underlining a term I mean the sum of all distinctly different terms of the form underlined, each taken only once, where the indices take all values from 1 to N, except that all indices must be different. [Thus if $N=3$, $\underline{p_i V(jk)}$ would mean (for $N=3$) $p_1V(23) + p_2V(13) + p_3V(12)$ since $V(ij) = V(ji)$ they are not distinct; while $\underline{p_i p_j p_k}$ is simply the one term $p_1 p_2 p_3$]

Next trying to find an I_3 by trying to start with the term $\underline{p_i p_j p_k}$ that is a constant of the motion (commutes with I_2) we find, as you discovered,

$$I_3 = \underline{p_i p_j p_k} - \underline{p_i V(jk)} \qquad (3)$$

provided V satisfies a special condition (which we call C_2):

$$C_2 : \quad \underline{V'(ij)V(jk)} = 0 \qquad (4)$$

$P(x)$

which you showed implies V is a Weierstrauss elliptic function ($V'(x) = dV(x)/dx$).

A simple special case is $V = 1/x^2$ and we can consider the general

Why is this system physically interesting?

Arises in various situations of physical interest: a 'Jack in the box'!

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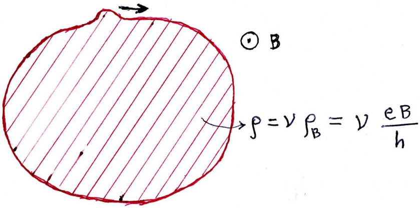
- Effective descriptions of fractional quantum Hall states



Edge states behave as one-dimensional Calogero particles

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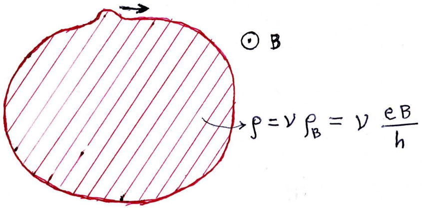

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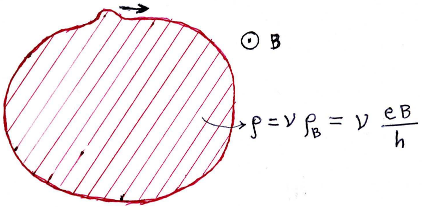
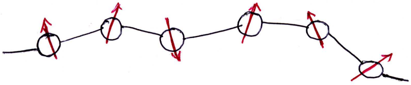


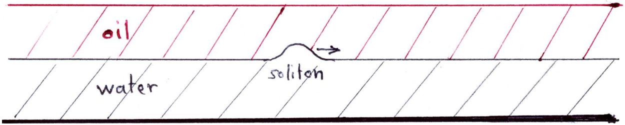
Edge states behave as one-dimensional Calogero particles

- Mapping to anyons
- Spin chains in macromolecules



Spin states behave like Calogero particles

- Interface dynamics in stratified fluids



Benjamin-(Davis-Acrivos)-Ono model
1967 - (1967) - 1975

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2}\int \frac{u(y)}{x-y}dy = 0$$

(Hilbert transform)

Admits 'soliton' solutions whose guiding centers behave like Calogero particles

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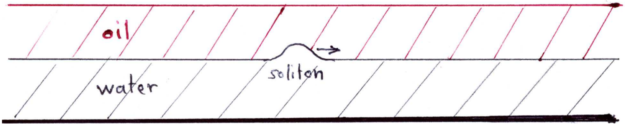

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Admits ‘soliton’ solutions whose guiding centers behave like Calogero particles

- Two-dimensional gauge field theories
- Matrix model descriptions of quantum gravity and string theory
- Etc...

...and mathematically interesting?

Plenty:

- Integrability
- Group theory
- Moment maps
- Differential-difference operators
- Symmetric functions
- Eigenvalue problems
- Soliton theory
- Self-adjoin extensions
- Others that I forget or ignore!

The Calogero system: a physical look

Nonrelativistic identical particles on the line with inverse-square two-body potentials (every word counts)

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- 'Borderline' short-distance stable potential in 2 or more dimensions (for attractive potentials)
- 'Borderline' long-distance potential for phase transition (fluid vs. crystal at low temperatures)
- $p^2 \sim \hbar^2/x^2$: only scale-free potential in QM
  Put $g = \hbar^2\alpha$, then α is *pure number*
  ($\hbar = 1$ from now on)

Stability, Statistics

- Classically stable: $g \geq 0$
- QM: uncertainty principle improves stability to $g = \ell(\ell - 1) \geq -\frac{1}{4}$ (Exercise in Landau & Lifshitz!)
- Wavefunction behaves as $\psi \sim x^{\ell}$ at coincidence points
- ℓ and $\ell' = 1 - \ell$, give the same g
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What about statistics?

- $1/x^2$ potential is QM impenetrable; therefore ordinary statistics (symmetry of wavefunction) **irrelevant**
- All physical observables of the system independent of statistics
- Dimensionless parameter ℓ subsumes all statistics of the model
- $\ell = 0$: free bosons; $\ell = 1$: free fermions; other values?

Fractional Statistics

- Calogero can be interpreted as free particles with fractional statistics
- Dimensionless parameter ℓ subsumes all statistics of the model!

One dimension: statistics a tricky concept

- Particles "bump" into each other
- No spin-statistics connection

Several ways to define statistics:

- Boundary condition at coincidence points: behavior of wavefunction
- Scattering: high energy limit of scattering phase shift
- Stat Mech: statistical properties of many-body systems

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Integrability

A classical Hamiltonian system possessing **conserved quantities** $I_n,\ n = 1, 2, \ldots N$ functions of coordinates and momenta not involving time explicitly is called integrable.

The I_n also must be (and always are) in involution: their Poisson brackets must vanish

$$\{I_n, I_m\} = 0$$

Obvious conserved quantities like total energy H and total momentum P are part of I_n but for $N > 2$ there must be more.

What are those for the Calogero system?
Equations of motion:

$$\dot{x}_i = p_i \ , \quad \dot{p}_i = 2 \sum_{j(\neq i)} \frac{g}{(x_i - x_j)^3}$$

The Lax pair

Consider $N \times N$ matrices

$$L_{jk} = p_j \delta_{jk} + (1 - \delta_{jk}) \frac{i\ell}{x_{jk}} \ , \quad A_{jk} = \ell \delta_{jk} \sum_{s \neq j} \frac{1}{x_{js}^2} + \ell(\delta_{jk} - 1) \frac{1}{x_{jk}^2}$$

where we defined the shorthand $x_{ij} = x_i - x_j$
Upon use of the equations of motion and some algebra we find

$$\frac{dL}{dt} = i[L, A]$$

and thus

$$L(t) = U(t)^{-1} L(0) U(t) \ , \quad U(t) = P e^{i \int_0^t A(\tau) d\tau}$$

So $L(t)$ evolves by unitary conjugation and its eigenvalues are constants of motion. Alternatively:

$$I_n = \mathrm{Tr} L^n$$

$I_1 = P$, $I_2 = 2H$; $I_3, \ldots$ new conserved quantities.

Properties of the classical system

As time goes to $\pm\infty$ particles fly away from each other and all coordinate terms in L and A vanish. Therefore

$$I_n = \sum_{i=1}^{N} k_i^n$$

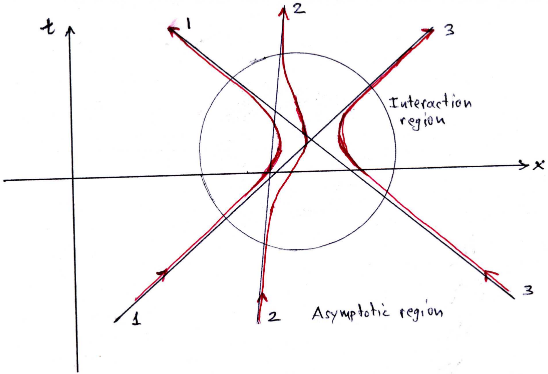
with k_i the asymptotic momenta.

- Asymptotic monenta k_i are conserved

Some more work reveals that the angle variables θ_n conjugate to I_n asymptotically become g-independent functions of the coordinates x_i and momenta k_i. Therefore, at $t \to \pm\infty$

$$x_i(t) = k_i t + a_i$$

- Impact parameters a_i are also conserved!



- After scattering, all particles resume their initial asymptotic momenta *and impact parameters*
- There is no time delay, just a permutation; therefore...
- System looks asymptotically free!
- Suggestive fact: action increases w.r.t. free system by $\Delta S = \pi\sqrt{g}$ per particle exchange

Quantum properties of the model

- Lax pair formulation extends quantum mechanically
- Classical limit is $\ell \to \infty$

Corollaries:

- Particles scatter and preserve asymptotic momenta
- Scattering phase shift is a constant:

$$\theta_{sc} = \frac{N(N-1)}{2}\pi\ell$$

(Compare with classical result $\Delta S = \pi\sqrt{g}$ per scattering)

- Since $\partial\theta_{sc}/\partial k_i = 0$ there is no time delay (again, compare with classical result)

Calogero particles can be interpreted as free particles with an extra phase shift of $\pi\ell$ when they exchange, that is,

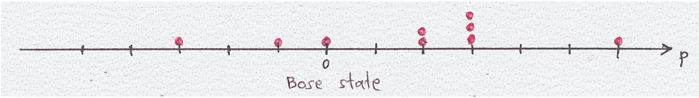
particles with exchange statistics of order ℓ (my entry, 1988)

The periodic model: particles, holes and duality

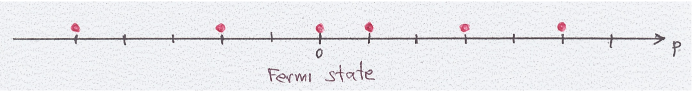
Particles on a circle of length L have discrete momenta $p = \frac{2\pi\hbar}{L} n = 0, \pm 1, \pm 2, \ldots$ upon choosing $\hbar = 1$ and $L = 2\pi$
Free particles would have energy eigenvalues

$$E = \sum_{i=1}^{N} \frac{1}{2} p_i^2$$

If the particles are indistinguishable only the set of values $\{p_1, p_2, \ldots, p_N\}$ matters, not their arrangement
Bose statistics: $p_1 \leq \cdots \leq p_N$ or $p_{i+1} - p_i \geq 0$



Fermi statistics: $p_1 < \cdots < p_N$ or $p_{i+1} - p_i \geq 1$ (no spin!)



For Calogero particles the energies turn out to be

$$E = \sum_{i=1}^{N} \tfrac{1}{2} p_i^2 + \ell \sum_{i<j} (p_j - p_i) + \ell^2 \frac{N(N^2-1)}{24}$$

with $p_i \leq p_{i+1}$ (boson rule)
(symmetry or antisymmetry of wavefunction is irrelevant)

Defining 'pseudomomenta' $\bar{p}_i = p_i + (\frac{N}{2} - i)\ell$ ('open the fan': a partial bosonization) the energy becomes *free*

$$E = \sum_{i=1}^{N} \tfrac{1}{2} \bar{p}_i^2$$

but now the $\bar{p}_i$ satisfy $\bar{p}_{i+1} - \bar{p}_i \geq \ell$

Fractional exclusion statistics (Haldane 1991)

For large N there are particle and hole excitations

Particles	Holes
• Exclude ℓ states	• Exclude one state
• Excited by one unit	• Excited by ℓ units

There is a particle-hole duality generalizing the one in free Fermi systems:

$$\ell \text{ holes} = -1 \text{ particle}$$

$$\text{particle} \leftrightarrow \text{hole}\ , \quad \ell \leftrightarrow \frac{1}{\ell}\ , \quad p \leftrightarrow \ell p$$

In the classical ($\ell \to \infty$) continuum ($N \to \infty$) limit:

Particles: Solitons   Holes: Waves

More formal: Particle symmetries and Statistics

Consider a set of first-quantized indistinguishable particles

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Consider a set of first-quantized indistinguishable particles

- Identical particles: S_N a dynamical symmetry
- Indistinguishable particles: all physical operators commute with S_N: discrete gauge symmetry
- Gauge invariants:

$$I(k,q) = \sum_{i=1}^{N} : e^{ik_i x_i + iqp_i} :$$

$$[I(k,q), I(k',q')] = 2i \sin \frac{kq' - k'q}{2} I(k+k', q+q')$$

- This is the 'sine' version of W_∞ algebra

The above algebra admits a host of representations, corresponding to the underlying particles being (para)bosons, (para)fermions...

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- Calogero: Scalar irreps; Casimir becomes coupling strength

Concentrate on the special case $N = 2$

- Center of mass variables X , P: gauge invariant
- Relative variables x, p: not gauge invariant - quadratic invariants

$$A = x^2 \ , \quad B = \frac{xp + px}{2} \ , \quad C = p^2$$

A, B, C commute with X, P and close to the $SL(2, R)$ algebra

$$[A, B] = 2iA \ , \quad [B, C] = 2iB \ , \quad [A, C] = 4iB$$

with Casimir

$$G = \frac{AC + CA}{2} - B^2$$

- Unitarity of $SL(2, R)$ mandates $G = \ell(\ell - 1)$
- A realization of the irrep with the above G is

$$A = x^2 \ , \quad B = \frac{xp + px}{2} \ , \quad C = p^2 + \frac{\ell(\ell - 1)}{x^2}$$

- The kinetic energy acquired an inverse-square potential part

The Calogero model! Coupling: Casimir (superselected)

Statistics $\leftrightarrow$ Superselection sectors $\leftrightarrow$ Calogero dynamics

The Hermitian Matrix Model

Formulate particle coordinates as eigenvalues of an $N \times N$ matrix

- S_N promoted to U_N
- Irreps of U_N define physical ilbert space

$$\mathcal{L} = \mathrm{Tr}\left\{\frac{1}{2}\dot{M}^2 - V(M)\right\}$$

$V(x)$ is a scalar potential evaluated for the matrix variable M.

- Time-translation invariance leads to conserved energy

$$H = \mathrm{Tr}\left\{\frac{1}{2}\dot{M}^2 + V(M)\right\}$$

- Invariance under $M \to UMU^{-1}$ leads to conserved

$$J = i[M, \dot{M}]$$

where [,] denotes ordinary matrix commutator

- J are 'gauge charges' that will determine the realization ('statistics') of the indistinguishable particle system

Reduction to particles

Parametrize $M = U\Lambda U^{-1}$ and define $A = -U^{-1}\dot{U}$. Then

$$\begin{aligned} \dot{M} &= U\left(\dot{\Lambda} + [\Lambda, A]\right) U^{-1} := U L U^{-1} \\ J &= iU\left([\Lambda, [\Lambda, A]]\right) U^{-1} := U K U^{-1} \end{aligned}$$

The matrix elements of A and K are related

$$K_{jk} = i\,[\Lambda, [\Lambda, A]]_{jk} = i(x_j - x_k)^2 A_{jk}\ , \quad K_{ii} = 0$$

Solving for A_{jk} and putting into H we obtain

$$H = \sum_i \tfrac{1}{2} p_i^2 + \tfrac{1}{2} \sum_{i \neq j} \frac{K_{ij} K_{ji}}{(x_i - x_j)^2} + \sum_i V(x_i)$$

- The K_{ij} are still dynamical
- Models obtain by choosing 'sectors' for J and thus K
- $J = 0 \Rightarrow K = 0$: free particles

Let there be spinning Calogeros

Realize J in terms of n vectors v_s $J = \ell(\sum_{s=1}^n v_s^\dagger v_s - 1)$

and thus
$$K = \ell(\sum_{s=1}^n u_s^\dagger u_s - 1)\ , \quad u_s = U^{-1} v_s$$

From $K_{ii} = 0$
$$\sum_{s=1}^n u_{si}^* u_{si} = 1 \quad (\text{no sum over } i)$$

So
$$K_{ij} K_{ji} = \sum_{s,r} \ell u_{si} u_{sj}^* \ \ell u_{rj} u_{ri}^* = \ell^2 \sum_{s,r} S_i^{sr} S_j^{rs} \quad (i \neq j)$$

with $S_i^{sr} = u_{si} u_{ri}^*$ and H becomes
$$H = \sum_i \tfrac{1}{2} p_i + \sum_{i<j} \frac{\ell^2 \mathrm{Tr}(S_i S_j)}{(x_i - x_j)^2} + \sum_i V(x_i)$$

The Calogero model with dynamical particle 'spins': $\mathrm{Tr} S_i^2 = 1$

- The strength $g = \ell^2$ is related to the conserved charge ℓ
- For $n = 1$ we recover spinless Calogero
- External potential $V(x)$ arbitrary at this stage

Integrability through matrix technology

Consider the free matrix model: $V(x) = 0$ and thus $\dot{M} = 0$
Therefore $I_n = \mathrm{Tr}\dot{M}^n = \mathrm{Tr}(U\dot{M}U^{-1})^n = \mathrm{Tr}L^n$ are conserved

- $\dot{M}$ is the canonical momentum matrix
- Its elements have vanishing Poisson brackets

Therefore, the I_n are in involution: $\{I_m, I_n\} = 0$

$$L_{jk} = (U\dot{M}U^{-1})_{jk} = \delta_{jk}\dot{x}_j - (1-\delta_{jk})\frac{iK_{jk}}{x_j - x_k}$$

The Lax matrix!

$$= \delta_{jk}\,\dot{x}_j - (1-\delta_{jk})\frac{i\ell\sum_s u_{sj}u^*_{sk}}{x_j - x_k}$$

- In $\mathrm{Tr}L^n$ products $u_{s_1 i_1} u^*_{s_1 i_2}\, u_{s_2 i_2} u^*_{s_2 i_3} \cdots = \mathrm{Tr}(S_{i_1} \cdots S_{i_n})$
- I_n will involve only x_i, $\dot{x}_i = p_i$. S_i and the constant ℓ
- I_n are the conserved integrals of the Calogero model
- The actual motion of the model can be obtained explicitly

Farewel to the hermitian matrix model

- Quantum model: can find explicitly spectrum and states (matrix ladder operators and $U(N)$ reduction)

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The larger gauge symmetry may have anomalies!
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Gone! Matrix model fermionizes particles (renormalizes ℓ to $\ell+1$)
(the infamous Vandermonde determinant in the measure)

The unitary matrix model

Particles on unit circle (periodic): phases of eigenvalues of a unitary matrix U

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(U^{-1}\dot{U})^2 \;\Rightarrow\; \frac{d}{dt}\left(U^{-1}\dot{U}\right) = 0$$

- Invariant under $U \to VUW^{-1}$, V, W unitary
- Two conserved matrix angular momenta L and R:

$$\begin{aligned} U \to VU: \qquad L &= i\dot{U}U^{-1} \\ U \to UW^{-1}: \quad R &= -iU^{-1}\dot{U} \end{aligned}$$

- Unitary conjugation corresponds to $W = V$ with generator

$$J = L + R = i[\dot{U}, U^{-1}]$$

Similarly as in hemitian model, we recover the Sutherland inverse-sine-square model

$$H = \sum_i \tfrac{1}{2}\dot{x}_i^2 + \tfrac{1}{2}\sum_{i\neq j} \frac{K_{ij}K_{ji}}{4\sin^2\frac{x_i - x_j}{2}}$$

Integrable, solvable and quantizable by similar techniques

Farewell to Matrices

The Matrix Model provides us with:

- A realization of the scalar model but with a *quantized* Calogero statistics parameter $\ell + 1$
- A realization of Calogero models with particle spin degrees of freedom
- A systematic way of solving the above models.
- Integrability for $V(x)$ up to quartic polynomial (others?)

What the matrix model has *not* provided is

- A realization of the Calogero model for fractional values of ℓ
- A realization of spin-Calogero systems with the spins in arbitrary (non-symmetric or antisymmetric) representations.
- A control of the coupling strength of the potential for the spin-Calogero models.

A new approach is needed!

Exchange operator formalism

Operators M_{ij} permute the *coordinate* DOF of N particles in one dimension. They satisfy the permutation algebra (symmetric group)

$$M_{ij} = M_{ij}^{-1} = M_{ij}^{\dagger} = M_{ji}$$
$$[M_{ij}, M_{kl}] = 0 \quad \text{if } i, j, k, l \text{ distinct}$$
$$M_{ij}M_{jk} = M_{ik}M_{ij} \quad \text{if } i, j, k \text{ distinct}$$

One-particle operators: any A_i satisfying

$$\begin{aligned} M_{ij}A_k &= A_k M_{ij} \quad \text{if } i, j, k \text{ distinct} \\ M_{ij}A_i &= A_j M_{ij} \end{aligned}$$

Construct the exchange-momenta one-particle (Dunkl) operators

$$\pi_j = p_j + \sum_{k(\neq j)} i\, W(x_j - x_k) M_{jk} := p_j + \sum_{k(\neq j)} i\, W_{jk} M_{jk}$$

For π_i to be Hermitian the prepotential $W(x)$ should satisfy

$$W(-x) = -W(x)^*$$

'Free' Hamiltonian in π_i would be $H = \sum_j \frac{1}{2}\pi_j^2$
Contains terms linear in p_j: to eliminate them

$$W(-x) = -W(x) = \mathrm{real}$$

Commutators of π_i and Hamiltonian become

$$[\pi_i, \pi_j] = \sum_k W_{ijk}(M_{ijk} - M_{jik})$$

$$H = \sum_i \frac{1}{2}p_i^2 + \sum_{i<j} \left(W_{ij}^2 + W_{ij}' M_{ij}\right) + \sum_{i<j<k} W_{ijk} M_{ijk}$$

where $M_{ijk} = M_{ij} M_{jk}$ is cyclic permutation of (i,j,k) and

$$W_{ijk} = W_{ij} W_{jk} + W_{jk} W_{ki} + W_{ki} W_{ij}$$

- Goal: commutator zero or a constant
- This leads to functional equation for $W(x)$:

$$W(x)W(y) - W(x+y)\,[W(x) + W(y)] = \mathrm{const}(= W_{ijk})$$

Can be solved and we will list its solutions (up to scaling of x)

a) $W_{ijk} = 0 \quad \Rightarrow \quad W(x) = \ell/x$
b) $W_{ijk} = -\ell^2 < 0 \quad \Rightarrow \quad W(x) = \ell \cot x$
c) $W_{ijk} = +\ell^2 > 0 \quad \Rightarrow \quad W(x) = \ell \coth x$

Case a) $$\pi_j = p_j + \sum_{k \neq j} \frac{i\ell}{x_{jk}} M_{jk} \ , \quad [\pi_j, \pi_k] = 0$$

$$H = \sum_i \tfrac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{x_{ij}^2}$$

- Calogero-like model with exchange interactions
- Trivially integrable: $$I_n = \sum_i \pi_i^n$$

Assume particles are bosons or fermions: $M_{ij} = \pm 1$ on states

- The model becomes the standard Calogero model
- Projected integrals $I_{n,\pm}$ commute (by locality)

We proved quantum integrability of Calogero model in one sweep!
Model with harmonic potential can also be solved this way

Let there be spin (again!)

Assume particles carry a number q of discrete internal states

- σ_{ij} exchanges the internal states of particles i and j
- Total particle permutation operator is $T_{ij} = M_{ij}\,\sigma_{ij}$

Assume states bosonic or fermionic under total particle exchange

$$T_{ij}\psi_{B,F} = \pm\psi_{B,F} \quad \Rightarrow \quad M_{ij}\psi_{B,F} = \pm\sigma_{ij}\psi_{B,F}$$

Exchange Calogero and Sutherland Hamiltonians become

$$H_c = \sum_i \tfrac{1}{2}p_i^2 + \sum_i \tfrac{1}{2}\omega^2 x_i + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{x_{ij}^2}$$

$$H_s = \sum_i \tfrac{1}{2}p_i^2 + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{\sin^2 x_{ij}} - \ell^2 \left(\frac{N(N-1)}{2} + \sum_{i<j<k} \sigma_{ijk} \right)$$

Calogero-Sutherland models with spin-exchange interactions

Fundamental $SU(q)$ generators T^a satisfy completeness relation

$$\sum_{a=1}^{q^2-1} T^a_{\alpha\beta} T^a_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{q}\delta_{\alpha\beta}\delta_{\gamma\delta} \quad \text{or} \quad \sum_a T_1 T_2 = T_{12} - \frac{1}{q}$$

Therefore
$$\sigma_{ij} = \vec{S}_i \cdot \vec{S}_j + \frac{1}{q}$$

where S^a_i is T^a acting on internal states of particle i
Calogero-Sutherland interaction coefficient becomes

$$\ell(\ell \mp \sigma_{ij}) = \ell \left(\mp \vec{S}_i \cdot \vec{S}_j + \ell \mp \frac{1}{q} \right)$$

- Ferro/Antiferromagnetic spin interaction models (as in Matrix)
- Arbitrary coefficient strength (ℓ)
- Spins necessarily in the fundamental of $SU(q)$
- Ferro $\to$ Antiferro: B $\to$ F or $\ell \to -\ell$
- $\ell = 1$: Matrix and Exchange models agree
  (B $\leftrightarrow$ fundamental, F $\leftrightarrow$ antifundamental)

Asymptotic Bethe Ansatz approach

ABA: physically lucid method, works well for periodic models

Consider distinguishable exchange-Calogero particles without external potential coming in with asymptotic momenta k_i 5

- Key fact: particles 'go through' each other, no backscattering
- Impenetrable $1/x^2$ potential became completely penetrable!

[Proof: simultaneous eigenstate of π_i becomes asymptotically an eigenstate of p_i at both $t \to \pm\infty$: no shuffling of p_i]

(Puzzle: what happens with the correspondence principle? The interaction coefficient is $\ell(\ell - \hbar M_{ij})$. How can $\hbar$ produce such a dramatic effect, particles going through each other?)

- Bottom line: free particles obeying generalized selection rules
- Particles with spin: spectrum and degeneracies the same as those of free particles with spin distributed among n_i

Sidebar: Yang-Baxter equation and integrability

The fact that particles go through each other means that their scattering trivially satisfies the Yang-Baxter relation:

$$S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}$$

- YB equation is considered a hallmark of integrability
- It is actually necessary for integrablity but not sufficient

YB condition can be viewed as absence of Aharonov-Bohm effects around triple coincidence points

- For zero-range interactions it is enough:
- Lieb-Liniger: triple points of measure zero
- Heisenberg and XXZ: triple points do not exist

In general, 3-particle effects introduce additional scattering that may spoil integrability. I will give an example in private.

But in Calogero land all is well.

The freezing trick

Take the strength of interaction to grow large: $\ell \to \infty$

- System 'freezes' around the position of classical equilibrium
- Low-lying excitations are vibrational modes (phonons) and spin excitation modes (spinons)
- Particle distances $x_i - x_j$ have negligible fluctuations, so spin and coordinates decouple into a spinless Calogero model and a spin chain model
- Both vibrational and spin excitation energies of order ℓ
- Energy states can be found by 'modding' the spin-Calogero states by the spinless Calogero states: $Z_s = Z_{sC}/Z_C$
- From spin-Sutherland model: The Haldane-Shastry model

$$H_{HS} = \mp \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j}{\sin^2 \frac{\pi(i-j)}{N}}$$

From harmonic spin-Calogero model we obtain a spin chain model with spins sitting on an inhomogeneous lattice

$$H_P = \mp \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j}{\bar{x}_{ij}^2}$$

- $\bar{x}_i$ minimize classical potential and satisfy

$$\bar{x}_i - \sum_{j(\neq i)} \frac{1}{(\bar{x}_i - \bar{x}_j)^3} = 0 \quad \Leftrightarrow \quad \bar{x}_i - \sum_{j(\neq i)} \frac{1}{\bar{x}_i - \bar{x}_j} = 0$$

- $\bar{x}_i$ are the roots of the N-th Hermite polynomial.
- Spectrum is equidistant – spin content is nontrivial
- Antiferromagnetic end of the spectrum is a $c = 1$ conformal field theory (perturbatively exactly in $1/N$)

Generalizations to spin chains for other spin representations (e.g., $SU(p|q)$ 'supersymmetric' chain) also exist

Parting words

Exchange operator formulation gave us

- Neat way to treat the system directly at the QM domain
- Spinless and spin-particle models with arbitrary, non-quantized coupling strength
- Correspondence to free particles and generalized statistics
- Spin chain models through the freezing trick

...while Matrix formulation gave us

- Arbitrary-size spin representations

No formulation gives “everything"

Desideratum: arbitrary spin (e.g., $SU(2)$ spin 1) and arbitrary strength to use freezing trick and obtain spin-1 chain and thus validate (or disprove) Haldane's mass gap conjecture

Still an open question...

Noncommutative Elliptic Spin-Calogero Model

Most exotic model: "Fold" spin-Calogero model with two noncommuting complex translations and spin rotations

Define spins $(\tilde{S}_i)^{pq}_{\vec{\alpha}}$ with Poisson brackets

$$\{(\tilde{S}_i)^{pq}_{\vec{\alpha}}, (\tilde{S}_j)^{rs}_{\vec{\beta}}\} = i\delta_{ij}\left[\omega^{\frac{\vec{\alpha}\times\vec{\beta}}{2}}\,\delta_{ps}\,(\tilde{S}_i)^{rq}_{\vec{\alpha}+\vec{\beta}} - \omega^{-\frac{\vec{\alpha}\times\vec{\beta}}{2}}\,\delta_{rq}\,(\tilde{S}_i)^{ps}_{\vec{\alpha}+\vec{\beta}}\right]$$

$U(k)$-extended Moyal (star-commutator) algebra
Elliptic model with spin couplings

$$W = \sum_{i,j}\sum_{\vec{\alpha};p,q}(\tilde{S}_i)^{pq}_{\vec{\alpha}}\;(\tilde{S}_j)^{qp}_{-\vec{\alpha}}\;W^{pq}_{\vec{\alpha}}(x_i - x_j)$$

- Elliptic model with noncommutative spin twists
- Two-body spin couplings $W(x_i - x_j)$ and spin self-coupling $\tilde{W}$

$*$ Practically nothing is known about this model

- Dense collection of particles: hydrodynamic description
- Exact in $N \to \infty$ limit, $1/N$ corrections
- Particle system is integrable, hydro system should be as well
- Use $\rho(x,t)$ and $v(x,t)$ as fundamental variables

$$\partial_t \rho + \partial_x(\rho v) = 0 \ , \quad \partial_t v + v \partial_x v = F(\rho, x)$$

Euler description is Hamiltonian

$$\{\rho(x), v(y)\} = \delta'(x-y) \ , \quad H = \int dx \left[\tfrac{1}{2}\rho v^2\right] + U$$

Suffices to find the potential energy of the system in terms of ρ
For external potential $V_o(x)$ and interaction $V(x-y)$ we expect

$$U = \int dx \rho(x) V_o(x) + \int dxdy\, \rho(x)\, \rho(y)\, V(x-y)$$

First part always OK. Not so for the second: singular interaction

Can find result either through Collective Field Theory or through careful regularization

$$U = \int dx \left[\frac{\pi^2 \ell^2}{6} \rho^3 + \frac{\pi \ell (\ell - 1)}{2} \rho \, \partial_x \rho^H + (\ell - 1)^2 \frac{(\partial_x \rho)^2}{8\rho} \right]$$

(ρ^H Hilbert transform)

- Terms $\sim \rho^3, (\partial_x \rho)^2/\rho$: short-distance contributions
- Term $\sim \rho \partial_x \rho^H$: regularized 'naive' continuum term
- Leading term: fermions with $\hbar \to \ell\hbar$: fractional statistics
- Classical limit: $\ell(\ell-1), (\ell-1)^2 \to \ell^2$
- Perturbatively exact. Nonperturbative effects: depletion
- Trigonometric or hyperbolic potentials: change kernel of Hilbert transform

$$\rho^H(x) = \fint K(x-y) \rho(y) dy \ , \quad K(x) = \frac{1}{x}, \cot x, \coth x$$

- Trigonometric amounts to choosing $\rho(x)$ periodic

Solitons...

Solitons/waves: $\rho(x,t) = \rho(x - \mathrm{v}t)$, $v(x,t) = v(x - \mathrm{v}t)$

- EOM admit solitons of rational type

$$\rho(x) = \rho_o \left[1 + \frac{g(\mathrm{v}^2 - \mathrm{v}_s^2)}{g\mathrm{v}_s^2 + (\mathrm{v}^2 - \mathrm{v}_s^2)^2 x^2} \right]$$

- Solitons can only have speed $|\mathrm{v}| > \mathrm{v}_s$
- Become highly peaked as $\mathrm{v} \gg \mathrm{v}_s$
- Particle number Q, momentum P and energy E of soliton are

$$Q = 1 \ , \quad P = \mathrm{v} \ , \quad E = \tfrac{1}{2}\mathrm{v}^2$$

- Same as those of a free particle at speed v
- Soliton can be identified as a particle 'going through' the system in a 'Newton's cradle' fashion

...and finite amplitude Waves

Summing periodic copies of (modified) soliton solutions: waves

- Can also be considered as solitons of Sutherland model
- Wave speed v can be both above and below v_s
- Nonlinear, amplitude-dependent dispersion relation
- Small amplitude: sound waves; large amplitude: solitons

Description in terms of classical pseudo-Fermi sea

- QM πFermi sea of k_j: $\hbar \to 0$, $\hbar\ell \to g$
- Pseudo-Fermi level $k_F = \mathrm{v}_s$
- Sound waves: holes – small gaps inside the πFermi sea
- Solitons: particles flying over the πFermi sea
- Large amplitude waves: finite gaps inside πFermi sea

Waves and solitons are dual descriptions of the system

Recent results: Dual formulation, Generating Function

with Manas Kulkarni, 2017 and ongoing

Consider x_a, $a = 1, \ldots, n$ particle coordinates with first order EOM

$$m_a \dot{x}_a = \partial_a \Phi$$

Φ a function of the x_a; m_a a set of constant "masses"

$$\begin{aligned} m_a \ddot{x}_a &= \sum_b \partial_b \partial_a \Phi \, \dot{x}_b = \sum_b \partial_a \partial_b \Phi \frac{1}{m_b} \partial_b \Phi \\ &= -\frac{\partial V}{\partial x_a} \, , \quad V = -\sum_a \frac{1}{2m_a} (\partial_a \Phi)^2 \end{aligned}$$

Choose $$\Phi = \tfrac{1}{2} \sum_{a \neq b} m_a m_b F_{ab}(x_a - x_b) + \sum_a m_a W_a(x_a)$$

(factors of m_a are for later convenience)

$$\partial_a \Phi = \sum_b m_a m_b f_{ab} + m_a w_a$$

where $$f_{ab} = F'_{ab}(x_a - x_b) \, , \quad w_a = W'_a(x_a)$$

After some algebra and symmetrizations of indices we obtain

$$
\begin{aligned}
-V &= \frac{1}{4}\sum_{b\neq c} m_b m_c (m_b + m_c) f_{bc}^2 \\
&+ \frac{1}{6}\sum_{b\neq c\neq d} m_b m_c m_d \left[f_{bc}f_{bd} + f_{cb}f_{cd} + f_{db}f_{dc}\right] \\
&+ \tfrac{1}{2}\sum_{b\neq c} m_b m_c (w_b - w_c) f_{bc} + \tfrac{1}{2}\sum_{b} m_b w_b^2
\end{aligned}
$$

Look for one- and two-body relative potentials

Conditions:

$$f_{bc}f_{bd} + f_{cb}f_{cd} + f_{db}f_{dc} = g_{bc} + g_{bd} + g_{cd}$$

for all distinct b, c, d, for some functions $g_{ab}(x_a - x_b)$, and

$$(w_b - w_c)\, f_{bc} = u_{bc} + v_b + v_c$$

for some functions $u_{ab}(x_a - x_b)$ and $v_a(x_a)$

Functional equations admitting families of solutions

Table: Solutions to functional equations for relative potentialsl

C_{abc}	$f_{ab}(x_{ab})$	$F_{ab}(x_{ab})$	$-V_{ab}(x_{ab})$
0	g/x_{ab}	$g\, m_a m_b \log\lvert x_{ab}\rvert$	$\frac{g\, m_a m_b(m_a+m_b)}{4\, x_{ab}^2}$
$-g^2$	$g \cot x_{ab}$	$g\, m_a m_b \log\lvert \sin x_{ab}\rvert$	$\frac{g^2 m_a m_b(m_a+m_b)}{4 \sin^2 x_{ab}}$
$+g^2$	$g \coth x_{ab}$	$g\, m_a m_b \log\lvert \sinh x_{ab}\rvert$	$\frac{g^2 m_a m_b(m_a+m_b)}{4 \sinh^2 x_{ab}}$

Table: Solutions to functional equations for external potential

$f_{ab}(x_{ab})$	$w_a(x_a)$	$-2V_a(x_a)$
g/x_{ab}	$c_0 + c_1 x_a + c_2\, x_a^2$	$m_a w_a^2 + g(m_{tot} - m_a)m_a(c_2\, x_a + \frac{3}{2} c_3\, x_a^2)$
$g \cot x_{ab}$	$c_0 + c_1 \cos 2x_a + c_2 \sin 2x_a$	$m_a w_a^2 + g(m_{tot} - m_a)m_a(c_2 \cos 2x_a - c_1 \sin 2x_a)$
$g \coth x_{ab}$	$c_0 + c_1 \cosh 2x_a + c_2 \sinh 2x_a$	$m_a w_a^2 + g(m_{tot} - m_a)m_a(c_2 \cosh 2x_a + c_1 \sinh 2x_a)$

- Recover standard Calogero interactions (rational, trigo, hyper)
- Arbitrary particle masses apparently allowed
- External potentials up to quartic

However

- Potential is negative definite: instability
- To cure it: make all f_{ab}, w_a imaginary
- First-order equations become complex

To have a real system

- A subset of particles x_j, $j = 1, \ldots N$, must be real
- The remaining z_α, $\alpha = 1, \ldots M$ $(M + N = n)$ can be complex: we will call them solitons
- x_j and z_α must decouple in the second-order equations
- This imposes: $m_j = -m_\alpha = m$ (put $= 1$)
- Solitons in this formulation are negative mass particles!

The dual equations

$$
\begin{aligned}
\dot{x}_j &= i \sum_{k(\neq j)} f(x_j - x_k) - i \sum_{\alpha} f(x_j - z_\alpha) + i w(x_j) \\
\dot{z}_\alpha &= i \sum_{\beta(\neq \alpha)} f(z_\alpha - z_\beta) - i \sum_{k} f(z_\alpha - x_k) + i w(z_\alpha)
\end{aligned}
$$

- Coupled, complex equations
- By construction, second order equations decouple
- Choosing x_j, $\dot{x}_j$ real at $t = 0$ they will remain real
- This implies

$$
\sum_{k(\neq j)} f(x_j - x_k) + w(x_j) = \mathrm{Re}\left(\sum_{\alpha} f(x_j - z_\alpha) \right)
$$

$$
\dot{x}_j = \mathrm{Im}\left(\sum_{\alpha} f(x_j - z_\alpha) \right)
$$

The values of z_α determine x_j, $\dot{x}_j$

For the simplest case of harmonic Calogero we have the dual system

$$\dot{x}_j - i\omega x_j = -ig \sum_{k\neq j}^{N} \frac{1}{x_j - x_k} + ig \sum_{\alpha=1}^{M} \frac{1}{x_j - z_\alpha}$$

$$\dot{z}_\alpha - i\omega z_\alpha = ig \sum_{\beta\neq\alpha}^{M} \frac{1}{z_\alpha - z_\beta} - ig \sum_{j=1}^{N} \frac{1}{z_\alpha - x_j}$$

- For *any* M, N the above equations imply:

$$\ddot{x}_j = -g^2 \sum_{k\neq j}^{N} \frac{1}{(x_j - x_k)^3} - \omega^2 x_j$$

$$\ddot{z}_\alpha = -g^2 \sum_{\beta\neq\alpha}^{M} \frac{1}{(z_\alpha - z_\beta)^3} - \omega^2 x_\alpha$$

- For $M \geq N$ system reproduces full Calogero dynamics
- For $M < N$ the above reproduces restricted Calogero solutions
- Similarly for other Calogero systems and external potentials

Deriving hydrodynamics from dual system

Take the limit of $N \to \infty$ (but M can remain small)
Device: Introduce one more spectator particle $x_0 = x$ with mass m_0

- Full system of $N + M + 1$ particles remains Calogero-like
- Must take $m_0 \to 0$ in order not to disturb the system
- Spectator particle is a "pilot fish": it becomes "trapped" and follows particles as they move

$$\frac{du}{dt} = -\partial_x \left[\sum_j \frac{1}{2} f(x - x_j)^2 + \sum_\alpha \frac{1}{2} f(x - z_\alpha)^2 \right. \\ \left. + (N - M) v(x) + \frac{1}{2} w(x)^2 \right]$$

Final twist: "Sprincle" the system with many spectator particles at various positions x ("sawdust on a stream")

- The spectator velocity $u(t)$ is promoted to a field $u(x, t)$ (x dependence arises from its dependence on initial position x)

Therefore $$\frac{du}{dt} = \partial_t u + \partial_x \left(\tfrac{1}{2} u^2\right)$$

- Need to express terms in $\frac{du}{dt}$ involving x_j and z_α in terms of u
- Necessitates splitting $u = u^+ + u^-$

$$\begin{aligned} u^+(x) &= -i \sum_\alpha f(x - z_\alpha) + i w(x) \\ u^-(x) &= i \sum_j f(x - x_j) \end{aligned}$$

Eventually, we obtain (with $V(x)$ the external potential)

$$\partial_t u + \partial_x \left[\tfrac{1}{2} u^2 + \frac{ig}{2} \partial_x (u^+ - u^-) + V \right] = 0$$

- This is a bi-chiral version of the Benjamin-Ono equation
- Equation valid for all N and M (even before $N \to \infty$)

$u^+(x)$ essentially determines everything through analyticity and its pole structure

Go to hydro limit

- Hydro obtained by expressing u^+ and u^- in terms of ρ and v

$$u^+(x) = v(x) - i\pi g \rho^H(x) + ig\,\partial_x \ln \sqrt{\rho(x)}$$
$$u^-(x \pm i0) = \mp \pi g \rho(x) + i\pi g \rho^H(x)$$

Finally some generalized solitons in external potentials

One-soliton solution: a single $z_\alpha = Z$ satisfying the equation

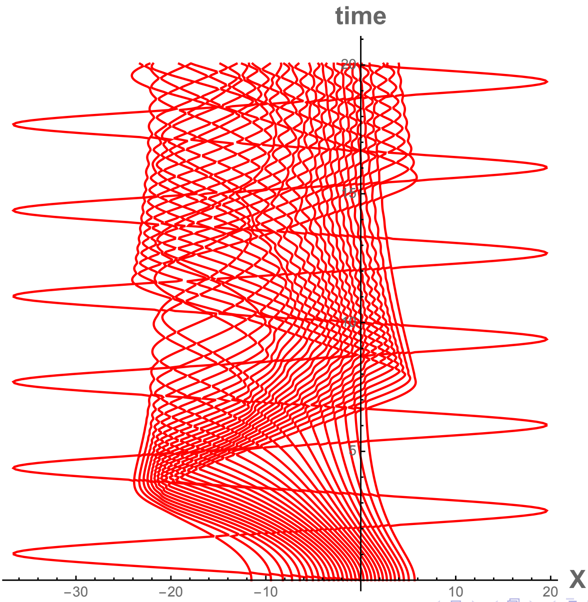
$$\ddot{z} + V'(z) = 0$$

Fixes both $x_j(t)$ in N-body system and $u^\pm(x,t)$ in hydro limit

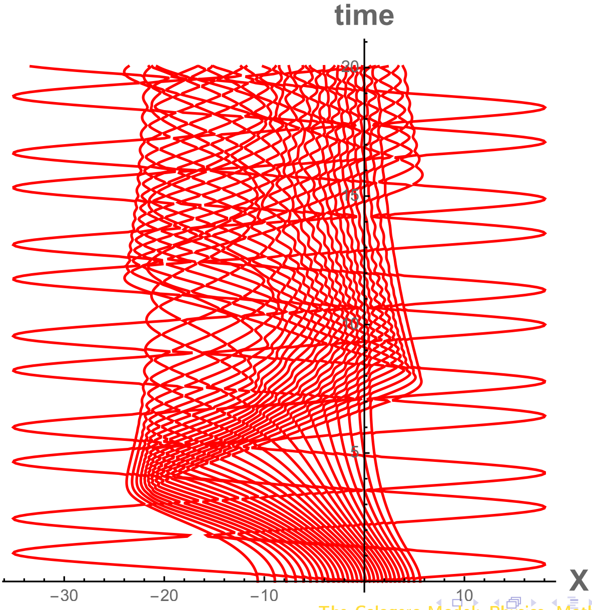
$$u^+(x,t) = \frac{ig}{x - z(t)} + iw(x)$$
$$v - ig(\pi\rho^H - \partial_x \log\sqrt{\rho}) = \frac{ig}{x - z} + iw(x)$$

- Above equations can in principle be solved for x_j, ρ and v
- Numerical solutions easily obtainable
- Et voilà some pictures:

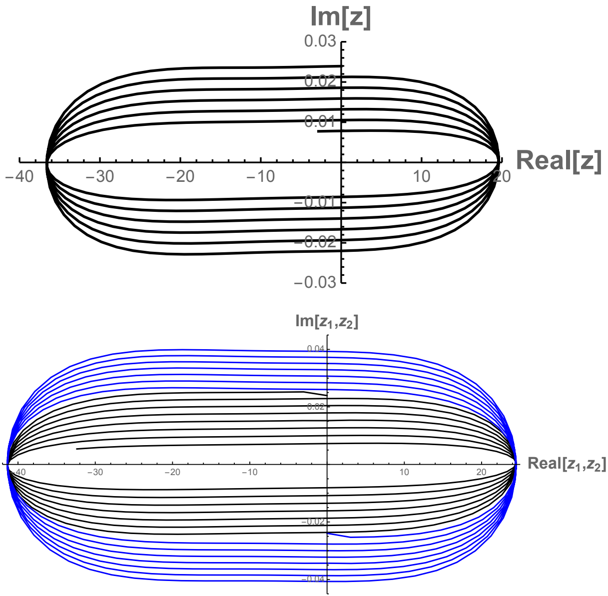
One soliton solution for quartic potential



Two soliton solution for quartic potential



$z(t)$ on the complex plane for one and two solitons



And at last... Anyons!

with Stéphane Ouvry, 2018 and ongoing...

Fundamental question:

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Fundamental question:

Aren't you sick of me already??

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- *Never underestimate the pleasure of the audience listening to what they already know*

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Fundamental question:

Aren't you sick of me already??

Didn't you have enough Calogero yet??

Will build upon two **important principles**:

- *Never underestimate the pleasure of the audience listening to what they already know*
- *Nobody ever hated a speaker for finishing early*
  (Now as for those who go overtime...)

Indications of relation between these models

- Similarity of Calogero and anyon LLL wavefunctions
- Real one-dimensional coordinate maps to complex coordinate on the plane

Already some results:

- Alternative realizations of operators in two systems Brink *et al.* 1993
- Matrix model noncommutative realization of FQH states AP 2001

We would like an **explicit** mapping rather than a formal one

Why is such a mapping useful?

- Instrinsically interesting
- Calculation of quantities in either side may be easier
- E.g., density correlation calculations
- Exploit particle $\leftrightarrow$ hole duality in either system

In the sequel we will derive an N-body kernel that maps Calogero to anyon wavefunctions

- N nonrelativistic anyons of unit mass and unit charge
- Anyonic statistical parameter α
- Constant magnetic field $B = 2\omega_c$
- Weak confining harmonic trap of frequency ω_o

Singular gauge: anyon statistics encoded in the monodromy of the wavefunction

- LLL states are analytic in z_i
- (unnormalized) 1-body eigenstates are $z_i^{\ell_i}$
- Many-body LLL eigenstates are

$$\psi_{\text{free}} = \prod_{i<j}(z_i - z_j)^{\alpha} e^{-\omega \sum_{i=1}^{N} z_i \bar{z}_i/2} \sum_{\pi \in S_N} \prod_{i=1}^{N} z_{\pi(i)}^{\ell_i}$$

with the spectrum

$$E_N = (\omega_t - \omega_c)\left[\sum_{i=1}^{N} \ell_i + \frac{1}{2}N(N-1)\alpha + N\right] + N\omega_t$$

The Calogero model

- Same spectrum as N-body 1d harmonic Calogero model with $\omega = \omega_t - \omega_o$ and $\ell = \alpha$ (up to a global $N\omega_t 2$ shift)
- We will seek mapping between anyon and Calogero wavefunctions of same energy (from now on set $\omega = 1$)

N-body ground state of harmonic Calogero Hamiltonian

$$\psi_0 = \prod_{i>j} (x_i - x_j)^g \, e^{-\sum_{i=1}^N x_i^2/2} \; := \; \Delta_x{}^g \, e^{-[x^2]/2}$$

where

$$\Delta_x = \prod_{i>j} (x_i - x_j) \, , \qquad [x^2] = \sum_{i=1}^N x_i^2$$

- ψ_0 is bosonic and above holds in the "wedge" $x_1 < \cdots < x_N$
- Use g instead of ℓ to avoid confusion with ℓ_i

The kernel

Consider the scattering Calogero N-body Hamiltonian is

$$\bar{H}_g = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}\frac{g(g-1)}{(x_i - x_j)^2}$$

Define $h_g[x, z]$ to be the scattering Calogero eigenstate with asymptotic momenta $z_1 < \cdots < z_N$

$$\bar{H}_g\, h_g[x, z] = \tfrac{1}{2}[z^2]\, h_g[x, z]$$

- $h_g[x, z]$ behaves as $(x_i - x_j)^g$ when $x_i \to x_j$
- Asymptotically combination of scattered plane waves

$$e^{i\sum_i x_i z_i} \to e^{-i\pi N(N-1)g/4} \sum_{\pi \in S_N} e^{i\pi g\, c(\pi)}\, e^{i\sum_i x_{\pi(i)} z_i}$$

- $c(\pi)$ is the number of particle crossings in $\{x_{\pi(i)}\}$
- Phase chosen symmetric for incident wave $e^{i(x_1 z_N + \cdots + x_N z_1)}$ and scattered wave $e^{i(x_1 z_1 + \cdots + x_N z_N)}$

The appropriate kernel is

$$k_g[x,z] = e^{[z^2]/4} e^{-[x^2]}\, h_g[x,z]$$

- Scattering momenta z_i become anyon coordinates

The proof is rather long (but oh, so beautiful... read all about it!)

Basic steps:

- Generate Calogero states through action of ladder operators
- Conjugate ladder operators to their free form
- Use (anti)hermiticity to make them act on $h_g[x,z]$
- Turn their action into eigenvalues of scattering conserved quantities to produce ${z_i}^{\ell_j}$ part
- Show that remaining integral reproduces Δ_z^g

The last step is the most challenging one

- Achieved only for integer g (For $N > 2$)
- Should be true in general by analytic continuation

And after a lot of work and twists...

Putting everything together we obtain our final result

$$\int e^{[z^2]/4} e^{-[x^2]/2} h_g[x,z]\, \psi_\ell[x]\, [dx] = \Delta_z{}^g \sum_{\pi \in S_N} \prod_{i=1}^{N} z_{\pi(i)}^{\ell_i}$$

$$\text{Calogero} \quad \rightarrow \quad \text{Anyons}$$

Putting everything together we obtain our final result

$$\int e^{[z^2]/4} e^{-[x^2]/2} h_g[x,z]\, \psi_\ell[x]\, [dx] = \Delta_z{}^g \sum_{\pi \in S_N} \prod_{i=1}^{N} z_{\pi(i)}^{\ell_i}$$

Calogero $\rightarrow$ Anyons

Our desired mapping relation!

Open questions

- Sutherlad to what? (Anyons on cylinder? Maybe on sphere??)
- Elliptic to what? (Anyons on torus???)
- Interesting applications????

Conclusions and Outlook

A grand tour in Calogero land, but we left out many sights:

- Foldings and new systems obtained through them (anisotropic, noncommutative etc)
- Density correlations and their analytic challenges (Jack polynomials, integral expressions etc.)
- Connection with Yang-Mills and Chern-Simons field theories
- "Relativistic" Calogero systems (Ruijsenaars-Schneider model)
- Duality and its manifestations; etc. etc.

Many open questions remaining

- Formulation encompassing Matrix and Operator ones
- Correlations for irrational values of ℓ
- "Continuous" expression of density correlations
- Full solution of Elliptic system
- Anything at all on the noncommutative model
- (Use your imagination...)

...sometimes doing things for fun rather than profit is the most profitable path

Thank You!