Local Anyons on the Circle leading to 2D Cone-local Anyons

Mathematical physics of anyons and topological states of matter

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- **Anyons:**
  ▷ Non-standard statistics $\rightarrow$ arbitrary phase factor in commutation relations

$$\phi_1\phi_2 \sim e^{i\pi\lambda}\phi_2\phi_1$$

  ▷ Non-trivial behavior under 2π-rotation $\rightarrow$ real valued spin $S_q \in \mathbb{R}$

$$U(2\pi) \sim e^{2\pi i S_q} \qquad \text{(here: } q \text{ ... } U(1)\text{-charge)}$$

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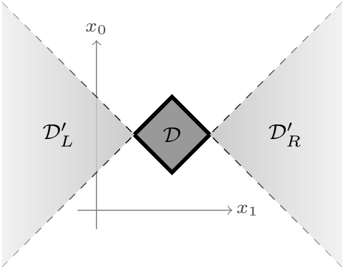
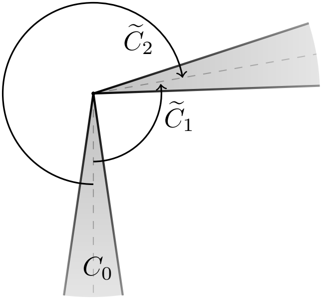
- Possibility for Anyon quantum fields depends on the dimension and on the localization properties of the fields

- Best possible localization of Anyons:
  ▷ in **1D**: Compact localization (double cones, intervals)
  ▷ in **2D**: localization in (generalized) cones *[Buchholz/Fredenhagen]*



disjoint causal complement



$\widetilde{C}_1$ and $\widetilde{C}_2$ differing by 2π-rotation

Field theories for Anyons often involve gauge theories with a Chern-Simons term $\mathcal{L}_{CS} \sim A \wedge dA$ and/or a Higgs term $\sim \lambda\phi^4$
(although not necessary for Anyon-statistics $\rightarrow$ cf. *[Liguori/Mintchev/Rossi]*)

Construction of fields often indirect: Lagrangian $\rightarrow$ Schwinger functions $\rightarrow$ O.S. reconstruction (e.g. *[Fröhlich/Marchetti]*);
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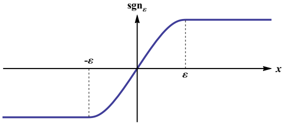
More explicit Constructions:

- *In d=1+1*: Operators multiplying with a kink-function as implementable Bogoliubov transformations $\to$ disorder operators



$\to$ "Left-right" dependent statistics

$$\phi(x_1)\phi(x_2) \sim e^{i\pi\lambda\,\mathrm{sgn}(x_1-x_2)}\phi(x_2)\phi(x_1)$$

- *On the circle:* Similar construction as in $d = 1 + 1 \rightarrow$ implementers of "Blip functions" (changes phase by 2π in small interval) *[Carey/Langmann]*
  Phase factor is again of the form $e^{i\pi\lambda\,\mathrm{sgn}(x_1 - x_2)}$ with $x_1, x_2 \in S_1$
- *In d=2+1:* Constructions also often lead to something like

$$\Psi(\vec{x})\Psi(\vec{y}) \sim e^{i\pi\lambda\,\mathrm{sgn}(\arg(\vec{x}-\vec{y}))}\Psi(\vec{y})\Psi(\vec{x})$$

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$\Rightarrow$ Want explicit field net localized in cones + dependence on winding number
Here: For simplicity first construct local operators on the (universal covering of the) circle $\rightarrow$ Non-relativistic net localized in cones on $\mathbb{R}^2$

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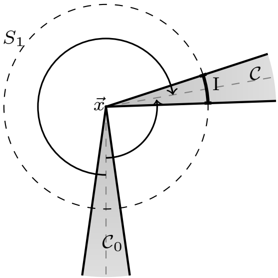
Difficulty: No-Go theorem for free Anyons: *[Bros/Mund]*

> *A covariant cone-local field net for Anyons (satisfying Reeh-Schlieder) always has non-trivial S-matrix $\Rightarrow$ There are no one-particle generators for Anyons*

For the definition of a quantum field net for anyons we first need the concept of generalized cones (or "paths of cones").

Usual cone C on $\mathbb{R}^2$ specified by two ingredients:
- ▹ *point* $\vec{x} \in \mathbb{R}^2$ and
- ▹ *interval* $I \subset S_1$ on the circle

(with S_1 serving as the set of directions)



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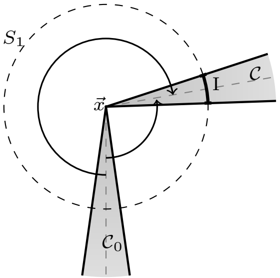

Taking the universal covering $\widetilde{S_1}$ of S_1 and generalized intervals $\tilde{I} \subset \widetilde{S_1}$ a *generalized cone* $\tilde{C}$ can be defined as the pair

$$\tilde{C} \mathrel{\widehat{=}} (\vec{x}, \tilde{I})$$

$\rightarrow$ Depends on the "winding number" of $\tilde{I}$!

$\Rightarrow$ Not invariant under 2π-rotations!

Definition: An anyonic quantum field net is a map $\tilde{C} \mapsto \mathcal{F}(\tilde{C})$, assigning to every generalized cone $\tilde{C}$ a $*$-algebra $\mathcal{F}(\tilde{C})$ of operators acting on a Hilbert space $\mathcal{H}$, which satisfies the following axioms:

1. **Isotony:** The map $\tilde{C} \mapsto \mathcal{F}(\tilde{C})$ preserves inclusions, i.e. $\mathcal{F}(\tilde{C}_1) \subset \mathcal{F}(\tilde{C}_2)$ if $\tilde{C}_1 \subset \tilde{C}_2$.

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2. **Charged Sectors:** The Hilbert space splits into a direct sum of charge eigenspaces

$$\mathcal{H} = \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_q \ , \qquad \text{vacuum } \Omega \in \mathcal{H}_0$$

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3. **Basic Fields:** In every local algebra $\mathcal{F}(\tilde{C})$ there exist charge shifting “fields” $\Phi, \Phi^* \in \mathcal{F}(\tilde{C})$ such that
$$\Phi \mathcal{H}_q \subset \mathcal{H}_{q+1} \ , \quad \Phi^* \mathcal{H}_q \subset \mathcal{H}_{q-1}.$$

4 **Covariance and Spin:** On $\mathcal{H}$ there is a representation U of $\widetilde{E(2)} \simeq \mathbb{R}^2 \rtimes \mathbb{R}$ with

$$U(2\pi) = \bigoplus_{q \in \mathbb{Z}} e^{2\pi i S_q} P_q,$$

where $S_q \in \mathbb{R}$ is the spin of the sector $\mathcal{H}_q$, such that

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$$\boxed{\Phi_1 \, \Phi_2 = e^{2\pi i s (2N(\tilde{C}_1, \tilde{C}_2)+1)} \; \Phi_2 \, \Phi_1,} \qquad (\star)$$

where $s \in \mathbb{R}$ is a real parameter and $N(\tilde{C}_1, \tilde{C}_2) \in \mathbb{Z}$ denotes the relative winding number of $\tilde{C}_1$ w.r.t. $\tilde{C}_2$.

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The commutation relations $(\star)$, together with $S_0 = 0$ and $S_{-q} = S_q$ already lead to

Lemma (Spin addition rule)

$$S_q = sq^2$$

Model on the **circle** $\rightarrow$ simple enough to construct explicitly (implementable multiplication operators) but still capable of showing non-trivial rotational behavior!

$\Rightarrow$ Use representation of $\widetilde{U(1)}$ of the form

$$U(\omega) = e^{i\omega s Q^2} U_0(\omega)$$

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Basic idea of construction:

- Implementer of one-particle multiplication operators $\rightarrow$ Auxiliary field $\hat{\Phi}$ on $S_1 \simeq [0, 2\pi)$, covariant under $U_0 = \Gamma(U_1)$
- Lift to covering space $\widetilde{S_1}$ by using the full representation U

$$\boxed{\Phi_\omega := e^{i\omega s Q^2} \hat{\Phi}_\omega e^{-i\omega s Q^2}}\,, \quad \hat{\Phi}_\omega = U_0(\omega) \hat{\Phi}_0 U_0(-\omega)$$

- **Hilbert space:** $\mathcal{H}_1 = L^2(S_1) \simeq l^2(\mathbb{N}_0) \oplus l^2(\mathbb{N}_-) =: \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$
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  ▷ Free field $\phi(\varphi) := a^*(\varphi^+) + b(\overline{\varphi_-})$
  ▷ Implementer of diagonal unitary operator $U = U_{++} \oplus U_{--}$:

$$\hat{\Gamma}(U) := \Gamma(U_{++})\Gamma(\overline{U_{--}})$$

$\Rightarrow \hat{\Gamma}(e^{i\gamma}) = e^{i\gamma Q}$ and $\hat{\Gamma}(U)\phi(\varphi)\hat{\Gamma}(U)^* = \phi(U\varphi)$

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Theorem (Shale-Stinespring)

A unitary V on $\mathcal{H}_1$ has an implementer $\hat{\Gamma}(V)$ on Fock space $\mathcal{F}_a(\mathcal{H}_1)$ iff

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- **Charge shifts:** An implementer $\hat{\Gamma}(V)$ shifts the charge of a vector by the integer $q(V)$, the Fredholm index of V_{--}, i.e.

$$\boxed{q(V) := \dim\ker(V_{--}) - \dim\ker(V_{--}^*)}$$

Lemma (Properties of Implementers)

i) *For a self-adjoint A the unitary operator e^{itA} is implementable $\forall t \in \mathbb{R}$ iff*

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iii) *For a charge shift* V *and self-adjoint* A, B *such that* $[A, B] = [V, A] = [V, B] = 0$ *there holds*

$$\hat{\Gamma}(e^{iA})\hat{\Gamma}(V) = e^{i\langle \hat{\Gamma}(V)\Omega, d\hat{\Gamma}(A)\hat{\Gamma}(V)\Omega\rangle}\ \hat{\Gamma}(V)\hat{\Gamma}(e^{iA}),$$
$$\hat{\Gamma}(e^{iA})\hat{\Gamma}(e^{iB}) = e^{iS(A,B)}\ \hat{\Gamma}(e^{iB})\hat{\Gamma}(e^{iA})$$

with the Schwinger term $S(A, B) := iTr(A_{-+}B_{+-} - B_{-+}A_{+-})$.

Use **multiplication operators** in x-space:

$\Rightarrow$ All occurring operators *commute*
$\Rightarrow$ The *Schwinger term* takes a simple form

Lemma (Multiplication operators)

i) *For a smooth real-valued function $\alpha \in C^\infty(S_1, \mathbb{R})$ the multiplication operator $(\alpha\varphi)(x) := \alpha(x)\varphi(x)$ satisfies*

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Therefore $e^{it\alpha}$ is implementable $\forall t \in \mathbb{R}$ and moreover $q(e^{it\alpha}) = 0$.

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ii) *For* $\alpha, \beta \in C^\infty(S_1, \mathbb{R})$ *the Schwinger term for the corresponding multiplication operators turns out to be*

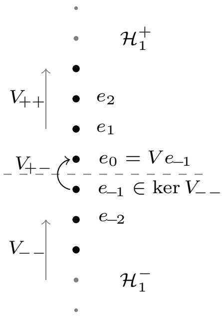
$$S(\alpha, \beta) = \frac{1}{2\pi} \int_0^{2\pi} dx\, \alpha(x)\beta'(x) = \frac{1}{4\pi} \int_0^{2\pi} dx\, \big(\alpha(x)\beta'(x) - \alpha'(x)\beta(x)\big)$$

Consider the following **shift operator**:

$$\boxed{(V\varphi)(x) := e^{ix}\varphi(x)}$$

W.r.t. the basis $e_n(x) := \frac{1}{\sqrt{2\pi}}e^{inx}$ it acts as

$$Ve_n = e_{n+1}.$$



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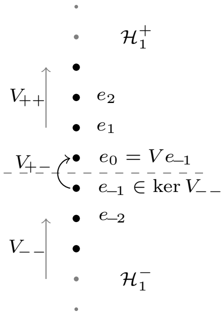

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i) *V_{+-} is Hilbert-Schmidt and $V_{-+} = 0$*

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$$\ker V_{--} = \{\lambda e_{-1} \mid \lambda \in \mathbb{R}\}\ , \quad \ker V_{--}^* = \emptyset$$

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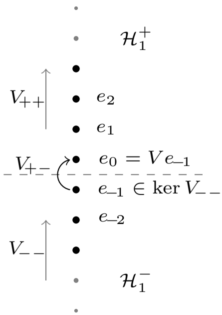

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From $V_\omega := U_1(\omega)VU_1(\omega)^* = e^{-i\omega}V$ it also follows that

$$\hat{\Gamma}(V_\omega) = \hat{\Gamma}(V)e^{i\omega Q} \implies \hat{\Gamma}(V_{\omega_1})\hat{\Gamma}(V_{\omega_2}) = e^{-i(\omega_1-\omega_2)}\hat{\Gamma}(V_{\omega_2})\hat{\Gamma}(V_{\omega_1})$$

A smooth function $\alpha \in C^{\infty}(S_1)$ (defined later) and real parameter $\lambda \in \mathbb{R}$ then lead to

Definition (Auxiliary field)

$$\hat{\Phi}_\omega := \hat{\Gamma}(V_\omega)\hat{\Gamma}(e^{i\lambda\alpha_\omega})$$
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$\rightarrow$ **Commutation relations:**

$$\Phi_{\omega_1}\Phi_{\omega_2} = e^{i\left[(1-2s)(\omega_2-\omega_1)+\lambda^2 S(\alpha_{\omega_1},\alpha_{\omega_2})\right]}\,\Phi_{\omega_2}\Phi_{\omega_1}$$

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With $\lambda^2 = (1-2s)$ and the winding number

$$2\pi N(\omega) := \omega - \hat{\omega} \qquad \hat{\omega} \text{ ... projection onto } S_1$$

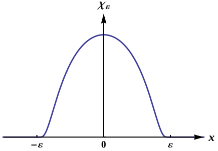
we need to find a smooth function α such that the Schwinger term satisfies

$$\boxed{S(\alpha_{\omega_1},\alpha_{\omega_2}) = (\widehat{\omega_1 - \omega_2}) - \pi}$$

Need multiplication operator of the form $e^{i\lambda\hat{x}}$ for $\lambda \in \mathbb{R}$
But: not smooth on the circle $\Rightarrow$ not implementable

Solution: Smearing
with a bump function $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$ with properties

- $\operatorname{supp}\chi_\varepsilon = [-\varepsilon, \varepsilon]$ with $0 < \varepsilon < \frac{\pi}{2}$
- $\chi_\varepsilon(-x) = \chi_\varepsilon(x) = \overline{\chi_\varepsilon(x)}$
- $\int dx\, \chi_\varepsilon(x) = 1$



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Solution: Smearing
with a bump function $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$ with properties

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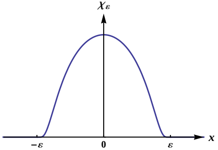
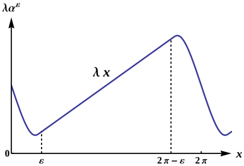


Now define a smooth 2π-periodic function α^ε according to

$$\boxed{\alpha^\varepsilon(x) := \int dy(\widehat{x-y})\chi_\varepsilon(y)}$$



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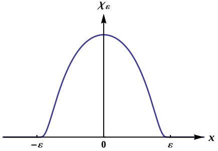

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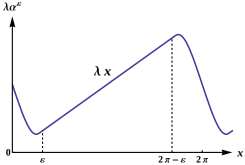

Lemma (Schwinger Term)

For $\omega_1, \omega_2 \in \mathbb{R}$ such that $\varepsilon_1 + \varepsilon_2 < (\widehat{\omega_1 - \omega_2}) < 2\pi - \varepsilon_1 - \varepsilon_2$ the Schwinger term satisfies

$$S(\alpha^{\varepsilon_1}_{\omega_1}, \alpha^{\varepsilon_2}_{\omega_2}) = (\widehat{\omega_1 - \omega_2}) - \pi\,.$$

Conclusion: For every admissible smearing function χ_ε with support in $I_0^\varepsilon = (-\varepsilon, \varepsilon)$ one can construct fields

$$\Phi_\omega^\varepsilon = \hat{\Gamma}(e^{i\lambda\alpha_\omega^\varepsilon})\ \hat{\Gamma}(V_\omega)e^{i\omega s(2Q+1)}$$

localized in the interval

$$\tilde{I}_\omega^\varepsilon := \{x \in \widetilde{S_1} \mid \omega - \varepsilon < x < \omega + \varepsilon\} \subset \widetilde{S_1},$$

satisfying anyonic commutation relations.

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$(\Phi_\omega^\varepsilon)^* \Longrightarrow$ Polynomial algebra $\Longrightarrow$ Local algebras $\mathcal{F}(\tilde{I}_\omega^\varepsilon)$

$\Longrightarrow$ **Local, covariant quantum field net for Anyons on the (covering of the) circle!**

Consider the free field

$$\Psi(f) := c^*(f) + c(\bar{f}) \ , \quad \{c(f), c^*(g)\} = \langle f, g \rangle$$

on the anti-symmetric Fock space $\mathcal{F}_a(L^2(\mathbb{R}^2))$. For $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ it satisfies

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Now define the composite field

$$\boxed{F^\varepsilon_\omega(f) := \Psi(f) \otimes \Phi^\varepsilon_\omega}$$

on the total Hilbert space $\mathcal{H} = \mathcal{F}_a(L^2(\mathbb{R}^2)) \otimes \mathcal{F}_a(L^2(S_1))$.

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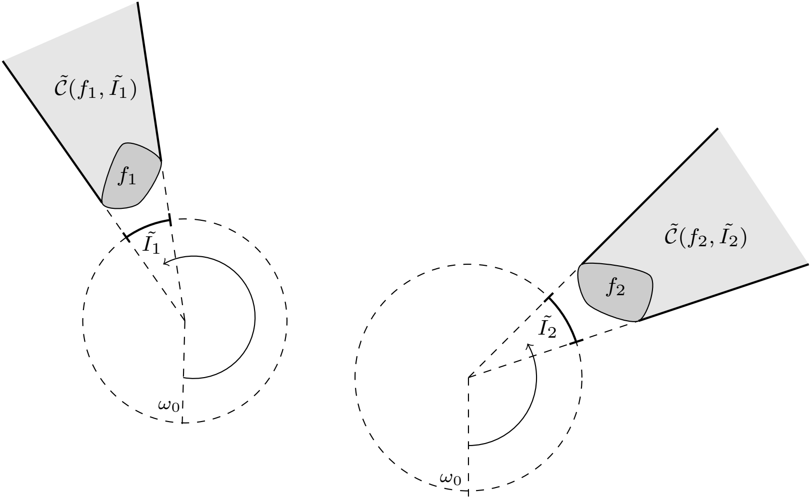
$\Rightarrow$ Localized in the (generalized) cone

$$C[f, I] := \operatorname{supp} f + \mathbb{R}_+ \bigcup_{\mu \in I^\varepsilon_\omega} \vec{n}_\mu \subset \mathbb{R}^2$$

$$\tilde{C}[f, \tilde{I}^\varepsilon_\omega] \leftrightarrow (\operatorname{supp} f, \tilde{I}^\varepsilon_\omega),$$



$\tilde{\mathcal{C}}(f_1, \tilde{I}_1)$
f_1
$\tilde{I}_1$
ω_0
$\tilde{\mathcal{C}}(f_2, \tilde{I}_2)$
f_2
$\tilde{I}_2$
ω_0

- Covariant under a representation of translations and rotations of the form

$$\hat{\Gamma}(\mathcal{U}(\vec{a},\omega)) \otimes e^{is\omega Q^2}\hat{\Gamma}(U_1(\omega))$$

- Satisfies anyonic commutation relations

$$F^{\varepsilon_1}_{\omega_1}(f_1)F^{\varepsilon_2}_{\omega_2}(f_2) = e^{2\pi is(2N(\tilde{C}_1,\tilde{C}_2)+1)}\ F^{\varepsilon_2}_{\omega_2}(f_2)F^{\varepsilon_1}_{\omega_1}(f_1),$$

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- Generalization to arbitrary field algebra $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ on $\mathbb{R}^{1+2}$

$$\mathcal{F}(\tilde{C}) = \bigcup_{\mathcal{O}\subset C} \mathcal{F}(\mathcal{O}) \otimes \mathcal{F}(\tilde{I}),$$

$\rightarrow$ field net localized in generalized cones $\tilde{C}$ with winding-number dependent anyonic commutation relations

Shortcomings and open questions:

- No boost covariance. Representation of the full (covering of the) Lorentz group:

  $$U(\tilde{\Lambda}) \sim e^{isQ\Omega(\tilde{\Lambda})}U_0(\Lambda)$$

  where $\Omega(\tilde{\Lambda})$ is now an operator on the Hilbert space [Wigner rotation $\Omega(\tilde{\Lambda}, p)$] $\rightarrow$ simple trick of lifting an auxiliary field to covering space only works for pure rotations
- Method of using multiplication operators as implementable Bogoliubov transformations leads to problems in more than one dimension (Hilbert-Schmidt property)
- Reeh-Schlieder property for intervals (or cones)?
- Direct (more "physical") construction in $d = 2 + 1$?

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