The Localization-Topology Correspondence: Periodic Systems and Beyond

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based on joint work with G. Marcelli, D. Monaco, M. Moscolari, A. Pisante, S. Teufel

Mathematical Physics of Anyons and Topological States of Matter

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Plan of the talk

- Intro: from the Transport-Topology Correspondence [TKNN] to the Localization-Topology Correspondence

I. Chern insulators vs ordinary insulators: the periodic case

II. Chern insulators vs ordinary insulators: the non-periodic case
    work in progress with G. Marcelli and M. Moscolari (preliminary results in one direction, conjecture in the other direction)
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- Retrospectively, one can rephrase the idea as follows:
  \[
  \text{periodic* Hamiltonian } \xrightarrow{BF} \text{ Fermi projector } \{P(k)\}_{k \in \mathbb{T}^d} \longrightarrow \text{ Bloch bundle } E_P
  \]

\[
\sigma_{xy}^{(Kubo)} = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr} \left( P(k) [\partial_{k_1} P(k), \partial_{k_2} P(k)] \right) dk = \frac{1}{2\pi} C_1(E_P)
\]

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- Recent generalizations to interacting electrons [GMP; BRF, MT].
- Other complementary explanations of the QHE are possible, but not covered here [Laughlin, Fröhlich, Halperin]. Some can be adapted to topological insulators.

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The predecessor: Transport-Topology Correspondence
Chern insulators and the Quantum Anomalous Hall effect

Left panel: transverse and direct resistivity in a Quantum Hall experiment

Right panel: transverse and direct resistivity in a Chern insulator (hysteresis cycle)

High-precision realization of robust quantum anomalous Hall state in a hard ferromagnetic topological insulator

Cui-Zu Chang\textsuperscript{1,}\textsuperscript{*}, Weiwei Zhao\textsuperscript{2,}\textsuperscript{*}, Duk Y. Kim\textsuperscript{3}, Haijun Zhang\textsuperscript{3}, Badih A. Assaf\textsuperscript{4}, Don Heiman\textsuperscript{4}, Shou-Cheng Zhang\textsuperscript{3}, Chaoxing Liu\textsuperscript{2}, Moses H. W. Chan\textsuperscript{2} and Jagadeesh S. Moodera\textsuperscript{1,}\textsuperscript{*}

Precise Quantization of the Anomalous Hall Effect near Zero Magnetic Field

A. J. Bestwick,\textsuperscript{1,}\textsuperscript{2} E. J. Fox,\textsuperscript{1,}\textsuperscript{2} Xufeng Kou,\textsuperscript{3} Lei Pan,\textsuperscript{3} Kang L. Wang,\textsuperscript{3} and D. Goldhaber-Gordon\textsuperscript{1,}\textsuperscript{2,}\textsuperscript{*}

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We report a nearly ideal quantum anomalous Hall effect in a three-dimensional topological insulator thin film with ferromagnetic doping. Near zero applied magnetic field we measure exact quantization in the Hall resistance to within a part per 10 000 and a longitudinal resistivity under 1 $\Omega$ per square, with chiral edge transport explicitly confirmed by nonlocal measurements. Deviations from this behavior are found to be caused by thermally activated carriers, as indicated by an Arrhenius law temperature dependence. Using the deviations as a thermometer, we demonstrate an unexpected magnetocaloric effect and use it to reach near-perfect quantization by cooling the sample below the dilution refrigerator base temperature in a process approximating adiabatic demagnetization refrigeration.
Part I

The Localization-Topology Correspondence:

the periodic case

Joint work with D. Monaco, A. Pisante and S. Teufel
How to measure localization of extended states?

► Our question: Relation between dissipationless transport, topology of the Bloch bundle and localization of electronic states.
How to measure localization of extended states?


What is the relevant notion of localization for periodic quantum systems?
How to measure localization of extended states?


- **Spectral type??** For ergodic random Schrödinger operators localization is measured by the spectral type $\left(\sigma_{pp}, \sigma_{ac}, \sigma_{sc}\right)$ [AW]. However, for periodic systems the spectrum is generically *purely absolutely continuous*. 

---


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- Kernel of the Fermi projector?? For gapped periodic 1-body Hamiltonians one has

$$|P_{\mu}(x, y)| \sim e^{-\lambda_{\text{gap}}|x-y|}$$

as a consequence of Combes-Thomas theory [AS$_2$, NN].

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Notice that crucially

$$|P_\mu(x, y)| \simeq e^{-\lambda_{\text{gap}}|x-y|} \not\Rightarrow \left\{ \begin{array}{l} \text{exist CWFs such that} \\
|w_{a,\gamma}(x)| \simeq e^{-c|x-\gamma|} \end{array} \right.$$
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\end{array} \right\} \not\Rightarrow \text{(TOPOLOGICAL TRIVIALITY)}$$

(GAP CONDITION) \not\Rightarrow \text{(TOPOLOGICAL TRIVIALITY)}
Theorem [MPPT]: The Localization Dichotomy for periodic systems

Under assumptions specified later, the following holds:

- **either** there exists $\alpha > 0$ and a choice of CWFs $\tilde{\mathbf{w}} = (\tilde{w}_1, \ldots, \tilde{w}_m)$ satisfying

$$\sum_{a=1}^{m} \int_{\mathbb{R}^d} e^{2\beta|x|} |\tilde{w}_a(x)|^2 \, dx < +\infty \quad \text{for every } \beta \in [0, \alpha);$$

- or for every possible choice of CWFs $\mathbf{w} = (w_1, \ldots, w_m)$ one has

$$\langle \mathbf{X}^2 \rangle_{\mathbf{w}} = \sum_{a=1}^{m} \int_{\mathbb{R}^d} |w_a(x)|^2 \, dx = +\infty.$$
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Intermediate regimes are forbidden!!
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  \]

The result is largely model-independent: it holds for tight-binding as well as continuum models.

Synopsis: symmetry, localization, transport, topology

Time-reversal symmetry
↓
Trivial Bloch bundle
⇔
Exponentially loc. Wannier functions
\exists \beta > 0 : e^{\beta|x|} w \in L^2(\mathbb{R}^d)
↓
Vanishing Hall current

Symmetry

Broken TR symmetry

Topology

Non-trivial Bloch bundle
⇔
Delocalized Wannier functions
\langle w, |x|^2 w \rangle = +\infty
↓
Non-zero Hall current

Localization

Transport
Chern insulators: For $d \in \{2, 3\}$, consider the magnetic Schrödinger operator

$$H_\Gamma = \frac{1}{2} (-i \nabla_x - A_\Gamma(x))^2 + V_\Gamma(x)$$

acting in $L^2(\mathbb{R}^d)$

where $A_\Gamma$ and $V_\Gamma$ are periodic with respect to $\Gamma = \text{Span}_\mathbb{Z} \{a_1, \ldots, a_d\}$.
Setting: Magnetic periodic Schrödinger operators

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Hence the modified Bloch-Floquet transform (Zak transform)

$$(U \psi)(k, y) := \sum_{\gamma \in \Gamma} e^{-i k \cdot (y - \gamma)} (T_{\gamma} \psi)(y), \quad y \in \mathbb{R}^d, k \in \mathbb{R}^d$$

provides a simultaneous decomposition of $H_{\Gamma}$ and $T_{\gamma}$:

$$U H_{\Gamma} U^{-1} = \int_\mathbb{B} dk \, H(k) \quad \text{with} \quad H(k) = \frac{1}{2} ( -i \nabla_y - A_{\Gamma}(y) + k)^2 + V_{\Gamma}(y).$$
Chern insulators: For \( d \in \{2, 3\} \), consider the magnetic Schrödinger operator

\[
H_\Gamma = \frac{1}{2} \left( -i \nabla_x - A_\Gamma(x) \right)^2 + V_\Gamma(x)
\]
acting in \( L^2(\mathbb{R}^d) \)

where \( A_\Gamma \) and \( V_\Gamma \) are periodic with respect to \( \Gamma = \text{Span}_\mathbb{Z} \{a_1, \ldots, a_d\} \).

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\]

The operator \( H(k) \) acts in \( L^2_{\text{per}}(\mathbb{R}^d) := \left\{ \psi \in L^2_{\text{loc}}(\mathbb{R}^d) : T_\gamma \psi = \psi \text{ for all } \gamma \in \Gamma \right\} \).
The Bloch bundle and its topology

For each fixed $k \in \mathbb{R}^d$, the operator $H(k)$ has compact resolvent, so pure point spectrum accumulating at $+\infty$.

Question: how can I read from the picture whether TR-symmetry is broken?

Spectrum of $H(k)$ as a function of $k_j$ (left).
Spectrum of $H_{\Gamma} = \int \oplus H(k) dk$ (right).
For each fixed $k \in \mathbb{R}^d$, the operator $H(k)$ has **compact resolvent**, so pure point spectrum accumulating at $+\infty$.

*An isolated family of J Bloch bands.*

*Notice that the spectral bands may overlap.*
The Bloch bundle and its topology

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An isolated family of J Bloch bands.

Notice that the spectral bands may overlap.

Topology is encoded in the orthogonal projections on an isolated family of bands

\[
P_\ast(k) = \frac{i}{2\pi} \oint_{C_\ast(k)} (H(k) - \varepsilon 1)^{-1} \, dz = \sum_{n \in I_\ast} |u_n(k, \cdot)\rangle \langle u_n(k, \cdot)|.
\]

where \( C_\ast(k) \) intersects the real line in \( E_-(k) \) and \( E_+(k) \) (GAP CONDITION).
Quantum Hall insulators: For $d \in \{2, 3\}$, consider $A_b(x) = \frac{b}{2c} x \wedge \hat{e}$ with $\hat{e}$ a unit vector

$$H_{\Gamma, b} = \frac{1}{2} \left( -i \nabla_x - A_b(x) \right)^2 + V_\Gamma(x)$$

acting in $L^2(\mathbb{R}^d)$ where $V_\Gamma$ is periodic with respect to $\Gamma = \text{Span}_\mathbb{Z} \{a_1, \ldots, a_d\}$.

Assume moreover the commensurability condition for the magnetic field $B = b\hat{e}$:

$$\frac{1}{c} B \cdot (\gamma \wedge \gamma') \in 2\pi \mathbb{Q} \quad \text{for all} \quad \gamma, \gamma' \in \Gamma$$

Ordinary translations are replaced by the magnetic translations [Zak64]

$$(T_\gamma^b \psi)(x) := e^{i \gamma \cdot A_b(x)} \psi(x - \gamma) \quad \gamma \in \Gamma.$$ 

These commute with $H_{\Gamma, b}$, but satisfy the pseudo-Weyl relations

$$T_\gamma^b T_{\gamma'}^b = e^{\frac{i}{e} B \cdot (\gamma \wedge \gamma')} T_{\gamma'}^b T_\gamma^b \quad \gamma, \gamma' \in \Gamma.$$ 

The $\mathbb{Z}^d$-symmetry is recovered at the price of considering a smaller lattice $\Gamma_b \subset \Gamma$. 
Assumptions

Consider $A = A_\Gamma + A_b$ (periodic + linear) with $A_b$ satisfying the commensurability condition, and set

$$H(\kappa) = \frac{1}{2} \left( - i \nabla_y - A(y) + \kappa \right)^2 + V_\Gamma(y) \quad \text{acting in } \mathcal{H}^b_f, \quad \kappa \in \mathbb{C}^d,$$

where $\mathcal{H}^b_f := \left\{ \psi \in L^2_{\text{loc}}(\mathbb{R}^d) : T^b_\gamma \psi = \psi, \text{ for all } \gamma \in \Gamma_b \right\}.$

Assumption 1: The magnetic potential $A = A_\Gamma + A_b$ and the scalar potential $V_\Gamma$ are such that the family of operators $H(\kappa)$ is an entire analytic family in the sense of Kato with compact resolvent.

[BS]
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acting in \( \mathcal{H}^b_f \), \( \kappa \in \mathbb{C}^d \),

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**Assumption 1:** The magnetic potential \( A = A_\Gamma + A_b \) and the scalar potential \( V_\Gamma \) are such that the family of operators \( H(\kappa) \) is an entire analytic family in the sense of Kato with compact resolvent.

If \( A = A_\Gamma \) is \( \Gamma \)-periodic, with fundamental unit cell \( Y \), then it is sufficient to assume either:

- \( A \in L^\infty(Y; \mathbb{R}^2) \) when \( d = 2 \) or \( A \in L^4(Y; \mathbb{R}^3) \) when \( d = 3 \), and \( \text{div } A, V_\Gamma \in L^2_{loc}(\mathbb{R}^d) \) when \( 2 \leq d \leq 3 \);
- \( A \in L^r(Y; \mathbb{R}^2) \) with \( r > 2 \) and \( V_\Gamma \in L^p(Y) \) with \( p > 1 \) when \( d = 2 \), or \( A \in L^3(Y; \mathbb{R}^3) \) and \( V_\Gamma \in L^{3/2}(Y) \) when \( d = 3 \) (compare [BS]).

Lemma: Let $P_*(k)$ be the spectral projector of $H(k)$ corresponding to a set $\sigma_*(k) \subset \mathbb{R}$ such that the gap condition is satisfied. Then the family $\{P_*(k)\}_{k \in \mathbb{R}^d}$ has the following properties:

$(P_1)$ the map $k \mapsto P_*(k)$ is analytic from $\mathbb{R}^d$ to $\mathcal{B}(\mathcal{H}_f^b)$;

$(P_2)$ the map $k \mapsto P_*(k)$ is $\tau$-covariant, i.e.

$$P_*(k + \lambda) = \tau(\lambda) P_*(k) \tau(\lambda)^{-1} \quad \forall k \in \mathbb{R}^d, \quad \forall \lambda \in \Gamma_b^*.$$  

where $\tau : \Gamma_b^* \cong \mathbb{Z}^d \longrightarrow \mathcal{U}(\mathcal{H}_f^b)$ is a unitary representation.
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For the sake of simpler slides, here we consider the case $\tau \equiv 1$. 
Lemma: Let $P_*(k)$ be the spectral projector of $H(k)$ corresponding to a set $\sigma_*(k) \subset \mathbb{R}$ such that the gap condition is satisfied. Then the family $\{P_*(k)\}_{k \in \mathbb{R}^d}$ has the following properties:

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Concretely, for the operator $H_\Gamma$

$$\tau(\lambda)f(y) = e^{i\lambda \cdot y}f(y)$$

for $f \in L^2_{\text{per}}(\mathbb{R}^d, dy) = \mathcal{H}_f^0$.

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G. Panati
La Sapienza
The Localization-Topology Correspondence
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where $\tau : \Gamma_b^* \simeq \mathbb{Z}^d \longrightarrow \mathcal{U}(\mathcal{H}_f^b)$ is a unitary representation.

In view of (P_1) and (P_2) the ranges of $P_*(k)$ define a (smooth) Hermitian vector bundle over $\mathbb{T}_*^d := \mathbb{R}^d / \Gamma_b$. For $d = 2$, it is characterized by the Chern number

$$c_1(P) := \frac{1}{2\pi i} \int_{\mathbb{T}_*^2} \text{Tr}_{\mathcal{H}_f^b} \left( P(k) \left[ \partial_{k_1} P(k), \partial_{k_2} P(k) \right] \right) \, dk_1 \wedge dk_2$$

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IDEA: Wannier functions provide a reasonable compromise between localization in energy and localization in position space, as far as compatible with the uncertainty principle.
Wannier functions and their localization

**IDEA:** Wannier functions provide a reasonable compromise between **localization in energy** and **localization in position space**, as far as compatible with the uncertainty principle.

They are associated at the energy window corresponding to \( P_\star(\cdot) \) via

**Definition:** A Bloch frame is a collection \( \{\phi_1, \ldots, \phi_m\} \subset L^2(\mathbb{T}^d, \mathcal{H}_f^b) \) such that:

\[
(\phi_1(k), \ldots, \phi_m(k)) \text{ is an orthonormal basis of } \text{Ran } P_\star(k) \text{ for a.e. } k \in \mathbb{R}^d
\]
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IDEA: Wannier functions provide a reasonable compromise between localization in energy and localization in position space, as far as compatible with the uncertainty principle.

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**Definition:** A Bloch frame is a collection $\{\phi_1, \ldots, \phi_m\} \subset L^2(\mathbb{T}_d^*, \mathcal{H}_f^b)$ such that:

$$(\phi_1(k), \ldots, \phi_m(k)) \text{ is an orthonormal basis of } \text{Ran } P_*(k) \text{ for a.e. } k \in \mathbb{R}^d$$

In general, a Bloch frame mixes different Bloch bands

$$\phi_a(k) = \sum_{n \in I^*} u_n(k) \underbrace{U_{na}(k)}_{\text{Bloch funct.}}$$

for some unitary matrix $U(k)$. 

The ambiguity in the choice is dubbed Bloch gauge ambiguity.
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Competition between regularity and periodicity is encoded by the Bloch bundle

\[ \mathcal{E} = (E \to \mathbb{T}_d^*). \]

Existence of continuous Bloch frames is topologically obstructed if \( c_1(P_*) \neq 0. \)
**Definition (CWFs):** The composite Wannier functions \( \{ w_1, \ldots, w_m \} \subset L^2(\mathbb{R}^d) \) associated to a Bloch frame \( \{ \phi_1, \ldots, \phi_m \} \) are defined as

\[
w_a(x) := (U^{-1} \phi_a)(x) = \frac{1}{|B_b|} \int_{B_b} dk \, e^{i \mathbf{k} \cdot \mathbf{x}} \phi_a(k, x).
\]

Localization of CWFs in position space $\rightarrow$ BF transform
Smoothness of Bloch frame in momentum space $\rightarrow$

\[
\int_{\mathbb{R}^d} \langle x \rangle^2 |w_a(\gamma(x))|^2 \, dx \leftrightarrow ||\phi_a||^2_{H^s(T^d, H^b_f)}
\]

Hence, by Sobolev theorem, for \( s > d/2 \) the existence of \( s \)-localized CWFs is topologically obstructed.

But, for \( 2 \leq d \leq 3 \), still there might exist 1-localized CWFs, i.e. \( \langle X_2 \rangle_{w} < +\infty \).

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\[
\int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 \, dx \quad \leftrightarrow \quad ||\phi_a||_{H^s(\mathbb{T}_*, \mathcal{H}_f^b)}^2
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Localization of CWFs in position space \( \xrightarrow{\text{BF transform}} \) Smoothness of Bloch frame in momentum space

\[
\int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 \, dx \xleftrightarrow{\text{Sobolev theorem}} \|\phi_a\|_{H^s(\mathbb{T}_*, \mathcal{H}_b^*)}^2
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The constructive theorem

Theorem 1 [Monaco, GP, Pisante, Teufel ’17]
Assume $d \leq 3$. Consider a magnetic periodic Schrödinger operator in $L^2(\mathbb{R}^d)$ satisfying the previous assumptions (Kato smallness + commensurability + gap). Then we construct a Bloch frame in $H^s(\mathbb{T}^d; \mathcal{H}^m)$ for every $s < 1$ and, correspondingly, a system of CWFs $\{w_{a,\gamma}\}$ such that

$$
\sum_{a=1}^{m} \int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 \, dx < +\infty \quad \forall \gamma \in \Gamma, \forall s < 1.
$$
Theorem 2 [Monaco, GP, Pisante, Teufel ’17]
Assume \( d \leq 3 \). Consider a magnetic periodic Schrödinger operator in \( L^2(\mathbb{R}^d) \) satisfying the previous assumptions (Kato smallness + commensurability + gap).

The following statements are equivalent:

- **Finite second moment:** there exist CWFs \( \{w_a, \gamma\} \) such that

\[
\sum_{a=1}^{m} \int_{\mathbb{R}^d} |\mathcal{N}_{a,\gamma}(x)|^2 \, dx < +\infty \quad \forall \gamma \in \Gamma;
\]

- **Exponential localization:** there exist CWFs \( \{w_a, \gamma\} \) and \( \alpha > 0 \) such that

\[
\sum_{a=1}^{m} \int_{\mathbb{R}^d} e^{2\beta|x|} |\mathcal{N}_{a,\gamma}(x)|^2 \, dx < +\infty \quad \forall \gamma \in \Gamma, \beta \in [0, \alpha);
\]

- **Trivial topology:** the family \( \{P_*(k)\}_k \) corresponds to a trivial Bloch bundle.
Synopsis: symmetry, localization, transport, topology

- **Time-reversal symmetry**
- **Symmetry**
- **Broken TR symmetry**
- **Topology**
- **Non-trivial Bloch bundle**
- **Localization**
- **Exponentially loc. Wannier functions**
  \[ \exists \beta > 0 : e^{\beta|x|} w \in L^2(\mathbb{R}^d) \]
- **Vanishing Hall current**
- **Transport**
- **Non-zero Hall current**
- **Delocalized Wannier functions**
  \[ \langle w, |x|^2 w \rangle = +\infty \]
### First column: existence of exponentially localized CWFs

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Our result applies to all \( d \leq 3 \) and is largely model-independent, since it covers both continuous and tight-binding models.
Grassmanian reinterpretation

\[ G_m(\mathcal{H}) := \{ P \in B(\mathcal{H}) : P^2 = P = P^*, \text{Tr} P = m \} \quad \text{(Grassmann manifold)} \]

\[ W_m(\mathcal{H}) := \{ J : \mathbb{C}^m \to \mathcal{H} \text{ linear isometry} \} \quad \text{(Stiefel manifold)} \]

Notice that \( J = \sum_{a=1}^{m} |\psi_a\rangle \langle \epsilon_a| \), where \( \{\psi_a\} \subset \mathcal{H} \) is an \( m \)-frame. There is a natural map \( \pi : W_m(\mathcal{H}) \to G_m(\mathcal{H}) \) sending each \( m \)-frame \( \Psi = \{\psi_1, \ldots, \psi_m\} \) into the orthogonal projection on its linear span, namely \( \pi : J \mapsto JJ^* = \sum_{a=1}^{m} |\psi_a\rangle \langle \psi_a| \).
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At least formally, \( W_m(\mathcal{H}) \) is a principal bundle over \( G_m(\mathcal{H}) \) with projection \( \pi \) and fiber \( \mathcal{U}(\mathbb{C}^m) \). The data \( P \) and \( \Phi \) correspond to a commutative diagram:

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\[
\begin{array}{ccc}
W_m(\mathcal{H}) & \xrightarrow{\Phi} & \mathbb{T}^d \\
\downarrow{\pi} & & \downarrow{P} \\
G_m(\mathcal{H}) & & G_m(\mathcal{H})
\end{array}
\]

where \( P \in C^\infty(\mathbb{T}^d; \mathcal{B}(\mathcal{H})) \) and \( \Phi \in H^s(\mathbb{T}^d; \mathcal{H}^m) \).

\[ c_1(P) = \frac{1}{2\pi i} \int_{\mathbb{T}^2} \text{Tr} H(P)(\partial_k^1 P(\mathbf{k}) [\partial_k^1 P(\mathbf{k}), \partial_k^2 P(\mathbf{k})]) d\mathbf{k} = \frac{1}{2\pi i} \int_{\mathbb{T}^2} m \sum_{a=1}^{m} \text{Im} \langle \partial_k^1 \phi_a(\mathbf{k}), \partial_k^2 \phi_a(\mathbf{k}) \rangle_{\mathcal{H}} d\mathbf{k}. \]
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\[
c_1(P) = \frac{1}{2\pi i} \int_{T^*_2} \text{Tr}_\mathcal{H} \left( P(k) [\partial_{k_1} P(k), \partial_{k_2} P(k)] \right) dk_1 \wedge dk_2
\]

\[
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Since the set of \( m \)-frames is not a linear space, the approximation of a given Sobolev map by smooth maps might be topologically obstructed.

where \( P \in C^\infty(T^d; \mathcal{B}(\mathcal{H})) \) and \( \Phi \in H^s(T^d; \mathcal{H}^m) \).
Approximation of a given Sobolev map

Theorem [Hang & Lin]
Let $2 \leq d \leq 3$. Consider a compact, boundaryless, smooth submanifold $M \subset \mathbb{R}^\nu$. If $d = 3$, assume moreover that the homotopy group $\pi_2(M)$ is trivial.
Then, every Sobolev map $\Psi \in H^1(\mathbb{T}^d, M)$ can be approximated by a sequence $\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^\infty(\mathbb{T}^d, M)$ such that $\Psi^{(\ell)} \xrightarrow{H^1} \Psi$ as $\ell \to \infty$.

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- The result can be applied to the (finite-dimensional) Stiefel manifold $W_m(\mathbb{C}^n)$, using that $\pi_2(W_m(\mathbb{C}^n)) = 0$ whenever $n \geq m + 2$ (here $\mathbb{C}^n$ is regarded as a Galerkin truncation of $\mathcal{H}$).

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$\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^\infty(\mathbb{T}^d, M)$ such that $\Psi^{(\ell)} \rightharpoonup \Psi$ as $\ell \to \infty$.

- The result can be applied to the (finite-dimensional) Stiefel manifold $W_m(\mathbb{C}^n)$, using that $\pi_2(W_m(\mathbb{C}^n)) = 0$ whenever $n \geq m + 2$ (here $\mathbb{C}^n$ is regarded as a Galerkin truncation of $\mathcal{H}$).
- One has to reduce to a finite-dimensional space $V_n \simeq \mathbb{C}^n$.
- One has to reduce covariance to periodicity, while staying in a $k$-independent fiber Hilber space [CHN].
- In our specific case, there is no obstruction to construct a Bloch frame with regularity $H^s(\mathbb{T}^d; \mathcal{H}^m)$ for every $s < 1$, essentially via parallel transport [construction in the paper].

---


Part II

The Localization-Topology Correspondence:
the non-periodic case

Work in progress with G. Marcelli and M. Moscolari
A modified paradigm

Recall the periodic TKNN paradigm:

\[ \sigma_{xy}^{(\text{Kubo})} = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr} \left( P(k) \left[ \partial_{k_1} P(k), \partial_{k_2} P(k) \right] \right) dk = \frac{1}{2\pi} C_1(\mathcal{E}_P) \]

Transport

Topology
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In a non-periodic setting, the decomposition \( \{P(k)\}_{k \in \mathbb{T}^d} \) over the Brillouin torus makes no sense, and so \( \partial_{k_j} \) and the integral should be reinterpreted.
A modified paradigm

- Recall the periodic TKNN paradigm:

\[
\sigma_{xy}^{(\text{Kubo})} = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr} \left( P(k) \left[ \partial_{k_1} P(k), \partial_{k_2} P(k) \right] \right) dk = \frac{1}{2\pi} C_1(\mathcal{E}_P)
\]

- In a non-periodic setting, the decomposition \( \{ P(k) \}_{k \in \mathbb{T}^d} \) over the Brillouin torus makes no sense, and so \( \partial_{k_j} \) and the integral should be reinterpreted.

- Inspired by [AS2], we can write

\[
\sigma_{xy}^{(\text{Kubo})} = \mathcal{T} \left( i P \left[ [X_1, P], [X_2, P] \right] \right)
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where the trace per unit volume is \( \mathcal{T}(A) = \lim_{\Lambda_n \nearrow \mathbb{R}^d} |\Lambda_n|^{-1} \text{Tr}(\chi_{\Lambda_n} A \chi_{\Lambda_n}) \).
A modified paradigm

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\]

where the trace per unit volume is \( \mathcal{T}(A) = \lim_{\Lambda_n \nearrow \mathbb{R}^d} |\Lambda_n|^{-1} \text{Tr}(\chi_{\Lambda_n} A \chi_{\Lambda_n}) \).
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Inspired by \([AS_2]\), we can write

\[ \sigma_{xy}^{(\text{Kubo})} = T \left( i P \left[ [X_1, P], [X_2, P] \right] \right) =: \frac{1}{2\pi} C_1(P) \]

where the trace per unit volume is \( T(A) = \lim_{\Lambda_n \to \mathbb{R}^d} |\Lambda_n|^{-1} \text{Tr}(\chi_{\Lambda_n} A \chi_{\Lambda_n}) \).

Analogy with the NCG approach to QHE \([Co, Be, BES, Ke, \ldots]\)

\[ C_1(p) \cong \tau \left( ip \left[ \partial_1(p), \partial_2(p) \right] \right) \]

for \( p \) a projector in the rotation \( C^*\)-algebra \( \cong \) NC torus. Here \( C_1(p) \in \mathbb{Z} \).
A modified paradigm

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Questions for experts in NCG:
- Is \( \mathcal{T}(\cdot) \) a tracial state over a \( C^* \)-subalgebra of \( B(L^2(\mathbb{R}^d)) \)? Which one? NCG?
A modified paradigm

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\[ \sigma_{xy}^{(\text{Kubo})} = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr} \left( P(k) \left[ \partial_{k_1} P(k), \partial_{k_2} P(k) \right] \right) dk = \frac{1}{2\pi} C_1(\mathcal{E}_P) \]

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Question we addressed:

Is there any relation between \( C_1(P) = 0 \) and the existence of a well-localized GWB??
Definition (Generalized Wannier basis) (compare with [NN93])

An orthogonal projector $P$ acting in $L^2(\mathbb{R}^2)$ admits a G-localized generalized Wannier basis (GWB) if there exist:

(i) a Delone set $\Gamma \subseteq \mathbb{R}^2$, i.e. a discrete set such that $\exists \ 0 < r < R < \infty$ s.t.

(a) $\forall x \in \mathbb{R}^2$ there is at most one element of $\Gamma$ in the ball of radius $r$ centred in $x$ (the set has no accumulation points);

(b) $\forall x \in \mathbb{R}^2$ there is at least one element of $\Gamma$ in the ball of radius $R$ centred in $x$ (the set is not sparse);

(ii) a localization function $G$ (typically $G(x) = (1 + |x|^2)^{s/2}$ for some $s \geq 1$), a constant $M > 0$ independent of $\gamma \in \Gamma$ and an orthonormal basis of $\text{Ran} \ P$, $\{\psi_{\gamma, a}\} \gamma \in \Gamma, 1 \leq a \leq m(\gamma) < \infty$ with $m(\gamma) \leq m^* \ \forall \gamma \in \Gamma$, satisfying

$$\int_{\mathbb{R}^2} G(|x-\gamma|)^2 |\psi_{\gamma, a}(x)|^2 \, dx \leq M.$$ 

We call each $\psi_{\gamma, a}$ a generalized Wannier function (GWF).

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Localization implies Chern triviality

Theorem - work in progress [Marcelli, Moscolari, GP]

Let $P_\mu$ be the Fermi projector of a reasonable Schrödinger operator in $L^2(\mathbb{R}^2)$.

Suppose that $P_\mu$ admits a generalized Wannier basis, $\{\psi_\gamma,a\}_{\gamma \in \Gamma, 1 \leq a \leq m(\gamma)}^{m_*}$, which is $s_*$-localized in the sense

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Then, if $s_* \geq 4$ [provisional hypothesis], one has that

$$\mathcal{T} \left( i P_\mu \left[ [X_1, P_\mu], [X_2, P_\mu] \right] \right) = 0.$$
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Clearly, the optimal statement would be for $s_* = 1$, as in the periodic case. Technical difficulties.

---

Technical details:

- In [NN93] it is proved that an exponentially localized GWB exists under general hypothesis ($\Gamma$ discrete): for $d = 1$, $P_\mu X P_\mu$ has discrete spectrum $\sigma_{\text{disc}}(P_\mu X P_\mu) =: \Gamma$, and a GWB is provided by its eigenfunctions $\{\psi_\gamma,a\}_{\gamma \in \Gamma, 1 \leq a \leq m(\gamma) < m_*}$. G. Panati, La Sapienza.
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\{\psi_\gamma,a\} \quad \gamma \in \sigma_{\text{disc}}(P_\mu XP_\mu) =: \Gamma, \ a \in \{1, \ldots, m(\gamma)\}.
$$
A simple but relevant observation

Let $\widetilde{X}_j := P_\mu X_j P_\mu$ be the reduced position operator. Then, by simple algebra

$$P_\mu [X_1, P_\mu], [X_2, P_\mu] = [\widetilde{X}_1, \widetilde{X}_2].$$

(1)

If $T(\cdot)$ were cyclic, one would conclude that $C_1(P)$ is always zero. (Hall transport would be always forbidden!).

Luckily for transport theory, $T(\cdot)$ is NOT cyclic in general. Cyclicity is recovered if it happens that $P_\mu = \sum \gamma, a |\psi_\gamma, a\rangle\langle \psi_\gamma, a|$

(2)

where $\{\psi_\gamma, a\}$ is a $s$-localized GWB, with $s$ sufficiently large.

Indeed, in this case the series obtained by plugging (2) in (1) converges absolutely, hence the series can be conveniently rearranged to obtain $C_1(P) = 0$.

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Thank you for your attention!!
Synopsis: symmetry, localization, transport, topology

- **Time-reversal symmetry**
  - Trivial Bloch bundle
    - Exponentially loc. Wannier functions: $\exists \beta > 0 : e^{\beta|x|} w \in L^2(\mathbb{R}^d)$
    - Vanishing Hall current
- **Symmetry**
  - Broken TR symmetry
    - Non-trivial Bloch bundle
      - Delocalized Wannier functions: $\langle w, |x|^2 w \rangle = +\infty$
      - Non-zero Hall current
- **Topology**
  - Localization
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  - Transport