The Localization-Topology Correspondence: Periodic Systems and Beyond

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based on joint work with G. Marcelli, D. Monaco, M. Moscolari, A. Pisante, S. Teufel

Mathematical Physics of Anyons and Topological States of Matter

NORDITA Stockholm, March 2019

Plan of the talk

◇ Intro: from the Transport-Topology Correspondence [TKNN] to the Localization-Topology Correspondence

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based on joint paper with D. Monaco, A. Pisante and S. Teufel
(Commun. Math. Phys. **359** (2018))

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I. Chern insulators vs ordinary insulators: the periodic case
based on joint paper with D. Monaco, A. Pisante and S. Teufel (Commun. Math. Phys. **359** (2018))

II. Chern insulators vs ordinary insulators: the non-periodic case
work in progress with G. Marcelli and M. Moscolari
(preliminary results in one direction, conjecture in the other direction)

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$$\underbrace{\sigma_{xy}^{(\mathrm{Kubo})}}_{\text{Transport}} = -\frac{\mathrm{i}}{(2\pi)^2} \int_{\mathbb{T}^2} \mathrm{Tr}\left(P(k)[\partial_{k_1} P(k), \partial_{k_2} P(k)] \right) \mathrm{d}k = \frac{1}{2\pi} \underbrace{C_1(\mathcal{E}_P)}_{\text{Topology}}$$

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- Recent generalizations to interacting electrons [GMP; BRF, MT].
- Other complementary explanations of the QHE are possible, but not covered here [Laughlin, Fröhlich, Halperin]. Some can be adapted to topological insulators.

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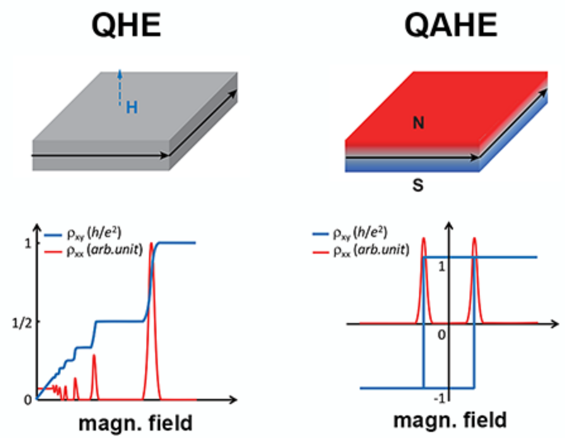
Chern insulators and the Quantum Anomalous Hall effect



Left panel: transverse and direct resistivity in a Quantum Hall experiment
Right panel: transverse and direct resistivity in a Chern insulator (histeresis cycle)

Picture: © A. J. Bestwick (2015)

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High-precision realization of robust quantum anomalous Hall state in a hard ferromagnetic topological insulator

Cui-Zu Chang[1*], Weiwei Zhao[2*], Duk Y. Kim[2], Haijun Zhang[3], Badih A. Assaf[4], Don Heiman[4], Shou-Cheng Zhang[3], Chaoxing Liu[2], Moses H. W. Chan[2] and Jagadeesh S. Moodera[1,5*]

Selected for a Viewpoint in *Physics*
PRL 114, 187201 (2015) PHYSICAL REVIEW LETTERS week ending 8 MAY 2015

Precise Quantization of the Anomalous Hall Effect near Zero Magnetic Field

A. J. Bestwick,[1,2] E. J. Fox,[1,2] Xufeng Kou,[3] Lei Pan,[3] Kang L. Wang,[3] and D. Goldhaber-Gordon[1,2,*]
[1]*Department of Physics, Stanford University, Stanford, California 94305, USA*
[2]*Stanford Institute for Materials and Energy Sciences, SLAC National Accelerator Laboratory, 2575 Sand Hill Road, Menlo Park, California 94025, USA*
[3]*Department of Electrical Engineering, University of California, Los Angeles, California 90095, USA*
(Received 16 January 2015; revised manuscript received 16 March 2015; published 4 May 2015)

We report a nearly ideal quantum anomalous Hall effect in a three-dimensional topological insulator thin film with ferromagnetic doping. Near zero applied magnetic field we measure exact quantization in the Hall resistance to within a part per 10 000 and a longitudinal resistivity under 1 Ω per square, with chiral edge transport explicitly confirmed by nonlocal measurements. Deviations from this behavior are found to be caused by thermally activated carriers, as indicated by an Arrhenius law temperature dependence. Using the deviations as a thermometer, we demonstrate an unexpected magnetocaloric effect and use it to reach near-perfect quantization by cooling the sample below the dilution refrigerator base temperature in a process approximating adiabatic demagnetization refrigeration.

Part I

The Localization-Topology Correspondence:

the periodic case

Joint work with D. Monaco, A. Pisante and S. Teufel

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What is the relevant notion of localization for periodic quantum systems?

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- Kernel of the Fermi projector?? For gapped periodic 1-body Hamiltonians one has

$$|P_\mu(x, y)| \simeq \mathrm{e}^{-\lambda_{\mathrm{gap}}|x-y|}$$

as a consequence of Combes-Thomas theory [AS_2, NN].

[AW] M. Aizenman, S. Warzel: *Random operators*, AMS (2015).
[AS_2] J. Avron, R. Seiler, B. Simon: *Commun. Math. Phys.* **159** (1994).
[NN] A. Nenciu, G. Nenciu: Phys. Rev B **47** (1993).

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$$|P_\mu(x,y)| \simeq \mathrm{e}^{-\lambda_{\mathrm{gap}}|x-y|} \qquad \not\Rightarrow \qquad \begin{cases} \text{exist CWFs such that} \\ |w_{a,\gamma}(x)| \simeq \mathrm{e}^{-c|x-\gamma|} \end{cases}$$

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(GAP CONDITION) $\not\Rightarrow$ (TOPOLOGICAL TRIVIALITY)

Theorem [MPPT]: The Localization Dichotomy for periodic systems

Under assumptions specified later, the following holds:

- either there exists $\alpha > 0$ and a choice of CWFs $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_m)$ satisfying

$$\sum_{a=1}^{m} \int_{\mathbb{R}^d} \mathrm{e}^{2\beta|x|} \, |\widetilde{w}_a(x)|^2 \mathrm{d}x < +\infty \qquad \text{for every } \beta \in [0, \alpha);$$

[MPPT] Monaco, Panati, Pisante, Teufel: Commun. Math. Phys. **359** (2018).

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$$\langle X^2 \rangle_w = \sum_{a=1}^{m} \int_{\mathbb{R}^d} |x|^2 \, |w_a(x)|^2 \mathrm{d}x = +\infty.$$

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Intermediate regimes are forbidden!!

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The result is largely model-independent: it holds for tight-binding as well as continuum models

[MPPT] Monaco, Panati, Pisante, Teufel: Commun. Math. Phys. **359** (2018).

Synopsis: symmetry, localization, transport, topology

Time-reversal symmetry	$\overset{Symmetry}{\longleftrightarrow}$	Broken TR symmetry
$\Downarrow$		
Trivial Bloch bundle	$\overset{Topology}{\longleftrightarrow}$	Non-trivial Bloch bundle
$\Updownarrow$		$\Updownarrow$
Exponentially loc. Wannier functions $\exists\, \beta > 0 : e^{\beta \lvert x \rvert} w \in L^2(\mathbb{R}^d)$	$\overset{Localization}{\longleftrightarrow}$	Delocalized Wannier functions $\langle w,\, \lvert x \rvert^2\, w \rangle = +\infty$
$\downarrow$		$\downarrow$
Vanishing Hall current	$\overset{Transport}{\longleftrightarrow}$	Non-zero Hall current

Setting: Magnetic periodic Schrödinger operators

Chern insulators: For $d \in \{2,3\}$, consider the magnetic Schrödinger operator

$$H_\Gamma = \tfrac{1}{2}\left(-\mathrm{i}\nabla_x - A_\Gamma(x)\right)^2 + V_\Gamma(x) \qquad \text{acting in } L^2(\mathbb{R}^d)$$

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Hence the modified Bloch-Floquet transform (Zak transform)

$$(\mathcal{U}\,\psi)(k, y) := \sum_{\gamma\in\Gamma} \mathrm{e}^{-\mathrm{i}k\cdot(y-\gamma)}\,(T_\gamma\,\psi)(y), \qquad y \in \mathbb{R}^d,\ k \in \mathbb{R}^d$$

provides a simultaneous decomposition of H_Γ and T_γ:

$$\mathcal{U} H_\Gamma \mathcal{U}^{-1} = \int_{\mathbb{B}}^{\oplus} dk\, H(k) \qquad \text{with} \quad H(k) = \tfrac{1}{2}\left(-\mathrm{i}\nabla_y - A_\Gamma(y) + k\right)^2 + V_\Gamma(y).$$

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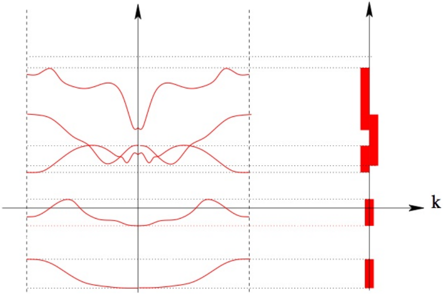
The operator $H(k)$ acts in $L^2_{\mathrm{per}}(\mathbb{R}^d) := \left\{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) : T_\gamma\psi = \psi \text{ for all } \gamma \in \Gamma\right\}$.

The Bloch bundle and its topology

For each fixed $k \in \mathbb{R}^d$, the operator $H(k)$ has compact resolvent, so pure point spectrum accumulating at $+\infty$.



Spectrum of $H(k)$ as a function of k_j (left).
Spectrum of $H_\Gamma = \int^{\oplus} H(k)\mathrm{d}k$ (right).

Question: how can I read from the picture whether TR-symmetry is broken?

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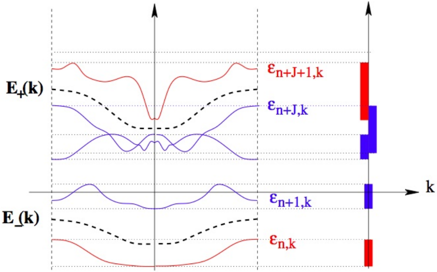

An isolated family of J Bloch bands.

Notice that the spectral bands may overlap.

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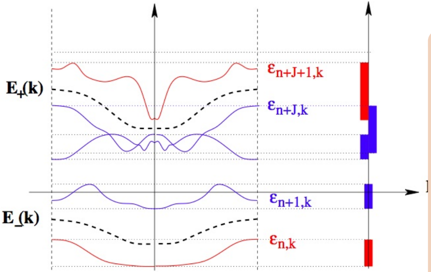

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Topology is encoded in the orthogonal projections on an isolated family of bands

$$P_*(k) = \frac{\mathrm{i}}{2\pi} \oint_{\mathcal{C}_*(k)} (H(k) - z1)^{-1} \, \mathrm{d}z = \sum_{n \in \mathcal{I}_*} |u_n(k,\cdot)\rangle \langle u_n(k,\cdot)| \, .$$

where $\mathcal{C}_*(k)$ intersects the real line in $E_-(k)$ and $E_+(k)$ (GAP CONDITION).

A model for Quantum Hall insulators

Quantum Hall insulators: For $d \in \{2,3\}$, consider $A_b(x) = \frac{b}{2c} x \wedge \hat{e}$ with $\hat{e}$ a unit vector

$$H_{\Gamma,b} = \tfrac{1}{2}\left(-\mathrm{i}\nabla_x - A_b(x)\right)^2 + V_\Gamma(x) \qquad \text{acting in } L^2(\mathbb{R}^d)$$

where V_Γ is periodic with respect to $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{a_1, \ldots, a_d\}$.
Assume moreover the commensurability condition for the magnetic field $B = b\hat{e}$:

$$\frac{1}{c} B \cdot (\gamma \wedge \gamma') \in 2\pi\mathbb{Q} \quad \text{for all} \quad \gamma, \gamma' \in \Gamma$$

Ordinary translations are replaced by the magnetic translations [Zak64]

$$(T^b_\gamma \psi)(x) := \mathrm{e}^{\mathrm{i}\gamma \cdot A_b(x)}\, \psi(x - \gamma) \qquad \gamma \in \Gamma.$$

These commute with $H_{\Gamma,b}$, but satisfy the pseudo-Weyl relations

$$T^b_\gamma T^b_{\gamma'} = \mathrm{e}^{\frac{\mathrm{i}}{c} B \cdot (\gamma \wedge \gamma')}\, T^b_{\gamma'} T^b_\gamma \qquad \gamma, \gamma' \in \Gamma.$$

The $\mathbb{Z}^d$-symmetry is recovered at the price of considering a smaller lattice $\Gamma_b \subset \Gamma$.

Assumptions

Consider $A = A_\Gamma + A_b$ (periodic + linear) with A_b satisfying the commensurability condition, and set

$$H(\kappa) = \frac{1}{2}\big(-\mathrm{i}\nabla_y - A(y) + \kappa\big)^2 + V_\Gamma(y) \qquad \text{acting in } \mathcal{H}_{\mathrm{f}}^b, \quad \kappa \in \mathbb{C}^d,$$

where $\mathcal{H}_{\mathrm{f}}^b := \left\{ \psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) : T_\gamma^b \psi = \psi, \text{ for all } \gamma \in \Gamma_b \right\}.$

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If $A = A_\Gamma$ is Γ-periodic, with fundamental unit cell Y, then it is sufficient to assume either:

- $A \in L^\infty(Y; \mathbb{R}^2)$ when $d = 2$ or $A \in L^4(Y; \mathbb{R}^3)$ when $d = 3$, and $\mathrm{div}\, A$, $V_\Gamma \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ when $2 \leq d \leq 3$;
- $A \in L^r(Y; \mathbb{R}^2)$ with $r > 2$ and $V_\Gamma \in L^p(Y)$ with $p > 1$ when $d = 2$, or $A \in L^3(Y; \mathbb{R}^3)$ and $V_\Gamma \in L^{3/2}(Y)$ when $d = 3$ (compare [BS]).

[BS] Birman, Suslina: Algebra i Analiz **11** (1999). St. Petersburg Math. J. **11**, (2000).

Lemma: Let $P_*(k)$ be the spectral projector of $H(k)$ corresponding to a set $\sigma_*(k) \subset \mathbb{R}$ such that the gap condition is satisfied. Then the family $\{P_*(k)\}_{k \in \mathbb{R}^d}$ has the following properties:

(P_1) the map $k \mapsto P_*(k)$ is analytic from $\mathbb{R}^d$ to $\mathcal{B}(\mathcal{H}_{\mathrm{f}}^b)$;

(P_2) the map $k \mapsto P_*(k)$ is τ-covariant, *i. e.*

$$P_*(k+\lambda) = \tau(\lambda)\, P_*(k)\, \tau(\lambda)^{-1} \qquad \forall k \in \mathbb{R}^d, \quad \forall \lambda \in \Gamma_b^*.$$

where $\tau : \Gamma_b^* \simeq \mathbb{Z}^d \longrightarrow \mathcal{U}(\mathcal{H}_{\mathrm{f}}^b)$ is a unitary representation.

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Concretely, for the operator H_Γ

$$\tau(\lambda) f(y) = \mathrm{e}^{\mathrm{i}\lambda \cdot y} f(y)$$

for $f \in L^2_{\mathrm{per}}(\mathbb{R}^d, dy) = \mathcal{H}_{\mathrm{f}}^0$.

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Lemma: Let $P_*(k)$ be the spectral projector of $H(k)$ corresponding to a set $\sigma_*(k) \subset \mathbb{R}$ such that the gap condition is satisfied. Then the family $\{P_*(k)\}_{k \in \mathbb{R}^d}$ has the following properties:

(P_1) the map $k \mapsto P_*(k)$ is analytic from $\mathbb{R}^d$ to $\mathcal{B}(\mathcal{H}_{\mathrm{f}}^b)$;

(P_2) the map $k \mapsto P_*(k)$ is τ-covariant, *i. e.*

$$P_*(k+\lambda) = \tau(\lambda)\, P_*(k)\, \tau(\lambda)^{-1} \qquad \forall k \in \mathbb{R}^d, \quad \forall \lambda \in \Gamma_b^*.$$

where $\tau : \Gamma_b^* \simeq \mathbb{Z}^d \longrightarrow \mathcal{U}(\mathcal{H}_{\mathrm{f}}^b)$ is a unitary representation.

In view of (P_1) and (P_2) the ranges of $P_*(k)$ define a (smooth) Hermitian vector bundle over $\mathbb{T}_*^d := \mathbb{R}^d / \Gamma_b$. For $d = 2$, it is characterized by the Chern number

$$c_1(P) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}_*^2} \mathrm{Tr}_{\mathcal{H}_{\mathrm{f}}^b}\Big(P(k)\, [\partial_{k_1} P(k), \partial_{k_2} P(k)] \Big)\, \mathrm{d}k_1 \wedge \mathrm{d}k_2$$

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Wannier functions and their localization

IDEA: Wannier functions provide a reasonable compromise between localization in energy and localization in position space, as far as compatible with the uncertainty principle.

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They are associated at the energy window corresponding to $P_*(\cdot)$ via

Definition: A Bloch frame is a collection $\{\phi_1, \ldots, \phi_m\} \subset L^2(\mathbb{T}^d_*, \mathcal{H}^b_{\mathrm{f}})$ such that:

$$(\phi_1(k), \ldots, \phi_m(k)) \text{ is an orthonormal basis of } \operatorname{Ran} P_*(k) \text{ for a.e. } k \in \mathbb{R}^d$$

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In general, a Bloch frame mixes different Bloch bands

$$\phi_a(k) = \sum_{n \in \mathcal{I}_*} \underbrace{u_n(k)}_{\text{Bloch funct.}} U_{na}(k) \qquad \text{for some unitary matrix } U(k).$$

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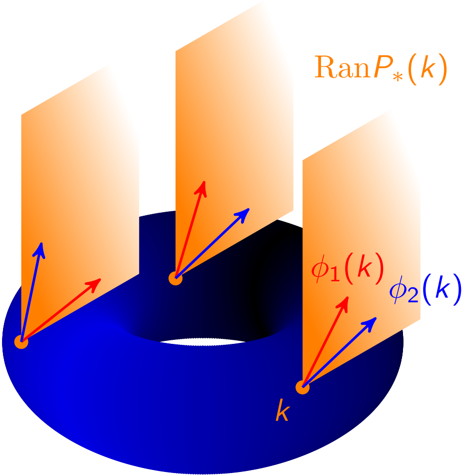
nitary matrix $U(k)$.

The ambiguity in the choice is dubbed Bloch gauge ambiguity.

The Bloch bundle



Competition between regularity and periodicity is encoded by the Bloch bundle

$$\mathcal{E} = (E \to \mathbb{T}^d_*).$$

Existence of continuous Bloch frames is topologically obstructed if $c_1(P_*) \neq 0$.

Definition (CWFs): The composite Wannier functions $\{w_1, \dots, w_m\} \subset L^2(\mathbb{R}^d)$ associated to a Bloch frame $\{\phi_1, \dots, \phi_m\}$ are defined as

$$w_a(x) := \left(\mathcal{U}^{-1}\phi_a\right)(x) = \frac{1}{|\mathbb{B}_b|}\int_{\mathbb{B}_b} \mathrm{d}k \, \mathrm{e}^{\mathrm{i}k\cdot x}\phi_a(k,x).$$

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Localization of CWFs in position space $\xrightarrow[\text{BF transform}]{}$ Smoothness of Bloch frame in momentum space

$$\int_{\mathbb{R}^d} \langle x\rangle^{2s} |w_{a,\gamma}(x)|^2 \mathrm{d}x \quad \longleftrightarrow \quad \|\phi_a\|^2_{H^s(\mathbb{T}^d_*, \mathcal{H}^b_{\mathrm{f}})}$$

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Hence, by Sobolev theorem, for $s > d/2$ the existence of s-localized CWFs is topologically obstructed.

But, for $2 \leq d \leq 3$, still there might exist 1-localized CWFs, *i. e.* $\langle X^2\rangle_w < +\infty$.

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NO! In the topologically non-trivial case, does NOT exist any system of 1-localized composite Wannier functions!

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The constructive theorem

Theorem 1 [Monaco, GP, Pisante, Teufel '17]
Assume $d \leq 3$. Consider a magnetic periodic Schrödinger operator in $L^2(\mathbb{R}^d)$ satisfying the previous assumptions (Kato smallness + commensurability + gap).
Then we construct a Bloch frame in $H^s(\mathbb{T}^d; \mathcal{H}^m)$ for every $s < 1$ and, correspondingly, a system of CWFs $\{w_{a,\gamma}\}$ such that

$$\sum_{a=1}^{m} \int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 \mathrm{d}x < +\infty \qquad \forall \gamma \in \Gamma, \forall s < 1.$$

The Localization-Topology Correspondence

Theorem 2 [Monaco, GP, Pisante, Teufel '17]

Assume $d \leq 3$. Consider a magnetic periodic Schrödinger operator in $L^2(\mathbb{R}^d)$ satisfying the previous assumptions (Kato smallness + commensurability + gap).

The following statements are equivalent:

- Finite second moment: there exist CWFs $\{w_{a,\gamma}\}$ such that

$$\sum_{a=1}^{m} \int_{\mathbb{R}^d} \langle x \rangle^2 |w_{a,\gamma}(x)|^2 \mathrm{d}x < +\infty \qquad \forall \gamma \in \Gamma;$$

- Exponential localization: there exist CWFs $\{w_{a,\gamma}\}$ and $\alpha > 0$ such that

$$\sum_{a=1}^{m} \int_{\mathbb{R}^d} \mathrm{e}^{2\beta|x|} |w_{a,\gamma}(x)|^2 \mathrm{d}x < +\infty \qquad \forall \gamma \in \Gamma, \beta \in [0, \alpha);$$

- Trivial topology: the family $\{P_*(k)\}_k$ corresponds to a trivial Bloch bundle.

Synopsis: symmetry, localization, transport, topology

Time-reversal symmetry	$\overset{\textit{Symmetry}}{\longleftrightarrow}$	Broken TR symmetry
$\Downarrow$		
Trivial Bloch bundle	$\overset{\textit{Topology}}{\longleftrightarrow}$	Non-trivial Bloch bundle
$\Updownarrow$		$\Updownarrow$
Exponentially loc. Wannier functions $\exists\, \beta > 0 : e^{\beta\|x\|} w \in L^2(\mathbb{R}^d)$	$\overset{\textit{Localization}}{\longleftrightarrow}$	Delocalized Wannier functions $\langle w, \, \|x\|^2 \, w \rangle = +\infty$
$\downarrow$		$\downarrow$
Vanishing Hall current	$\overset{\textit{Transport}}{\longleftrightarrow}$	Non-zero Hall current

References

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[Ko]	KOHN, W., *Phys. Rev.* **115** (1959)	($m = 1$, $d = 1$, even)
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Our result applies to all $d \leq 3$ and is largely model-independent, since it covers both continuous and tight-binding models.

Grassmanian reinterpretation

$$G_m(\mathcal{H}) := \quad \left\{ P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^*, \operatorname{Tr} P = m \right\} \qquad \text{(Grassmann manifold)}$$
$$W_m(\mathcal{H}) := \qquad \{ J : \mathbb{C}^m \to \mathcal{H} \text{ linear isometry} \} \qquad \text{(Stiefel manifold)},$$

Notice that $J = \sum_{a=1}^{m} |\psi_a\rangle \langle \mathfrak{e}_a|$, where $\{\psi_a\} \subset \mathcal{H}$ is an m-frame. There is a natural map $\pi \colon W_m(\mathcal{H}) \to G_m(\mathcal{H})$ sending each m-frame $\Psi = \{\psi_1, \ldots, \psi_m\}$ into the orthogonal projection on its linear span, namely $\pi : J \mapsto JJ^* = \sum_{a=1}^{m} |\psi_a\rangle \langle \psi_a|$.

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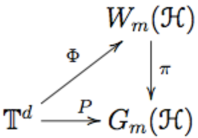
At least formally, $W_m(\mathcal{H})$ is a principal bundle over $G_m(\mathcal{H})$ with projection π and fiber $\mathcal{U}(\mathbb{C}^m)$. The data P and Φ correspond to a commutative diagram:



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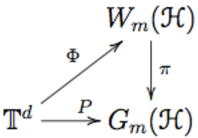

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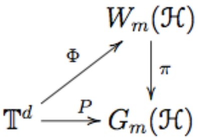

Since the set of m-frames is not a linear space, the approximation of a given Sobolev map by smooth maps might be topologically obstructed.

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Approximation of a given Sobolev map

Theorem [Hang & Lin]
Let $2 \leq d \leq 3$. Consider a compact, boundaryless, smooth submanifold $M \subset \mathbb{R}^{\nu}$. If $d = 3$, assume moreover that the homotopy group $\pi_2(M)$ is trivial.
Then, every Sobolev map $\Psi \in H^1(\mathbb{T}^d, M)$ can be approximated by a sequence $\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^{\infty}(\mathbb{T}^d, M)$ such that $\Psi^{(\ell)} \xrightarrow{H^1} \Psi$ as $\ell \to \infty$.

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Theorem [Hang & Lin]
Let $2 \leq d \leq 3$. Consider a compact, boundaryless, smooth submanifold $M \subset \mathbb{R}^\nu$. If $d = 3$, assume moreover that the homotopy group $\pi_2(M)$ is trivial.
Then, every Sobolev map $\Psi \in H^1(\mathbb{T}^d, M)$ can be approximated by a sequence $\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^\infty(\mathbb{T}^d, M)$ such that $\Psi^{(\ell)} \xrightarrow{H^1} \Psi$ as $\ell \to \infty$.

- The result can be applied to the (finite-dimensional) Stiefel manifold $W_m(\mathbb{C}^n)$, using that $\pi_2(W_m(\mathbb{C}^n)) = 0$ whenever $n \geq m+2$ (here $\mathbb{C}^n$ is regarded as a Galerkin truncation of $\mathcal{H}$).
- One has to reduce to a finite-dimensional space $V_n \simeq \mathbb{C}^n$
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- One has to reduce covariance to periodicity, while staying in a k-independent fiber Hilber space [CHN]
- In our specific case, there is no obstruction to construct a Bloch frame with regularity $H^s(\mathbb{T}^d; \mathcal{H}^m)$ for every $s < 1$, essentially via parallel transport [construction in the paper].

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Part II

The Localization-Topology Correspondence:

the non-periodic case

Work in progress with G. Marcelli and M. Moscolari

A modified paradigm

- Recall the periodic TKNN paradigm:

$$\underbrace{\sigma_{xy}^{(\mathrm{Kubo})}}_{\text{Transport}} = -\frac{\mathrm{i}}{(2\pi)^2}\int_{\mathbb{T}^2} \mathrm{Tr}\left(P(k)\left[\partial_{k_1}P(k), \partial_{k_2}P(k)\right]\right)\mathrm{d}k = \frac{1}{2\pi}\underbrace{C_1(\mathcal{E}_P)}_{\text{Topology}}$$

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$$\sigma_{xy}^{(\mathrm{Kubo})} = \mathcal{T}\left(\mathrm{i}P \Big[[X_1, P], [X_2, P] \Big] \right)$$

where the trace per unit volume is $\mathcal{T}(A) = \lim_{\Lambda_n \nearrow \mathbb{R}^d} |\Lambda_n|^{-1} \, \mathrm{Tr}(\chi_{\Lambda_n} A \, \chi_{\Lambda_n})$.

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- Analogy with the NCG approach to QHE [Co, Be, BES, Ke, . . .]

$$C_1(\mathfrak{p}) \simeq \tau\left(\mathrm{i}\mathfrak{p}\left[\partial_1(\mathfrak{p}), \partial_2(\mathfrak{p})\right]\right)$$

for $\mathfrak{p}$ a projector in the rotation C^*-algebra $\simeq$ NC torus. Here $C_1(\mathfrak{p}) \in \mathbb{Z}$.

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Questions for experts in NCG:

Is $\mathcal{T}(\cdot)$ a tracial state over a C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^d))$? Which one? NCG?

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Question we adressed:

Is there any relation between $C_1(P) = 0$ and the existence of a well-localized GWB??

Definition (Generalized Wannier basis) (compare with [NN93])

An orthogonal projector P acting in $L^2(\mathbb{R}^2)$ admits a G-localized generalized Wannier basis (GWB) if there exist:

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  (a) $\forall x \in \mathbb{R}^2$ there is at most one element of Γ in the ball of radius r centred in x (the set has no accumulation points);

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  - (b) $\forall x \in \mathbb{R}^2$ there is at least one element of Γ in the ball of radius R centred in x (the set is not sparse);
- (ii) a localization function G (typically $G(x) = (1 + |x|^2)^{s/2}$ for some $s \geq 1$), a constant $M > 0$ independent of $\gamma \in \Gamma$ and an orthonormal basis of Ran P, $\{\psi_{\gamma,a}\}_{\gamma\in\Gamma, 1\leq a \leq m(\gamma) < \infty}$ with $m(\gamma) \leq m_* \;\forall \gamma \in \Gamma$, satisfying

$$\int_{\mathbb{R}^2} G(|x-\gamma|)^2 |\psi_{\gamma,a}(x)|^2 \, \mathrm{d}x \leq M.$$

We call each $\psi_{\gamma,a}$ a generalized Wannier function (GWF).

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Localization implies Chern triviality

Theorem - work in progress [Marcelli, Moscolari, GP]

Let P_μ be the Fermi projector of a reasonable Schrödinger operator in $L^2(\mathbb{R}^2)$.
Suppose that P_μ admits a generalized Wannier basis, $\{\psi_{\gamma,a}\}_{\gamma\in\Gamma, 1\leq a\leq m(\gamma)<m_*}$, which is s_*-localized in the sense

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Then, if $s_* \geq 4$ [provisional hypothesis], one has that

$$\mathcal{T}\left(\mathrm{i} P_\mu \Big[[X_1, P_\mu], [X_2, P_\mu] \Big] \right) \; = \; 0.$$

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Clearly, the optimal statement would be for $s_* = 1$, as in the periodic case.
Technical difficulties.

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$d=1$ in [NN93] it is proved that an exponentially localized GWB exists under general hypothesis (Γ discrete): for $d=1$, $P_\mu X P_\mu$ has discrete spectrum **[!!!]**, and a GWB is provided by its eigenfunctions

$$\{\psi_{\gamma,a}\} \qquad\qquad \gamma\in\sigma_{\mathrm{disc}}(P_\mu X P_\mu) =: \Gamma,\, a\in\{1,\dots,m(\gamma)\}\,.$$

A simple but relevant observation

- Let $\widetilde{X}_j := P_\mu \, X_j \, P_\mu$ be the reduced position operator. Then, by simple algebra

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- Boring details. Our estimates are not optimal.

Thank you for your attention!!

Synopsis: symmetry, localization, transport, topology

Time-reversal symmetry	*Symmetry* $\longleftrightarrow$	Broken TR symmetry
$\Downarrow$		
Trivial Bloch bundle	*Topology* $\longleftrightarrow$	Non-trivial Bloch bundle
$\Updownarrow$		$\Updownarrow$
Exponentially loc. Wannier functions $\exists\, \beta > 0 : e^{\beta \lvert x \rvert} w \in L^2(\mathbb{R}^d)$	*Localization* $\longleftrightarrow$	Delocalized Wannier functions $\langle w, \, \lvert x \rvert^2 \, w \rangle = +\infty$
$\downarrow$		$\downarrow$
Vanishing Hall current	*Transport* $\longleftrightarrow$	Non-zero Hall current