Quadratic Forms for Two-Anyon Systems

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14/03/2019

Mathematical physics of anyons and topological states of matter

work in progress with M. Correggi (Rome)

1 Introduction.
- Intermediate statistics, magnetic gauge and the bosonization map;
- Motivation: well-posedness and self-adjointness of Hamiltonians in presence of a trap or an interaction potential;

2 Quadratic forms for 2 non-interacting anyons.
- Extraction of the center of mass and the "1-particle" system;
- Classification of all self-adjoint realizations of the Hamiltonian;
- Quadratic (energy) forms of the self-adjoint Hamiltonians: boundedness from below and closedness;

3 Quadratic forms for 2 interacting anyons.
- Generalization to Hamiltonians with an interaction potential;

4 Perspectives: Forms for N non-interacting anyons;

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The results on the quadratic forms for both noninteracting and interacting case can be found in:

M.Correggi, L.O., Hamiltonians for Two-Anyon Systems, Rend. Mat. Appl. 39, 277-292 (2018).

Let us consider a system of two identical particles with positions x_1, x_2 in the 2-dimensional euclidean space.
Since

$$|\psi(x_2, x_1)|^2 = |\psi(x_1, x_2)|^2,$$

by the indistinguishability, then we have

$$\psi(x_2, x_1) = e^{i\alpha\pi}\psi(x_1, x_2), \quad \alpha \in [0, 1]. \tag{1}$$

Fundamental particles

Bosons satisfy (1) with $\alpha = 0$,
Fermions satisfy (1) with $\alpha = 1$.

For $\alpha \in (0, 1)$ the wave function is multivalued!

Heuristic Introduction

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This multivaluedness of the wave functions leads to consider a new formulation of Quantum Mechanics which extends to non-simply connected configuration spaces.
In our case, the N-particles configuration space Γ_N is

$$\Gamma_N = \mathbb{R}^{2N} \setminus \bigcup_{i<j} \{X = (x_1, \dots, x_N) | x_i = x_j\} ,$$

whose 1st homotopy group is the braid group of N elements $\mathbb{B}_N$! The wave functions are sections of a fiber bundle over Γ_N, in this picture.

Bosonization Map

One can still choose to work with regular wave functions, via a suitable gauge transformation

Indeed, one can consider the so called bosonization operator which maps multivalued wave functions into conventional ones.

Definition

Let $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$. Set $z^j := x_j^1 + i x_j^2$, for $j = 1, \dots, N$. For any $\alpha \in (0, 1]$ we define the bosonization operator $\mathcal{U}_\alpha : L^2_{\alpha-any}(\mathbb{R}^{2N}) \to L^2_{sym}(\mathbb{R}^{2N})$ on α-anyonic function as

$$(\mathcal{U}_\alpha \psi)(\mathbf{X}) := \prod_{j<k} \frac{|\, z^j - z^k \,|^\alpha}{(z^j - z^k)^\alpha} \psi(\mathbf{X}) = \prod_{j<k} e^{-\imath\alpha \cdot \arg(z^j - z^k)} \psi(\mathbf{X}).$$

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This operator simplifies the phase space. On the other hand, the new Hamiltonian contains a singular magnetic interaction term.

Gauges and Hamiltonians

	Anyonic Gauge	Magnetic Gauge
Hilbert Space	$L^2_{\alpha-any}(\mathbb{R}^{2N})$	$L^2_{sym}(\mathbb{R}^{2N})$
Hamiltonian	$\sum_{j=1}^{N} -\Delta_j$	$\sum_{j=1}^{N} (-i\nabla_j + \mathbf{A}_j)^2$

where $\mathbf{A}$ is a multiplication operator associated to an *Aharonov-Bohm*-like potential. We set $\hbar = 1, m = \frac{1}{2}$.

Aharonov-Bohm Potential

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Definition (Aharonov-Bohm(AB) Potential)

The Aharonov-Bohm potential of intensity α centered in $\mathbf{x_0} \in \mathbb{R}^2$ is the function $\mathbf{A}_{\mathbf{x}_0} : \mathbb{R}^2 \setminus \{\mathbf{x}_0\} \longrightarrow \mathbb{R}^2$ defined as follows:

$$\mathbf{A}_{\mathbf{x}_0}(\mathbf{x}) := \alpha \frac{(\mathbf{x}-\mathbf{x}_0)^\perp}{|\mathbf{x}-\mathbf{x}_0|^2}.$$

The operator $\mathbf{A}_j$ is the multiplication operator which attaches to each particle an AB flux:

$$\mathbf{A}_j(\mathbf{x}) := \alpha \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{(\mathbf{x}-\mathbf{x}_k)^\perp}{|\mathbf{x}-\mathbf{x}_k|^2}$$

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Motivation

Well-posedness and self-adjointness of the Hamiltonian

$$H_N = \sum_{j=1}^{N} \left[(-i\nabla_j + \mathbf{A}_j(\mathbf{x}_j))^2 + \sum_{k>j} v(|\mathbf{x}_j - \mathbf{x}_k|) + V(\mathbf{x}_j) \right],$$

even considering just one of the two potentials, are still open questions.

Pairwise interaction potential

Trapping Potential

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Setting

Consider a system of N non relativistic spinless identical particles with anyonic statistics in two dimensions, consider the Hilbert Space

$$\mathfrak{H} := L^2_{sym}(\mathbb{R}^{2N}),$$

and the operator defined on smooth functions supported away from the coincidence hyperplanes

$$\begin{cases} \mathcal{D}(\mathcal{H}_\alpha) := C_c^\infty(\Gamma_N) \\ \mathcal{H}_\alpha := \sum_{j=1}^{N} (D_\alpha^j)^2, \end{cases}$$

where $D_\alpha^j := -i\nabla + \mathbf{A}_j$.

Since $\mathbb{R}^{2N} \setminus \Gamma_N$ has Lebesgue measure zero, $\mathcal{H}_\alpha$ is a densely defined symmetric operator $\Rightarrow$ it is closable.

Furthermore $\mathcal{H}_\alpha$ is positive and thus there $\exists$ self-adjoint extensions of $\mathcal{H}_\alpha$.
In particular, the Friedrichs extension can be considered, by taking the closure of the quadratic form associated to $\mathcal{H}_\alpha$.

The case with $N = 2$ can also be studied by means of Von Neumann's theory of the self-adjoint extensions of a symmetric densely defined operator.

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VN's Approach to the Extensions

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Set $N = 2$. Our configuration space is

$$\Gamma_2 = \mathbb{R}^4 \setminus \{\mathbf{x}_1 = \mathbf{x}_2\}\,. \tag{2}$$

In any inertial reference frame, with coordinates $\mathbf{x}_1 = (x_1^1, x_2^1)$, $\mathbf{x}_2 = (x_1^2, x_2^2)$, two particles are described by the operator

$$\mathcal{H}_\alpha = (-\imath\nabla_1 + \mathbf{A}_1)^2 + (-\imath\nabla_2 + \mathbf{A}_2)^2, \tag{3}$$

where $\mathbf{A}_1(\mathbf{x}) = \alpha(\mathbf{x}_1 - \mathbf{x}_2)^\perp|(\mathbf{x}_1 - \mathbf{x}_2)|^2$, $\mathbf{A}_2(\mathbf{x}) = \alpha\frac{(\mathbf{x}_2-\mathbf{x}_1)^\perp}{|(\mathbf{x}_1-\mathbf{x}_2)|^2}$.

Extraction of the Center of Mass

The extraction of the center of mass leads to a major simplification.
By changing coordinates to

$$\begin{cases} \mathbf{X} := \frac{\mathbf{x}_1+\mathbf{x}_2}{2} \\ \mathbf{r} := \mathbf{x}_1 - \mathbf{x}_2, \end{cases} \tag{4}$$

the operator splits

$$\mathcal{H}_\alpha = -\frac{\Delta \mathbf{x}}{2} + 2(-i\nabla_{\mathbf{r}} + \mathbf{A}_{\text{rel}}(\mathbf{r}))^2, \tag{5}$$

where

$$\mathbf{A}_{\text{rel}}(r) := \alpha \frac{\mathbf{r}^\perp}{|\mathbf{r}|^2}.$$

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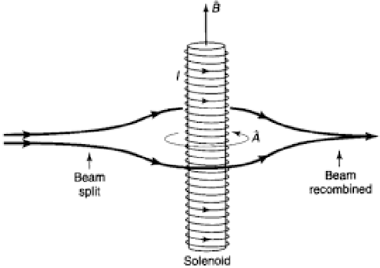
A Single Particle in an AB Flux

A particle subject to the Aharonov-Bohm potential

In [Adami,Teta, 1998] the problem of self-adjointness of the Hamiltonian of a particle on the plane subject to an AB potential is studied.



Phys.Rev. 115,485 1959

1-Particle Hamiltonian

The Hamiltonian reads

$$\begin{cases} \mathcal{D}(\mathcal{H}_\alpha) := C_c^\infty(\mathbb{R}^2 \setminus \{0\}) \\ \mathcal{H}_\alpha := -\Delta - 2i\alpha \dfrac{\mathbf{x}^\perp}{|\mathbf{x}|^2}\nabla + \dfrac{\alpha^2}{|\mathbf{x}|^2}, \end{cases}$$

where the α depends both on the charge of the particle and the flux generated by the solenoid. This operator defined on smooth functions is only symmetric and it admits a 4-parameter family of s.a. extensions.

2 Anyons

One has to take into account the symmetry constraints imposed by the indistinguishability of the two particles!

Von Neumann Theory

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We return to our problem.
The starting operator, in polar coordinates, reads

$$\begin{cases} \mathcal{D}(\mathcal{H}_\alpha) := C_c^\infty(\mathbb{R}^+) \otimes L^2_{even}([0,2\pi]), \\ \mathcal{H}_\alpha := -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\left(i\partial_\omega - \alpha\right)^2, \end{cases}$$

where $L^2_{even}([0,2\pi]) := span_{\mathbb{C}}\left\{e^{2in\omega}\right\}_{n\in\mathbb{N}}$.
This operator is densely defined and symmetric $\longrightarrow$ it is closable.
Let $\overline{\mathcal{H}_\alpha}$ be its closure.

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Let $\overline{\mathcal{H}_\alpha}$ be its closure.

The symmetry constraint actually leads to a different (smaller) family of self-adjoint realizations of the AB-like Hamiltonian.

One looks for solutions of the deficiency equations,

$$(\mathcal{H}_\alpha^* - i)\psi_+ = 0,$$
$$(\mathcal{H}_\alpha^* + i)\psi_- = 0.$$

which can be reduced to decoupled Bessel equations for the partial waves,

$$-w''(\rho) + \frac{1}{\rho^2}\left((2n+\alpha)^2 - \frac{1}{4}\right) w(\rho) = \pm i.$$

One finds that both deficiency indeces are $d_\pm = 1$. The normalized solutions are

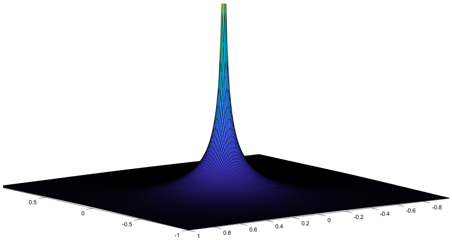
$$\psi_+ := N_\alpha K_\alpha(\rho e^{-i\frac{\pi}{4}}), \qquad \psi_- := N_\alpha e^{i\alpha\frac{\pi}{2}} K_\alpha(\rho e^{i\frac{\pi}{4}}), \tag{6}$$

where $N_\alpha := \frac{\sqrt{2\cos(\alpha\frac{\pi}{2})}}{\pi}$ and K_α is the modified *Bessel function* or *Macdonald function*.

Deficiency Functions

The deficiency functions have a singularity at the origin, their magnetic gradient D_α is not a square-integrable function. Indeed, when $z \to 0$, $\forall \alpha \notin -\mathbb{N}$:

$$K_\alpha(z) = \frac{\Gamma(\alpha)2^{\alpha-1}}{z^\alpha} - \frac{\Gamma(1-\alpha)z^\alpha}{\alpha 2^{\alpha+1}} + O(z^{2-\alpha}).$$



Plot of $|\psi_+|^2 = |N_{\frac{1}{4}} K_{\frac{1}{4}}(\rho e^{-i\frac{\pi}{4}})|^2$

Comparison between Different Statistics

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If we consider the relative Hamiltonian with the symmetry constraints

Bosons

$d_+ = d_- = 1 \qquad \psi_+ := N_0 K_0(\rho e^{-i\frac{\pi}{4}}), \quad \psi_- := N_0 K_0(\rho e^{i\frac{\pi}{4}}),$

Anyons

$d_+ = d_- = 1 \qquad \psi_+ := N_\alpha K_\alpha(\rho e^{-i\frac{\pi}{4}}), \quad \psi_- := N_\alpha e^{i\alpha\frac{\pi}{2}} K_\alpha(\rho e^{i\frac{\pi}{4}}),$

Fermions

$d_+ = d_- = 0$ (essential self-adjointness)

Hence we have that $\mathcal{H}_\alpha$ admits a 1-parameter family of s.a. extensions,

$$\begin{aligned}
\mathcal{D}(\mathcal{H}_{\alpha,\beta}) &= \mathcal{D}(\overline{\mathcal{H}}_\alpha) \oplus span_{\mathbb{C}}\{\psi_+ + e^{\imath\beta}\psi_-\} \\
&\equiv \{\psi = \phi + \gamma(\psi_+ + e^{i\beta}\psi_-) | \phi \in \mathcal{D}(\overline{\mathcal{H}}_\alpha), \gamma \in \mathbb{C}\}, \\
\mathcal{H}_{\alpha,\beta}\psi &= \overline{\mathcal{H}_\alpha}\phi + i\gamma\psi_+ - i\gamma e^{i\beta}\psi_- \\
&= \overline{\mathcal{H}}_\alpha\phi + i\gamma N_\alpha K_\alpha(\rho e^{-i\frac{\pi}{4}}) - i\gamma e^{i\left(\beta+\alpha\frac{\pi}{2}\right)} N_\alpha K_\alpha(\rho e^{i\frac{1}{4}\pi}),
\end{aligned}$$

with $\beta \in [-\pi, \pi]$.

The Friedrichs Extension

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In terms of the chosen parametrization, the Friedrichs extension is the one corresponding to $\beta = \pi$. (see below)

Since the symmetric operator $\mathcal{H}_\alpha$ is positive, the existence of the Friedrichs extension is guaranteed. One can find it among the others extensions. by imposing that its domain must be contained in the form domain. The form domain is the set of function which have a square-integrable magnetic gradient D_α. Let the form be

$$\begin{cases} \mathcal{D}\left(\mathcal{Q}_\alpha\right) = \{\phi \in L^2(\mathbb{R}^2) | D_\alpha \phi \in L^2(\mathbb{R}^2)^2\}, \\ \mathcal{Q}_\alpha[\phi] = \|D_\alpha \phi\|^2. \end{cases} \tag{7}$$

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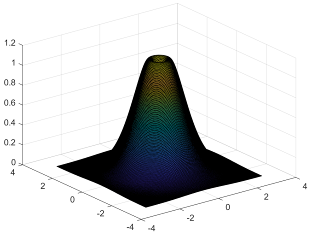
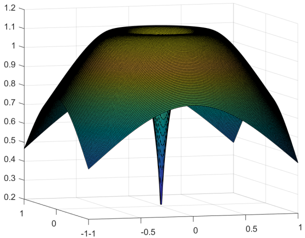
The first order terms in the expansion of K_α cancel out and it remains a function in the form domain $\mathcal{D}(\mathcal{Q}_\alpha)$.

$$D_\alpha(\psi_+ - i\psi_-) \in L^2$$



Plot of $|\psi_+ - i\psi_-|^2$



Plot of $|\psi_+ - i\psi_-|^2$ (closer look)

Def. of the Quadratic Forms

The expectation value of the operator $\mathcal{H}_{\alpha,\beta}$ suggests a possible expression of the quadratic form.
We write a parametrization which relies upon the knowledge of the asymptotics of the eigenvalue near the origin. We start from the form associated to the Friedrichs extension:

Friedrichs' Form

We define the quadratic form

$$\mathcal{F}_{\alpha,\mathrm{F}}[\psi] := \mathcal{F}_\alpha[\psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \left| \left(-i\nabla + \frac{\alpha \mathbf{r}^\perp}{r^2} \right) \psi \right|^2,$$

with domain

$$\mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F}}] = \overline{C_0^\infty(\mathbb{R}^2 \setminus \{0\})}^{\|\ \|_\alpha} \cap L^2_{\mathrm{even}}.$$

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Proposition (Friedrichs extension)

The quadratic form $\mathcal{F}_{\alpha,\mathrm{F}}$ is closed and positive on $\mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F}}]$ for any $\alpha \in [0,1]$. Furthermore, for any $\alpha \in (0,1)$,

$$\mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F}}] \subset H^1(\mathbb{R}^2).$$

The associated self-adjoint operator $H_{\alpha,\mathrm{F}}$ acts as H_α on the domain

$$\begin{aligned}\mathscr{D}\left(H_{\alpha,\mathrm{F}}\right) &= \left\{\psi \in \mathscr{D}\left[\mathcal{F}_{\alpha,\mathrm{F}}\right] \,\middle|\, H_\alpha \psi \in L^2\right\} \\ &= \left\{\psi \,\middle|\, \psi|_{\mathcal{H}_n} \in H^2(\mathbb{R}^2), \forall n \neq 0;\right. \\ &\left. \psi_0 \in H^2(\mathbb{R}^2 \setminus \{0\}) \cap H^1(\mathbb{R}^2),\ \psi_0(r) \underset{r \to 0^+}{\sim} r^\alpha + o(r)\right\}.\end{aligned}$$

All the other forms are defined decomposing the wave function in a regular part and in a singular one.
The family of quadratic forms $\mathcal{F}_{\alpha,\beta}[\psi]$, $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$, is defined as

$$\mathcal{F}_{\alpha,\beta}[\psi] := \mathcal{F}_{\alpha}[\phi_\lambda] - 2\lambda^2 \Re q \langle \phi_\lambda | G_\lambda \rangle + \left[\beta + (1-\alpha)\, c_\alpha \lambda^{2\alpha}\right] |q|^2$$

$$\mathscr{D}[\mathcal{F}_{\alpha,\beta}] = \left\{ \psi \in L^2_{\mathrm{even}} \,\middle|\, \psi = \phi_\lambda + q G_\lambda, \phi_\lambda \in \mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F}}], q \in \mathbb{C} \right\},$$

and G_λ , $\lambda \in \mathbb{R}^+$, is the defect function

$$G_\lambda(\mathbf{r}) := \lambda^\alpha K_\alpha(\lambda r),$$

with K_α the modified Bessel function of index α. The coefficient c_α is given by

$$c_\alpha := \frac{\lambda^{2-2\alpha} \|G_\lambda\|_2^2}{\alpha} = \frac{\pi^2}{\sin \pi\alpha} > 0$$

Closedness and Boundedness

The forms can be rewritten as

$$\mathcal{F}_{\alpha,\beta}[\psi] = \mathcal{F}_\alpha[\phi_\lambda] + \lambda^2 \|\phi_\lambda\|_2^2 - \lambda^2 \|\psi\|_2^2 + \left(\beta + c_\alpha \lambda^{2\alpha}\right) |q|^2 ,$$

The Friedrichs form $\mathcal{F}_{\alpha,\mathrm{F}}$ is included in the family and formally recovered for $\beta = +\infty$, in which case $q = 0$ and

$$\mathscr{D}[\mathcal{F}_{\alpha,+\infty}] = \mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F}}].$$

Closedness and Boundedness

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Closedness and Boundedness

Theorem (M.Correggi, L.O. '18)

For any $\alpha \in (0,1)$ and any $\beta \in \mathbb{R}$, the quadratic form $\mathcal{F}_{\alpha,\beta}$ is closed and bounded from below on the domain $\mathscr{D}[\mathcal{F}_{\alpha,\beta}]$. Furthermore,

$$\frac{\mathcal{F}_{\alpha,\beta}[\psi]}{\|\psi\|_2^2} \geqslant \begin{cases} 0, & \text{if } \beta \geqslant 0; \\ -\left(\frac{|\beta|\sin(\pi\alpha)}{\pi^2}\right)^{\frac{1}{\alpha}}, & \text{if } \beta < 0. \end{cases} \tag{8}$$

First, one drops the summands $\mathcal{F}_\alpha[\phi_\lambda]$ and $\lambda^2 \|\phi_\lambda\|_2^2$, then

$$\mathcal{F}_{\alpha,\beta}[\psi] \geqslant -\lambda^2 \|\psi\|_2^2 + \left(\beta + c_\alpha \lambda^{2\alpha}\right) |q|^2,$$

which implies that if $\beta \geqslant 0$ the form is positive when one takes λ arbitrarily small.

If $\beta < 0$, one can exploit the freedom in the choice of λ and get that if $\lambda = \left(\frac{|\beta|}{c_\alpha}\right)^{\frac{1}{2\alpha}}$,

$$\mathcal{F}_{\alpha,\beta}[\psi] \geqslant -\lambda^2 \|\psi\|_2^2 .$$

Sketch of the Proof: Closedness

We investigate the form $\widetilde{\mathcal{F}}_{\alpha,\beta}[\psi] := \mathcal{F}_{\alpha,\beta}[\psi] + \lambda^2 \|\psi\|_2^2$.

With the choice of λ made before, one has

$$\widetilde{\mathcal{F}}_{\alpha,\beta}[\psi_n - \psi_m] \geqslant \mathcal{F}_{\alpha,\mathrm{F}}[\phi_n - \phi_m] + \lambda^2 \|\phi_n - \phi_m\|_2^2 + C_{\alpha,\beta}|q_n - q_m|^2,$$

if we now take a sequence in $\mathscr{D}[\mathcal{F}_{\alpha,\beta}]$ s.t.

$$\lim_{n,m\to\infty} \widetilde{\mathcal{F}}_{\alpha,\beta}[\psi_n - \psi_m] = 0, \qquad \lim_{n,m\to\infty} \|\psi_n - \psi_m\|_2^2 = 0,$$

by positivity we find

$$\psi_n \xrightarrow[n\to\infty]{} \phi_\lambda + qG_\lambda \in \mathscr{D}[\mathcal{F}_{\alpha,\beta}],$$

ϕ, q being the limit respectively of $\{\phi_n\}, \{q_n\}$.

Corollary (Self-adjoint operators $H_{\alpha,\beta}$)

The one-parameter family of self-adjoint operators associated to the forms $\mathcal{F}_{\alpha,\beta}$, $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$, is given by

$$\left(H_{\alpha,\beta} + \lambda^2\right)\psi = \left(H_\alpha + \lambda^2\right)\phi_\lambda,$$

$$\mathscr{D}\left(H_{\alpha,\beta}\right) = \Bigg\{\psi \in L^2_{\mathrm{even}} \,\Bigg|\, \psi_n \in \mathscr{D}\left(H_{\alpha,\mathrm{F}}\right), \forall n \neq 0;\ \psi_0 = \phi_\lambda + qG_\lambda, \\ \phi_\lambda \in \mathscr{D}\left(H_{\alpha,\mathrm{F}}\right),\ q = -\frac{2^\alpha\Gamma(\alpha+1)}{\left(\beta + c_\alpha\lambda^{2\alpha}\right)} \lim_{r\to 0^+} \frac{\phi_\lambda(r)}{r^\alpha}\Bigg\},$$

where $\lambda > 0$ is free to choose provided $\beta + c_\alpha\lambda^{2\alpha} \neq 0$.
Furthermore, the operators $H_{\alpha,\beta}$ extend H_α, i.e., $H_\alpha|_{\mathscr{D}(H_\alpha)} = H_\alpha$, and, conversely, any self-adjoint extension of H_α is included in the family $H_{\alpha,\beta}$, $\beta \in \mathbb{R}$.

Spectral Properties

Proposition (Spectral properties of $H_{\alpha,\beta}$)

For any $\alpha \in (0,1)$ and any $\beta \in \mathbb{R}$,
$\sigma(H_{\alpha,\beta}) = \sigma_{pp}(H_{\alpha,\beta}) \cup \sigma_{ac}(H_{\alpha,\beta})$, with $\sigma_{ac}(H_{\alpha,\beta}) = \mathbb{R}^+$ and

$$\sigma_{pp}(H_{\alpha,\beta}) = \begin{cases} \left\{ -\left(\frac{|\beta| \sin(\pi\alpha)}{\pi^2} \right)^{\frac{1}{\alpha}} \right\}, & \text{if } \beta < 0; \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9}$$

The generalized eigenfunctions are

$$\varphi(k,\rho,\theta) = \sum_{m=-\infty}^{m=+\infty} J_{2m+\alpha}(k\rho) e^{im\theta}, \tag{10}$$

where J_α is the *first Bessel function* of order α.

Remarks

- *The domains of the self-adjoint operators are all different and no one is contained in any other.*
- *The domains of the forms are the same except for the Friedrichs form.*
- *$\beta < 0$ corresponds to an attractive interaction at the origin, while $\beta \geqslant 0$ to a repulsive one. In the former case there is a negative eigenvalue.*

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Interaction Potential

Let us specify the assumptions we make on the interaction:

Assumption (Interaction potential V)

Let $V = V(r)$ be a real radial function and let $V_\pm$ denote the positive and negative parts of V, respectively, i.e., $V = V_+ - V_-$. Then, we assume that

- $V_+ \in L^2_{\mathrm{loc}}(\mathbb{R}^+) \cap L^\infty([0, \varepsilon))$;
- $V_- \in L^\infty(\mathbb{R}^+)$.

Regular extension

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By analogy with the noninteracting case, we call the Friedrichs form the following quantity

$$\mathcal{F}_{\alpha,\mathrm{F},V}[\psi] := \mathcal{F}_{\alpha,\mathrm{F}}[\psi] + \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\, V(r)\, |\psi|^2 \,, \tag{11}$$

with domain

$$\mathscr{D}\left[\mathcal{F}_{\alpha,\mathrm{F},V}\right] := \overline{C_0^\infty(\mathbb{R}^2 \setminus \{0\})}^{\|\ \|_{\alpha,V}} \cap L^2_{\mathrm{even}}, \tag{12}$$

where $\|\psi\|^2_{\alpha,V} := \mathcal{F}_{\alpha,\mathrm{F},V}[\psi] + V_0\, \|\psi\|_2^2$, and

$$V_0 := \sup_{r\in\mathbb{R}^+} V_-(r). \tag{13}$$

We have the following

Proposition (Friedrichs extension)

Let the assumptions on V hold true. Then, the quadratic form $\mathcal{F}_{\alpha,\mathrm{F},V}$ is closed and bounded from below on $\mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F},V}]$ for any $\alpha \in [0,1]$. Furthermore, for any $\alpha \in (0,1)$,

$$\mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F},V}] \subset H^1(\mathbb{R}^2). \tag{14}$$

The associated self-adjoint operator $H_{\alpha,\mathrm{F},V}$ acts as $H_{\alpha,V}$ on the domain

$$\mathscr{D}\left(H_{\alpha,\mathrm{F},V}\right) = \left\{\psi \in \mathscr{D}\left(H_{\alpha,\mathrm{F}}\right) \,\middle|\, V\psi \in L^2\right\}. \tag{15}$$

We set for $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$

$$\boxed{\mathcal{F}_{\alpha,\beta,V}[\psi] = \mathcal{F}_{\alpha,V}[\phi_\lambda] + \lambda^2 \|\phi_\lambda\|_2^2 - \lambda^2 \|\psi\|_2^2 + \left(\beta + c_\alpha \lambda^{2\alpha}\right) |q|^2 \,,}$$

where ψ belongs to the domain

$$\mathscr{D}[\mathcal{F}_{\alpha,\beta,V}] = \left\{\psi \in L^2_{\text{even}} \;\middle|\; \psi = \phi_\lambda + qG_\lambda, \phi_\lambda \in \mathscr{D}[\mathcal{F}_{\alpha,\mathrm{F},V}], q \in \mathbb{C}\right\}.$$

Theorem (Closedness and boudedness from below of $\mathcal{F}_{\alpha,\beta,V}$)

Let the assumptions on V hold true. Then, for any $\alpha \in (0,1)$ and any $\beta \in \mathbb{R}$, the quadratic form $\mathcal{F}_{\alpha,\beta,V}$ is closed and bounded from below on the domain $\mathscr{D}[\mathcal{F}_{\alpha,\beta,V}]$. Furthermore,

$$\frac{\mathcal{F}_{\alpha,\beta,V}[\psi]}{\|\psi\|_2^2} \geqslant \begin{cases} -V_0, & \text{if } \beta \geqslant 0; \\ -\left(\frac{|\beta|}{c_\alpha}\right)^{\frac{1}{\alpha}} - V_0, & \text{if } \beta < 0. \end{cases} \tag{16}$$

Corollary (Self-adjoint operators $H_{\alpha,\beta,V}$)

Let Ass. 1 hold true. Then, the one-parameter family of self-adjoint operators associated to the forms $\mathcal{F}_{\alpha,\beta,V}$, $\alpha \in (0,1)$, $\phi_\lambda \in \mathscr{D}(H_{\alpha,\mathrm{F},V})$ and $\beta \in \mathbb{R}$, is given by

$$\left(H_{\alpha,\beta,V} + \lambda^2\right)\psi = \left(H_{\alpha,V} + \lambda^2\right)\phi_\lambda,$$

$$\mathscr{D}(H_{\alpha,\beta,V}) = \left\{\psi \,\middle|\, \psi_n \in \mathscr{D}(H_{\alpha,\mathrm{F},V}), \forall n \neq 0;\ \psi_0 = \phi_\lambda + qG_\lambda,\right.$$

$$\left. q = \frac{1}{\beta + c_\alpha \lambda^{2\alpha}}\left[\langle G_\lambda |V| \phi_\lambda\rangle - \Gamma(\alpha+1)2^\alpha \lim_{r\to 0^+} \frac{\phi_\lambda(r)}{r^\alpha}\right]\right\},$$

where $\lambda > 0$ is free to choose provided $\beta + c_\alpha \lambda^{2\alpha} \neq 0$. Furthermore, the operators $H_{\alpha,\beta,V}$ extend $H_{\alpha,V}$, i.e., $H_{\alpha,V}|_{\mathscr{D}(H_{\alpha,V})} = H_{\alpha,V}$.

For the N-anyon system the analysis is much more complicated.

We aim at finding extensions with the same asymptotics on the coincidence hyperplanes in the 2-particle channels.

The extensions are to be compared with the so called local extensions for fermions/bosons. The idea is to impose

$$\psi(X) = \frac{A_{i,j}}{|x_i - x_j|^\alpha} + B_{i,j}|x_i - x_j|^\alpha + o(|x_i - x_j|^\alpha), \tag{17}$$

when $x_i \to x_j$, where the coefficients $A_{i,j}$ are the same of the singular terms of the 2-particle wave function, while $B_{i,j}$ is more involved, since it must take all the other charges into account.

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Thanks for your attention!