Anyons in fractional quantum Hall models

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This work has in part been funded by the Villum Foundation.
The fractional quantum Hall effect

The effect has been observed in semi-conductor hetero-structures

**Conditions:**
- 2D electron gas
- High magnetic field (~ 10 T)
- Low temperature (~ 10 mK)
- High mobility (~ $10^7 \text{ cm}^2/\text{(Vs)}$)
- Low carrier density (~ $10^{11} \text{ cm}^{-2}$)

**Some features:**
- Good analytical trial wave functions
- Can host anyons
This talk: Fractional quantum Hall-like physics in lattice systems

Motivation:

- Fractional quantum Hall effect at no external magnetic field and high temperature?
- New features:
  - Bosonic systems
  - Lattice effects
- Implementation in optical lattices?
How can we obtain fractional quantum Hall-like physics in lattice systems?

Recipe 1: 
*Fractional Chern insulators*

\[ H_{\text{continuum}} \rightarrow H_{\text{lattice}} \]

Recipe 2: 
*Exact wave functions*

\[ \Psi_{\text{continuum}} \rightarrow \Psi_{\text{lattice}} \]

[Challenging] [Analytical] [Exact \( H \)]

[Kalmeyer & Laughlin]

[Schroeter, Kapit, Thomale & Greiter]
How can we obtain fractional quantum Hall-like physics in lattice systems?

**Recipe 1:**

*Fractional Chern insulators*

$H_{\text{continuum}} \rightarrow H_{\text{lattice}}$

**Recipe 2:**

*Exact wave functions*

$\Psi_{\text{continuum}}$

Analytical

$\Psi_{\text{lattice}}$

$H_{\text{continuum}} \rightarrow$ CFT is helpful!

Exact $H$

Nielsen, Cirac, Sierra, PRL 108, 257206 (2012)
Construction of lattice models with analytical ground states from conformal field theory
Interpolation between lattice and continuum

Tu, Nielsen, Cirac, Sierra, NJP 16, 033025 (2014); Glasser, Cirac, Sierra, Nielsen, PRB 94, 245104 (2016)
Construction of lattice FQH models

\[ |\psi\rangle = \sum_{n_1,n_2,\ldots,n_N} \psi(n_1, n_2, \ldots, n_N)|n_1, n_2, \ldots, n_N\rangle \]

\[ \Lambda_i |\psi\rangle = 0 \]

\[ \psi(n_1, n_2, \ldots, n_N) \propto \langle 0 | \phi_{n_1}(z_1) \phi_{n_2}(z_2) \cdots \phi_{n_N}(z_N) |0\rangle \]

\[ H = \sum_i \Lambda_i^\dagger \Lambda_i \]

1D:

\[ \text{Im}(z_i) \]

\[ \text{Re}(z_i) \]

2D:

\[ \text{Im}(z_i) \]

\[ \text{Re}(z_i) \]

Position

Internal state

\[ \phi_{n_i}(z_i) \]
Recipe to construct the Hamiltonian

1. Choose what $\phi_{n_i}(z_i)$ should be.
2. Find a null field $\chi(z_i)$.
3. Note that $\langle 0 | \phi_{n_1}(z_1) \phi_{n_2}(z_2) \cdots \chi(z_i) \cdots \phi_{n_N}(z_N) | 0 \rangle = 0$.
4. Deform the integration contour.
5. Use operator product expansions.
6. Do the integrals.
7. Rewrite such that the final expression is an operator acting on the initial wavefunction.
### Construction of lattice FQH models

**1D:**

\[
\phi_{n_j}(z_j) = e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}: \quad q = \text{integer} \quad n_j \in \{0,1\} \quad \text{Critical models (Haldane-Shastry for } q = 2) 
\]

**2D:**

\[
\phi_{n_j}(z_j) = e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}: \quad q = \text{integer} \quad n_j \in \{0,1\} \quad \text{Laughlin state with } q \text{ flux units per particle}
\]

\[
\phi_{n_j}(z_j) = \chi(z_j)^{n_j(2-n_j)} e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}: \quad q = \text{integer} \quad n_j \in \{0,1\} \text{ or } n_j \in \{0,1,2\} \quad \text{Moore-Read state with } q \text{ flux units per particle}
\]

\[
\phi_{n_j}(z_j) = \kappa_{n_j} e^{i\sqrt{2} \bar{m}_{n_j} \cdot \vec{\varphi}(z_j)}: \quad n_j \in \{1,2, \ldots, n\} \quad \bar{m}_{n_j} \text{ are vectors of numbers, see NPB 886, 328 (2014)} \quad \text{Halperin states}
\]

\(\chi\): chiral part of Majorana fermion field  \quad  \phi\): chiral part of massless free boson field  \quad  \kappa_{n_j}\): Klein factor  \quad  \eta\): number that determines the number of particles per lattice site
Construction of lattice FQH models

2D:

\[ \phi_{n_j}(z_j) = e^{i(qn_j - \eta)\varphi(z_j)/\sqrt{q}} : \]

\[ \phi_+(w_k) = e^{i\varphi(w_k)/\sqrt{q}} : \]

Laughlin states with quasiholes

2D:

\[ \phi_{n_j}(z_j) = \chi(z_j)^{n_j(2-n_j)} : e^{i(qn_j - \eta)\varphi(z_j)/\sqrt{q}} : \]

\[ \phi_+(w_k) = \sigma(w_k) : e^{i\varphi(w_k)/(2\sqrt{q})} : \]

Moore-Read states with quasiholes

\( \chi \): chiral part of Majorana fermion field

\( \phi \): chiral part of massless free boson field

\( \sigma \): spin field of the chiral Ising CFT

\( \eta \): number that determines the number of particles per lattice site
Laughlin states with quasiholes
Lattice Laughlin state with quasiholes

\[ H(w_1, w_2, w_3) \]
\[ \Psi(w_1, w_2, w_3) \]

\[ \langle n_i \rangle_{\text{with}} - \langle n_i \rangle_{\text{without}} \]

Nielsen, PRB 91, 041106(R) (2015)
Exchange properties

\[ H(w_1, w_2) \]
\[ \Psi(w_1, w_2) \]
\[ |\Psi\rangle \rightarrow e^{i\theta} |\Psi\rangle \]
\[ \frac{\theta}{2\pi} = -32 + \frac{2}{3} - 0.0017(5) \]

Nielsen, PRB 91, 041106(R) (2015)
Rodriguez, Nielsen, PRB 92, 125105 (2015)
Moore-Read states with quasiholes
Anyons in lattice Moore-Read states

\[ \langle n_i \rangle_{\text{with}} - \langle n_i \rangle_{\text{without}} \]

\[ \langle n_i \rangle_{\text{with}} - \langle n_i \rangle_{\text{without}} \]
Laughlin and Moore-Read states with quasielectrons
How do we obtain quasielectrons?

\[ |\psi_q\rangle = \sum_{n_1,n_2,...,n_N} \psi(n_1,n_2,...,n_N)|n_1,n_2,...,n_N\rangle \]

\[ \psi(n_1,n_2,...,n_N) \propto \langle 0|\phi_+(w_1)\cdots\phi_+(w_Q)\phi_{n_1}(z_1)\phi_{n_2}(z_2)\cdots\phi_{n_N}(z_N)|0\rangle \]

\[ \phi_+(W_k) = e^{i\varphi(z_j)/\sqrt{q}} \]

\[ \phi_{n_i}(Z_i) = e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}} \]

Position

Internal state
How do we obtain quasielectrons?

\[ |\psi_q\rangle = \sum_{n_1, n_2, \ldots, n_N} \psi(n_1, n_2, \ldots, n_N) |n_1, n_2, \ldots, n_N\rangle \]

\[ \psi(n_1, n_2, \ldots, n_N) \propto \langle 0 | \phi_+ (w_1) \cdots \phi_- (w_Q) \phi_{n_1} (z_1) \phi_{n_2} (z_2) \cdots \phi_{n_N} (z_N) |0 \rangle \]

\[ \phi_{\pm} (w_k) = e^{\pm i \varphi(z_j)/\sqrt{q}} : \]

\[ \phi_{n_i} (z_i) = e^{i(qn_j - \eta) \varphi(z_j)/\sqrt{q}} : \]
How do we obtain quasielectrons?

\[ |\psi_q\rangle = \sum_{n_1,n_2,...,n_N} \psi(n_1,n_2,...,n_N)|n_1,n_2,...,n_N\rangle \]

\[ \psi(n_1,n_2,...,n_N) \propto \langle 0| \phi_+ (w_1) \cdots \phi_- (w_Q) \phi_{n_1}(z_1) \phi_{n_2}(z_2) \cdots \phi_{n_N}(z_N)|0\rangle \]

\[ \phi_{n_i}(z_i) = e^{i(\eta n_j - \eta)\varphi(z_j)/\sqrt{q}} : \]

\[ \phi_{\pm}(W_k) = e^{\pm i\varphi(z_j)/\sqrt{q}} : \]

Continuum
How do we obtain quasielectrons?

\[ |\psi_q\rangle = \sum_{n_1, n_2, \ldots, n_N} \psi(n_1, n_2, \ldots, n_N)|n_1, n_2, \ldots, n_N\rangle \]

\[ \psi(n_1, n_2, \ldots, n_N) \propto \langle 0 | \phi_+(w_1) \cdots \phi_-(w_Q) \phi_{n_1}(z_1) \phi_{n_2}(z_2) \cdots \phi_{n_N}(z_N) |0\rangle \]

Same story for Moore-Read quasielectrons!
Charge distributions of Laughlin anyons

$\frac{1}{3}$ Laughlin state

Lattice filling $= \frac{1}{2}$
Laughlin Quasiholes <-> Laughlin Quasielectrons

\[ \frac{1}{3} \text{ Laughlin state} \]

\[ \langle n_i \rangle_{\text{with}} - \langle n_i \rangle_{\text{without}} \]

\[ \Omega_p(r) \equiv - \sum_{i \text{ inside circle}} (\langle n_i \rangle_{\text{with}} - \langle n_i \rangle_{\text{without}}) \]

Nielsen, Glasser, Rodriguez, NJP 20, 033029 (2018)
Quasiholes and quasielectrons in lattice Moore-Read states

Exchange properties are also as desired

Manna, Wildeboer, Nielsen, PRB 99, 045147 (2019)
Examples of exact parent Hamiltonians
Bosonic Laughlin model from CFT

\[ \phi_{s_j}(z_j) = e^{i\pi (j-1)(s_j+1)/2} : e^{is_j\varphi(z_j)}/\sqrt{2} : \quad s_j \in \{-1,1\} \]

CFT: SU(2)_1 WZW

\[ H = \frac{1}{2} \sum_{i \neq j} |w_{ij}|^2 - \frac{2i}{3} \sum_{i \neq j \neq k} \bar{w}_{ij}w_{ik} S_i \cdot (S_j \times S_k) \]
\[ + \frac{2}{3} \sum_{i \neq j} |w_{ij}|^2 S_i \cdot S_j + \frac{2}{3} \sum_{i \neq j \neq k} \bar{w}_{ij}w_{ik} S_j \cdot S_k \]
\[ w_{ij} = \frac{g(z_i)}{z_i - z_j} + h(z_i) \]

\[ S_i = (S_i^x, S_i^y, S_i^z), \quad [S_i^a, S_j^b] = i\delta_{ij} \varepsilon_{abc} S_i^c \]

Generalization of the Kalmeyer-Laughlin state to arbitrary lattices.
Hamiltonian for Laughlin states

\[ \phi_{n_j}(z_j) = e^{i\pi(j-1)n_j} : e^{i(qn_j-1)\varphi(z_j)/\sqrt{q}} : \]

\[ H = \sum_i \Lambda_i^\dagger \Lambda_i \quad \Lambda_i = \sum_{j \neq i} \frac{1}{z_i - z_j} \left( d_j - d_i (qn_j - 1) \right) \]
Can we truncate the Hamiltonians to obtain local models?
Local Hamiltonian on a square lattice

Approach: Truncate Hamiltonian, adjust, optimize.

Works also on kagome lattice, but not on triangular lattice.

\[ H = J_2 \sum_{\langle n,m \rangle} \hat{S}_n \cdot \hat{S}_m + J'_2 \sum_{\langle\langle n,m \rangle\rangle} 2\hat{S}_n \cdot \hat{S}_m - J_3 \sum_{\langle n,m,p \rangle} 4\hat{S}_n \cdot (\hat{S}_m \times \hat{S}_p) \]

Nielsen, Sierra, Cirac, Nat. Commun. 4, 2864 (2013).
Local Hamiltonian

\[ H = J_2 \sum_{\langle n,m \rangle} 2 \hat{S}_n \cdot \hat{S}_m + J'_2 \sum_{\langle n,m \rangle} 2 \hat{S}_n \cdot \hat{S}_m - J_3 \sum_{\langle n,m,p \rangle} 4 \hat{S}_n \cdot (\hat{S}_m \times \hat{S}_p) \]

\[ J_2 = \cos(\phi_1)\cos(\phi_2), \quad J'_2 = \sin(\phi_1)\cos(\phi_2), \quad J_3 = \sin(\phi_2) \]

Overlap with CFT state on 4 \times 5 lattice

Total Chern number of ground states on the torus

\[ J_2 = 1, \quad J'_2 = 0, \quad J_3 = 1/2 \]

Nielsen, Sierra, Cirac, Nat. Commun. 4, 2864 (2013).
Is the ground state degeneracy correct?

Nielsen, Sierra, Cirac, Nat. Commun. 4, 2864 (2013).
Can we do something more well-defined?

\[ H = \sum_i \Lambda_i^\dagger \Lambda_i \quad \Lambda_i = \sum_{j(\neq i)} \frac{1}{z_i - z_j} \left( d_j - d_i (q \eta_j - 1) \right) \]

Approach: Truncate \( \Lambda_i \) operator directly.

Advantages:

- The \( \Lambda_i \) operator is simpler to truncate than the Hamiltonian.
- The result of the truncation does not depend on the number of sites in the lattice.
- It is clear how to obtain models with periodic boundary conditions.
- No optimization needed.

Nandy, Srivatsa, Nielsen, arXiv:1902.09017
Results for $q = 2$ and $q = 4$

### Square lattice

<table>
<thead>
<tr>
<th>$L_x \times L_y$</th>
<th>$q = 2$</th>
<th>$q = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta$</td>
<td>$\Delta^{1/N}$</td>
</tr>
<tr>
<td>$3 \times 4$</td>
<td>0.8679</td>
<td>0.9883</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>0.9692</td>
<td>0.9980</td>
</tr>
<tr>
<td>$4 \times 5$</td>
<td>0.9239</td>
<td>0.9961</td>
</tr>
<tr>
<td>$4 \times 6$</td>
<td>0.9226</td>
<td>0.9966</td>
</tr>
<tr>
<td>$5 \times 6$</td>
<td>0.9164</td>
<td>0.9971</td>
</tr>
</tbody>
</table>

### Triangular lattice

<table>
<thead>
<tr>
<th>$L_x \times L_y$</th>
<th>$q = 2$</th>
<th>$q = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta$</td>
<td>$\Delta^{1/N}$</td>
</tr>
<tr>
<td>$3 \times 4$</td>
<td>0.8400</td>
<td>0.9856</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>0.9507</td>
<td>0.9968</td>
</tr>
<tr>
<td>$4 \times 5$</td>
<td>0.9098</td>
<td>0.9953</td>
</tr>
<tr>
<td>$4 \times 6$</td>
<td>0.8913</td>
<td>0.9952</td>
</tr>
<tr>
<td>$5 \times 6$</td>
<td>0.8210</td>
<td>0.9934</td>
</tr>
</tbody>
</table>

Nandy, Srivatsa, Nielsen, arXiv:1902.09017
Is the ground state degeneracy correct?

Nandy, Srivatsa, Nielsen, arXiv:1902.09017
Conclusion
Conclusion

• We have constructed families of lattice models with analytical ground states and few-body Hamiltonians and investigated their properties.

• We have shown how to construct trial wavefunctions and parent Hamiltonians for Laughlin and Moore-Read quasiholes and quasielectrons in lattices.

• We have investigated different ways to use the exact Hamiltonians as a starting point to find local, few-body Hamiltonians with almost the same ground states.
Thank you!

MPQ collaborators:
Ignacio Cirac
Ivan Glasser
Hong-Hao Tu
Benedikt Herwerth
Ivan D. Rodriguez

UAM collaborators:
German Sierra

Sourav Manna  
Dillip K. Nandy

Julia Wildeboer  
Srivatsa N.S