Anyons in fractional quantum Hall models

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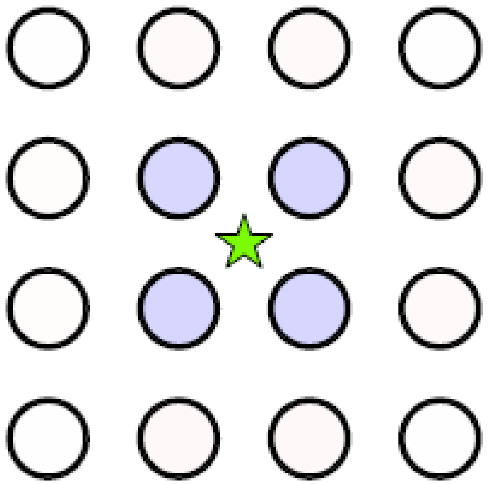
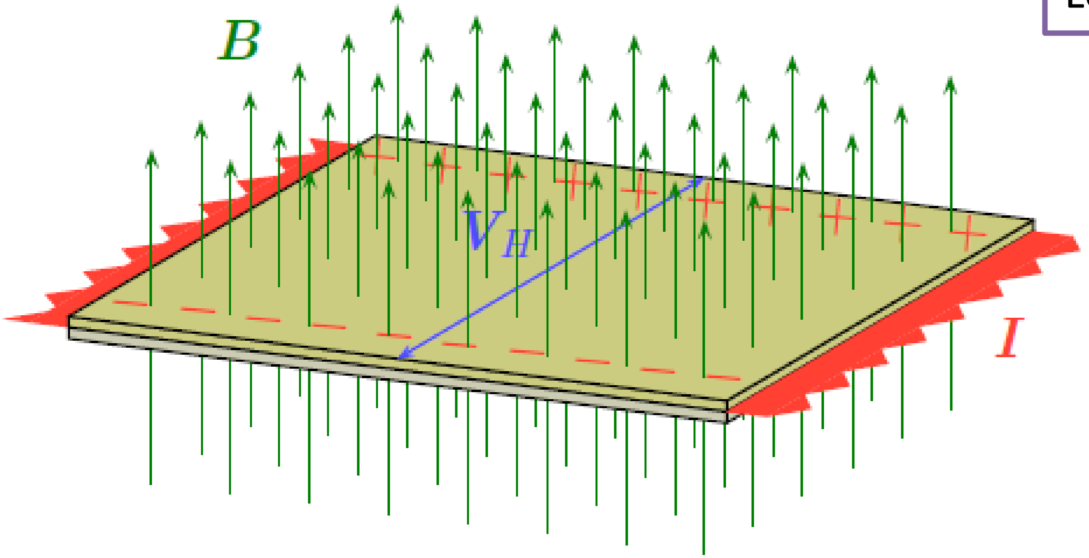


This work has in part been funded by the Villum Foundation.

The fractional quantum Hall effect

The effect has been observed in semi-conductor hetero-structures

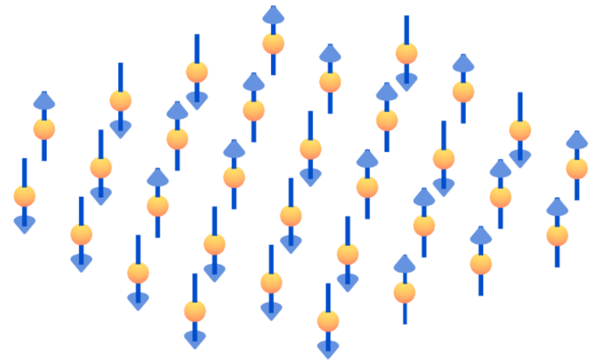


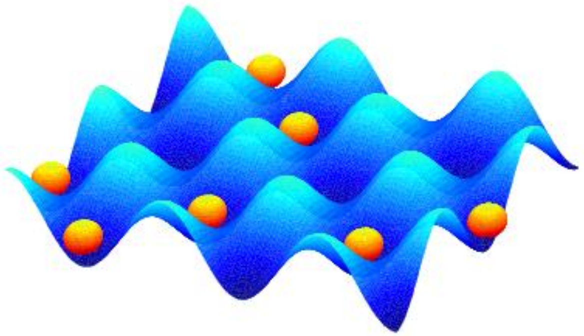
Conditions:
2D electron gas
High magnetic field (~ 10 T)
Low temperature (~ 10 mK)
High mobility ($\sim 10^7\ \mathrm{cm}^2/(\mathrm{Vs})$)
Low carrier density ($\sim 10^{11}\ \mathrm{cm}^{-2}$)

Some features:

- Good analytical trial wave functions
- Can host anyons

This talk: Fractional quantum Hall-like physics in lattice systems





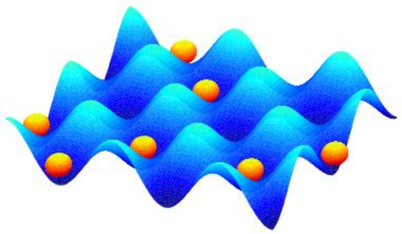
Motivation:

Fractional quantum Hall effect at no external magnetic field and high temperature?

New features

Bosonic systems
Lattice effects
...

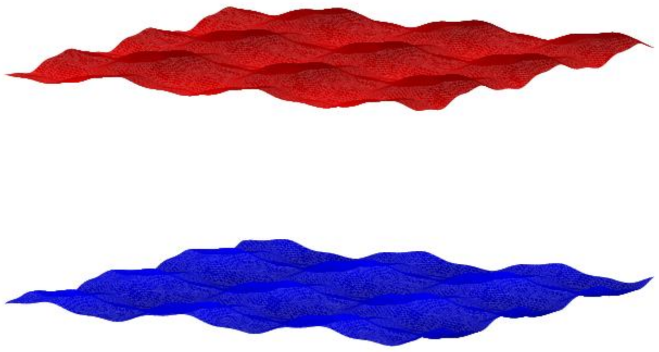
Implementation in optical lattices?



How can we obtain fractional quantum Hall-like physics in lattice systems?

Recipe 1:

Fractional Chern insulators



$H_{\text{continuum}} \rightarrow H_{\text{lattice}}$

Recipe 2:

Exact wave functions

$\Psi_{\text{continuum}}$

↓ [Kalmeyer & Laughlin]

Analytical ← Ψ_{lattice}

↓ Challenging

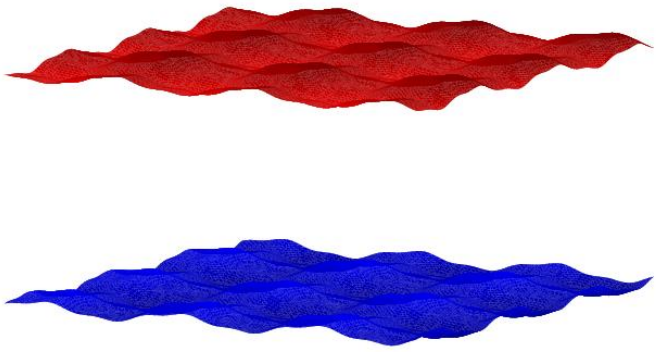
Exact H

[Schroeter, Kapit, Thomale & Greiter]

How can we obtain fractional quantum Hall-like physics in lattice systems?

Recipe 1:

Fractional Chern insulators



$H_{\text{continuum}} \rightarrow H_{\text{lattice}}$

Recipe 2:

Exact wave functions

$\Psi_{\text{continuum}}$

↓

Analytical ← Ψ_{lattice}

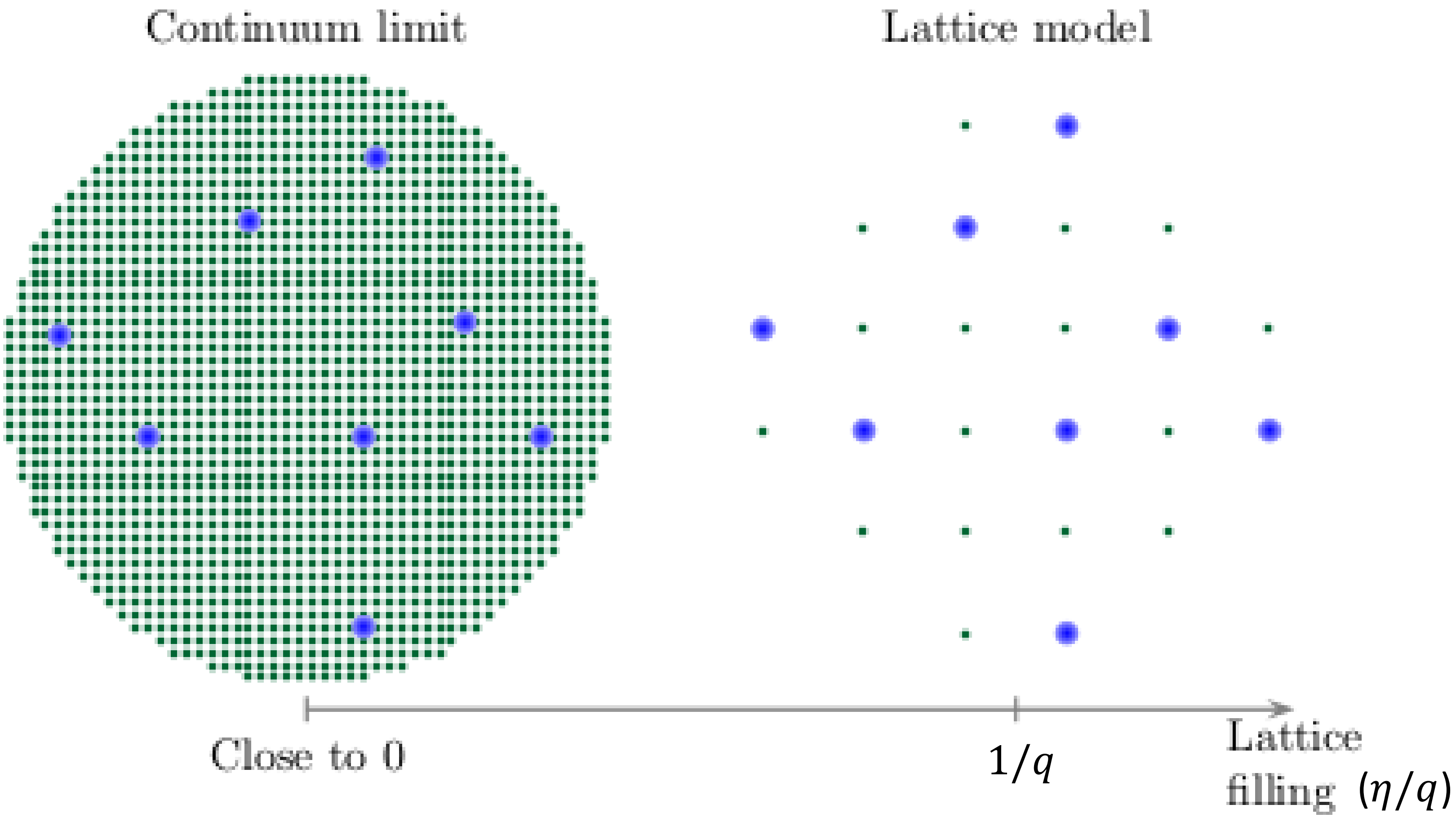
↓ CFT is helpful!

Exact H

Nielsen, Cirac, Sierra, PRL **108**, 257206 (2012)

Construction of lattice models with analytical ground states from conformal field theory

Interpolation between lattice and continuum



Tu, Nielsen, Cirac, Sierra, NJP **16**, 033025 (2014); Glasser, Cirac, Sierra, Nielsen, PRB **94**, 245104 (2016)

Construction of lattice FQH models

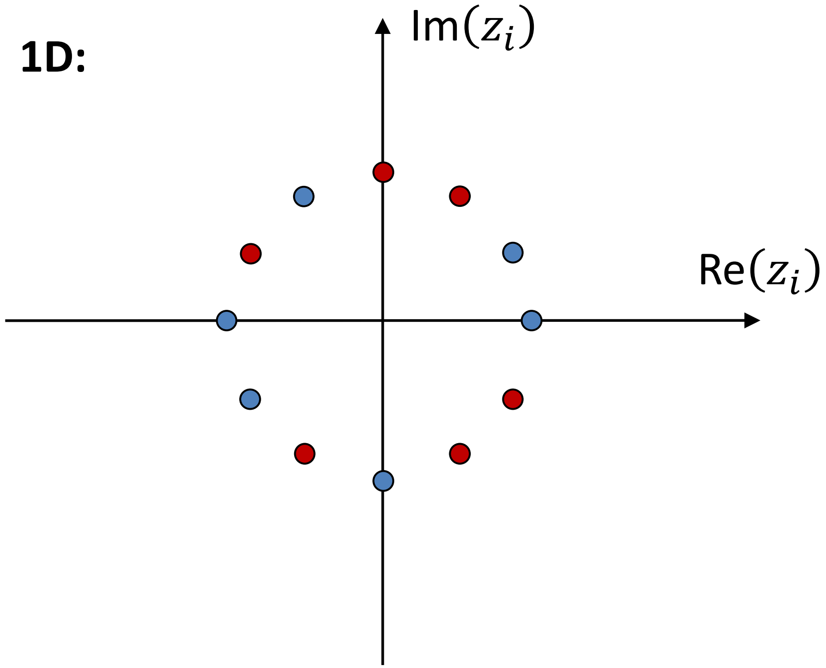
$$|\psi\rangle = \sum_{n_1,n_2,\dots,n_N} \psi(n_1,n_2,\dots,n_N)|n_1,n_2,\dots,n_N\rangle$$

$$\Lambda_i|\psi\rangle = 0$$

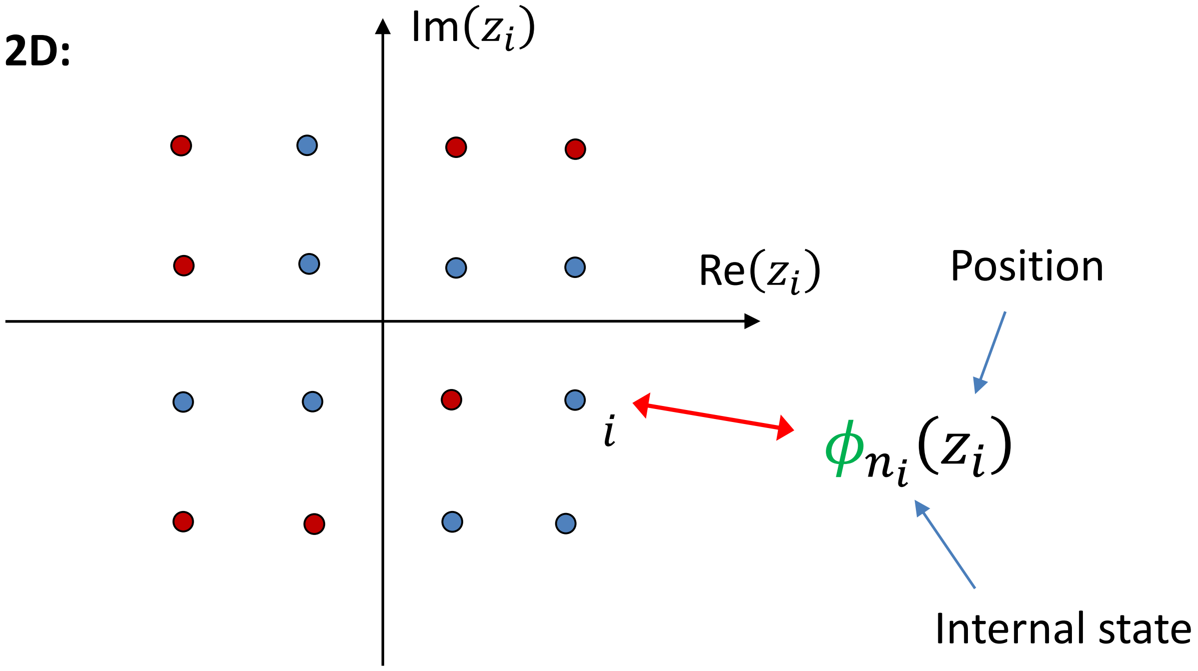
$$\psi(n_1,n_2,\dots,n_N) \propto \left\langle 0\middle|\phi_{n_1}(z_1)\phi_{n_2}(z_2)\cdots\phi_{n_N}(z_N)\middle|0\right\rangle$$

$$H = \sum_i \Lambda_i^\dagger \Lambda_i$$

1D:



2D:



Recipe to construct the Hamiltonian

1. Choose what $\phi_{n_i}(z_i)$ should be.
2. Find a null field $\chi(z_i)$.
3. Note that $\langle 0 \big| \phi_{n_1}(z_1)\phi_{n_2}(z_2)\cdots\chi(z_i)\cdots\phi_{n_N}(z_N)\big| 0\rangle = 0$.
4. Deform the integration contour.
5. Use operator product expansions.
6. Do the integrals.
7. Rewrite such that the final expression is an operator acting on the initial wavefunction.

Construction of lattice FQH models

1D:

$$\phi_{n_j}(z_j) = :e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}:$$

$q =$ integer
$n_j \in \{0,1\}$ → Critical models (Haldane-Shastry for $q = 2$)

2D:

$$\phi_{n_j}(z_j) = :e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}:$$

$q =$ integer
$n_j \in \{0,1\}$ → Laughlin state with q flux units per particle

$$\phi_{n_j}(z_j) = \chi(z_j)^{n_j(2-n_j)} :e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}:$$

$q =$ integer
$n_j \in \{0,1\}$ or
$n_j \in \{0,1,2\}$ → Moore-Read state with q flux units per particle

$$\phi_{n_j}(z_j) = \kappa_{n_j} :e^{i\sqrt{2}\,\vec{m}_{n_j}\cdot\vec{\varphi}(z_j)}:$$

$n_j \in \{1,2,\ldots,n\}$
$\vec{m}_{n_j}$ are vectors of numbers, see NPB **886**, 328 (2014) → Halperin states

---

χ: chiral part of Majorana fermion field

ϕ: chiral part of massless free boson field

κ_{n_j}: Klein factor

η: number that determines the number of particles per lattice site

Construction of lattice FQH models

2D:

$$\phi_{n_j}(z_j) = \,:e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}:$$

$$\phi_+(w_k) = \,:e^{i\varphi(w_k)/\sqrt{q}}:$$

q = integer
$n_j \in \{0,1\}$

→ Laughlin states with quasiholes

2D:

$$\phi_{n_j}(z_j) = \chi(z_j)^{n_j(2-n_j)} :e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}:$$

$$\phi_+(w_k) = \sigma(w_k) :e^{i\varphi(w_k)/(2\sqrt{q})}:$$

q = integer
$n_j \in \{0,1\}$ or
$n_j \in \{0,1,2\}$

→ Moore-Read states with quasiholes

χ: chiral part of Majorana fermion field

ϕ: chiral part of massless free boson field

σ: spin field of the chiral Ising CFT

η: number that determines the number of particles per lattice site

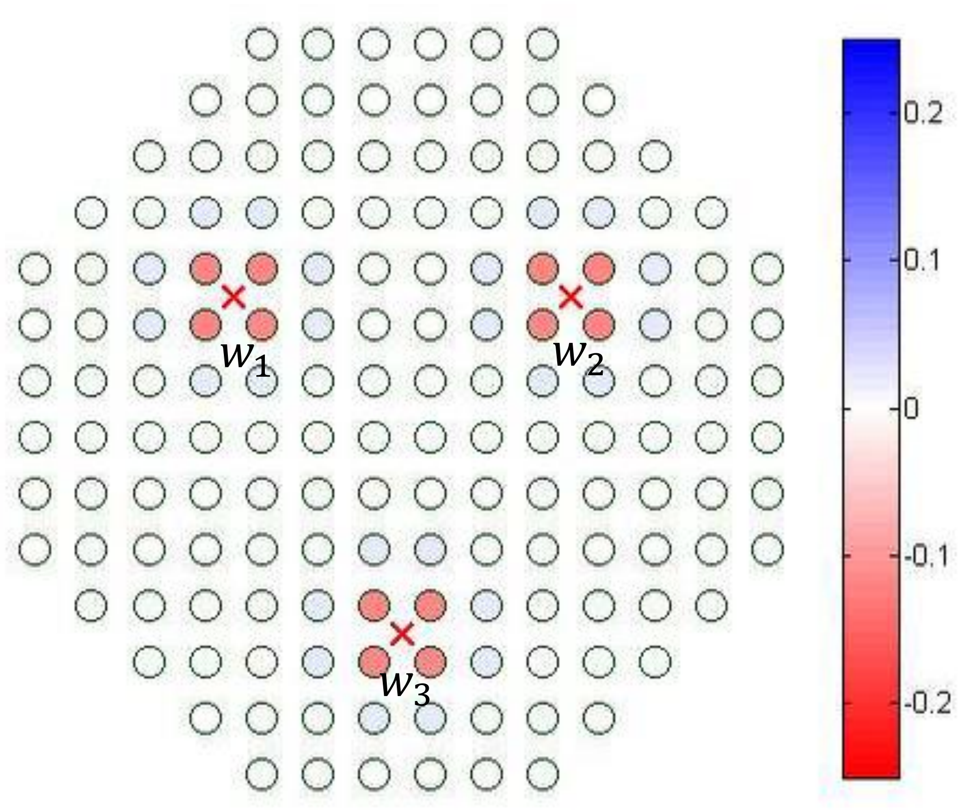
Laughlin states with quasiholes

Lattice Laughlin state with quasiholes

$H(w_1, w_2, w_3)$

$\Psi(w_1, w_2, w_3)$

$\langle n_i \rangle_{with} - \langle n_i \rangle_{without}$

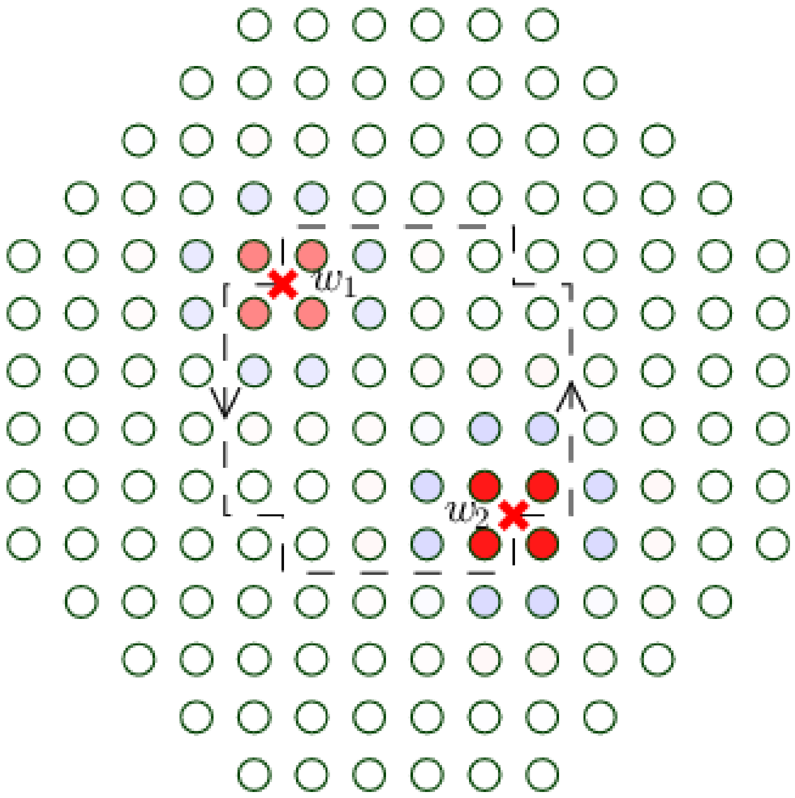


Nielsen, PRB **91**, 041106(R) (2015)

Exchange properties

$H(w_1, w_2)$

$\Psi(w_1, w_2)$



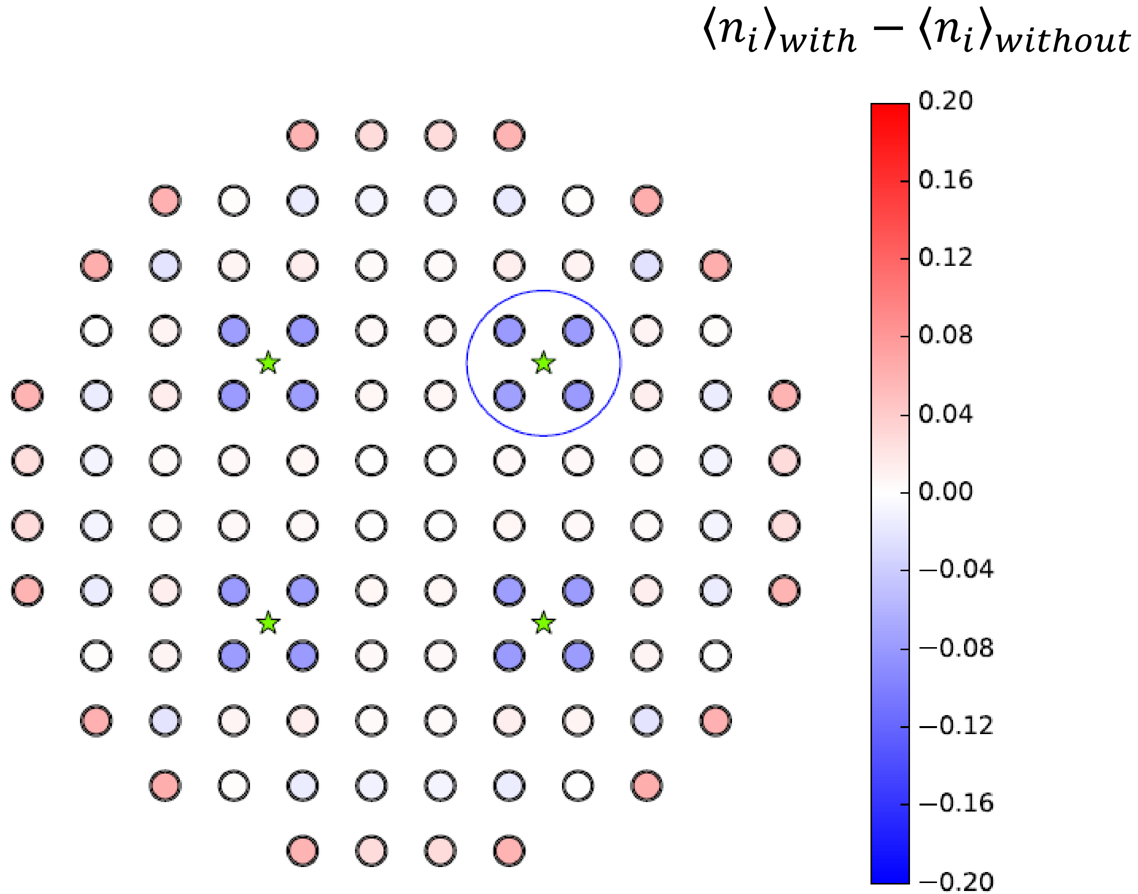
$|\Psi\rangle \to e^{i\theta}|\Psi\rangle$

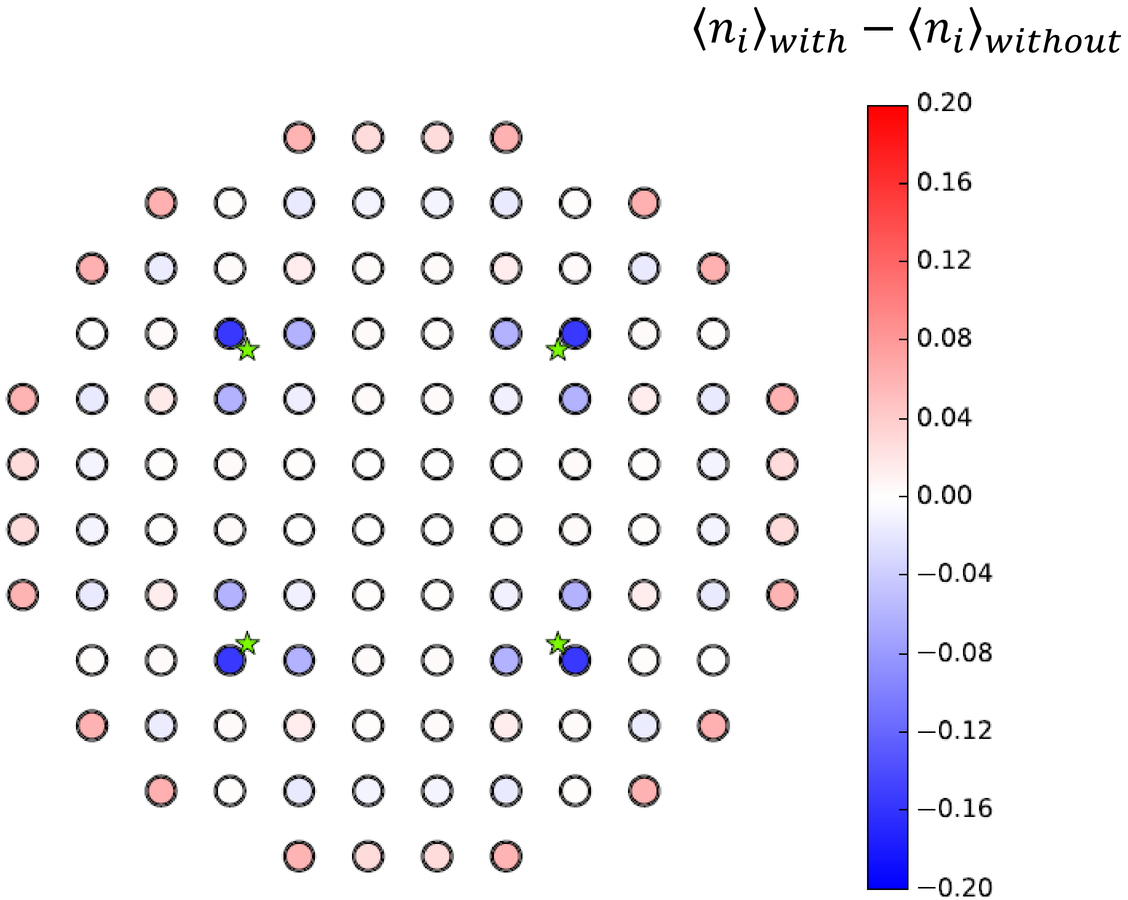
$$\frac{\theta}{2\pi} = -32 + \frac{2}{3} - 0.0017(5)$$

Nielsen, PRB **91**, 041106(R) (2015)
Rodriguez, Nielsen, PRB **92**, 125105 (2015)

Moore-Read states with quasiholes

Anyons in lattice Moore-Read states



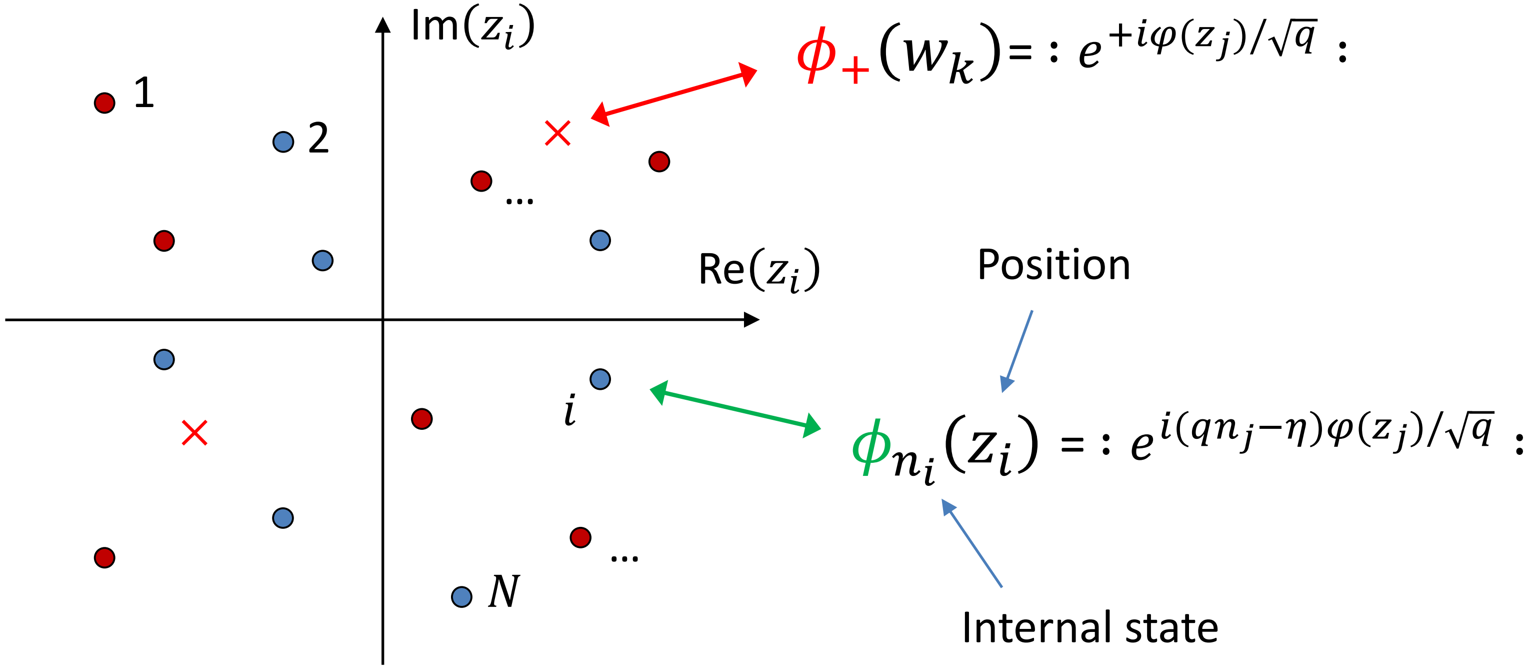


Laughlin and Moore-Read states with quasielectrons

How do we obtain quasielectrons?

$$|\psi_q\rangle = \sum_{n_1,n_2,\ldots,n_N} \psi(n_1, n_2, \ldots, n_N)|n_1, n_2, \ldots, n_N\rangle$$

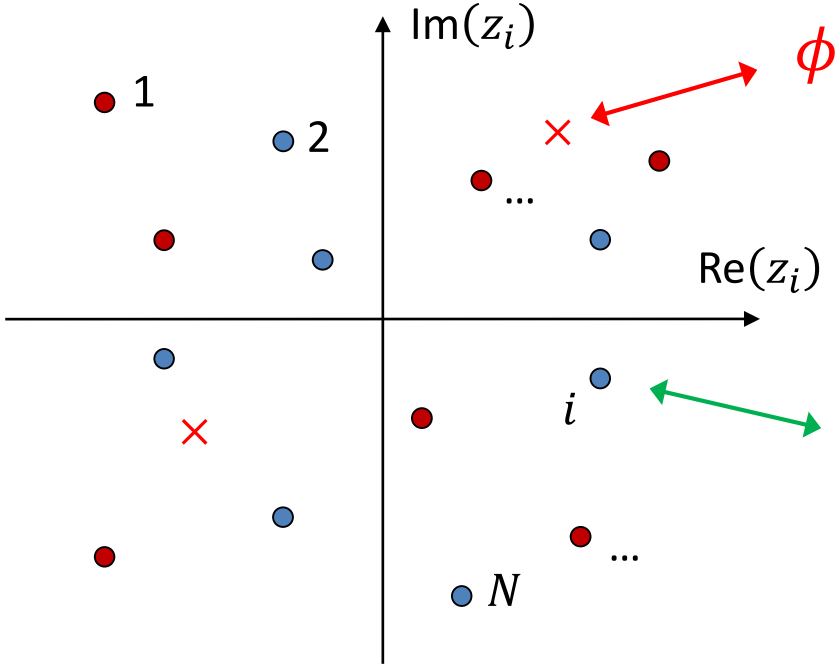
$$\psi(n_1, n_2, \ldots, n_N) \propto \langle 0|\phi_+(w_1)\cdots\phi_+(w_Q)\phi_{n_1}(z_1)\phi_{n_2}(z_2)\cdots\phi_{n_N}(z_N)|0\rangle$$



How do we obtain quasielectrons?

$$|\psi_q\rangle = \sum_{n_1, n_2, \ldots, n_N} \psi(n_1, n_2, \ldots, n_N) |n_1, n_2, \ldots, n_N\rangle$$

$$\psi(n_1, n_2, \ldots, n_N) \propto \langle 0 | \phi_+(w_1) \cdots \phi_-(w_Q) \phi_{n_1}(z_1) \phi_{n_2}(z_2) \cdots \phi_{n_N}(z_N) | 0 \rangle$$



$$\phi_\pm(w_k) = :e^{\pm i\varphi(z_j)/\sqrt{q}}:$$

Position

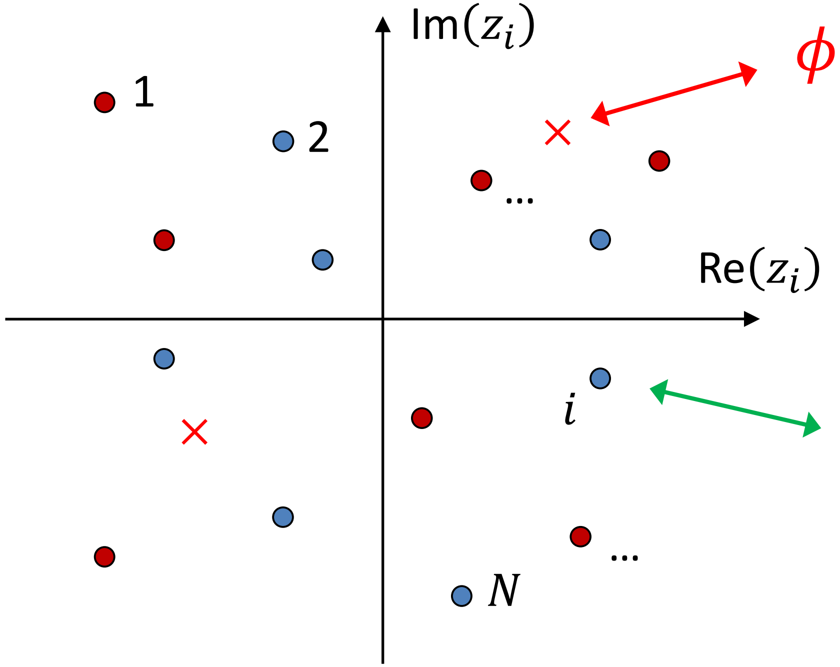
$$\phi_{n_i}(z_i) = :e^{i(qn_j - \eta)\varphi(z_j)/\sqrt{q}}:$$

Internal state

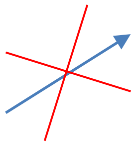
How do we obtain quasielectrons?

$$|\psi_q\rangle = \sum_{n_1,n_2,\dots,n_N} \psi(n_1,n_2,\dots,n_N)|n_1,n_2,\dots,n_N\rangle$$

$$\psi(n_1,n_2,\dots,n_N) \propto \langle 0|\phi_+(w_1)\cdots\phi_-(w_Q)\phi_{n_1}(z_1)\phi_{n_2}(z_2)\cdots\phi_{n_N}(z_N)|0\rangle$$



$$\phi_\pm(w_k) = :e^{\pm i\varphi(z_j)/\sqrt{q}}:$$



Position

$$\phi_{n_i}(z_i) = :e^{i(qn_j-\eta)\varphi(z_j)/\sqrt{q}}:$$

Internal state

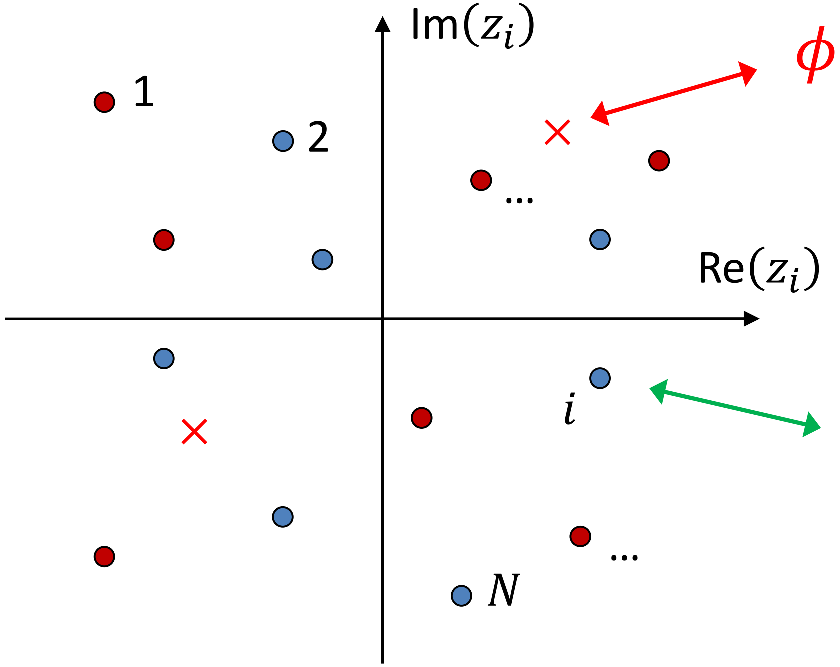
How do we obtain quasielectrons?

$$|\psi_q\rangle = \sum_{n_1, n_2, \ldots, n_N} \psi(n_1, n_2, \ldots, n_N)|n_1, n_2, \ldots, n_N\rangle$$

Same story for Moore-Read quasielectrons!

$$\psi(n_1, n_2, \ldots, n_N) \propto \langle 0|\phi_+(w_1)\cdots\phi_-(w_Q)\phi_{n_1}(z_1)\phi_{n_2}(z_2)\cdots\phi_{n_N}(z_N)|0\rangle$$

Continuum



$$\phi_\pm(w_k) = :e^{\pm i\varphi(z_j)/\sqrt{q}}:$$

Lattice

Position

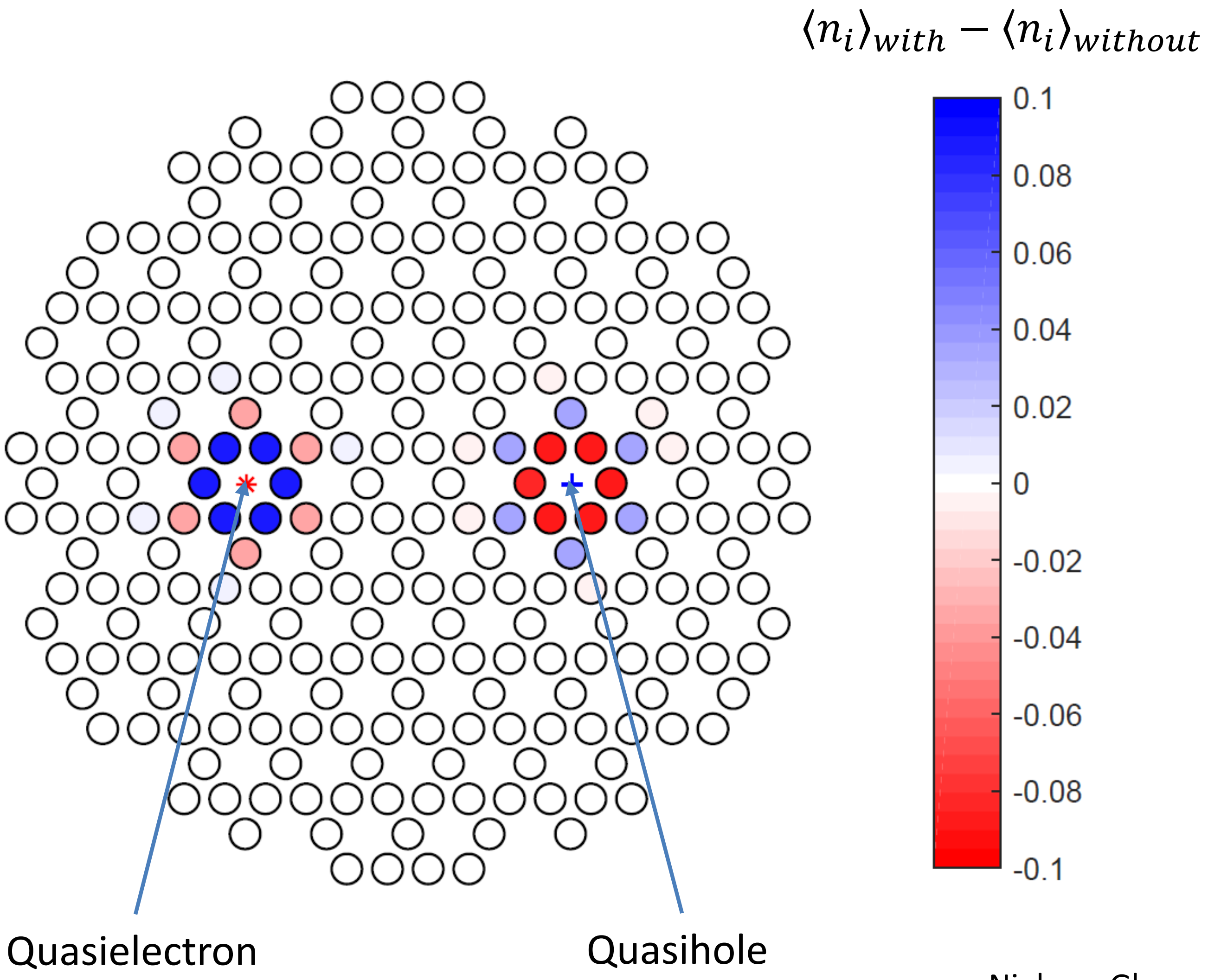
$$\phi_{n_i}(z_i) = :e^{i(qn_j - \eta)\varphi(z_j)/\sqrt{q}}:$$

Internal state

Charge distributions of Laughlin anyons

$\frac{1}{3}$ Laughlin state

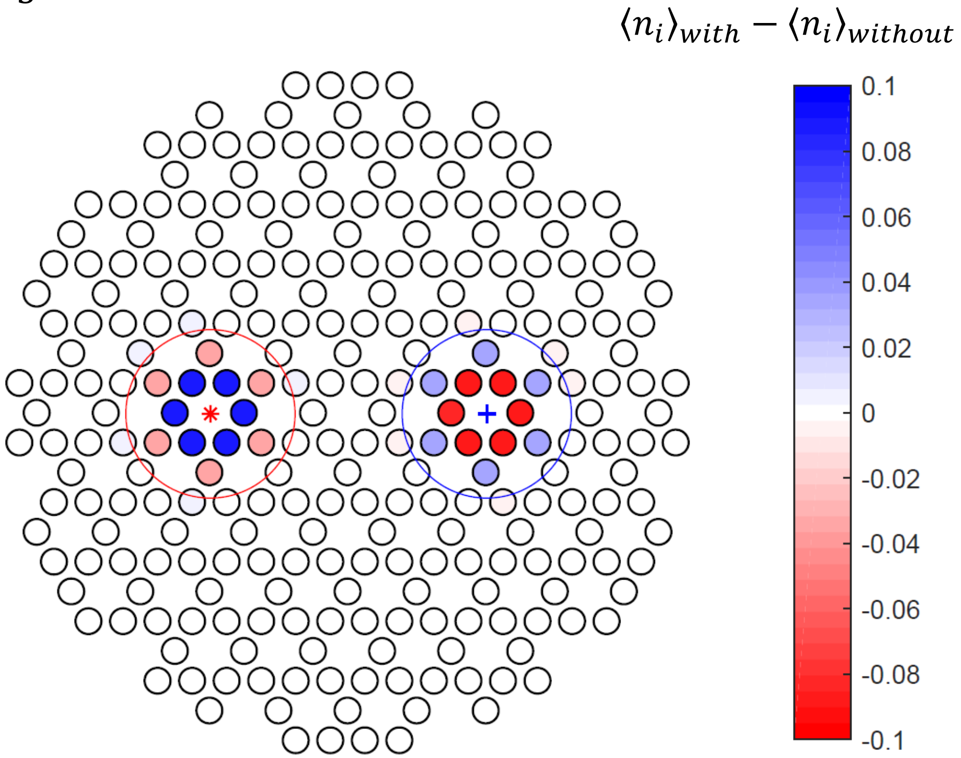
Lattice filling $= \frac{1}{2}$

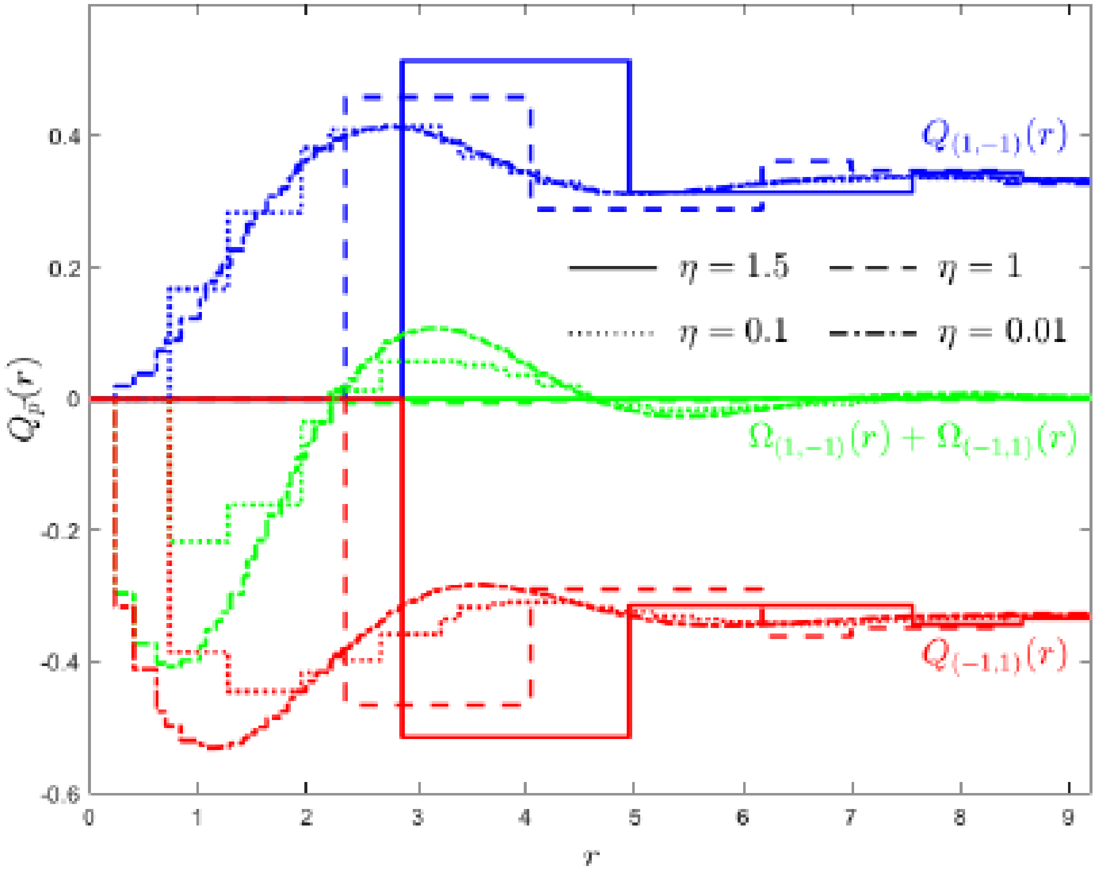


Nielsen, Glasser, Rodriguez, NJP **20**, 033029 (2018)

Laughlin Quasiholes <--> Laughlin Quasielectrons

$\frac{1}{3}$ Laughlin state



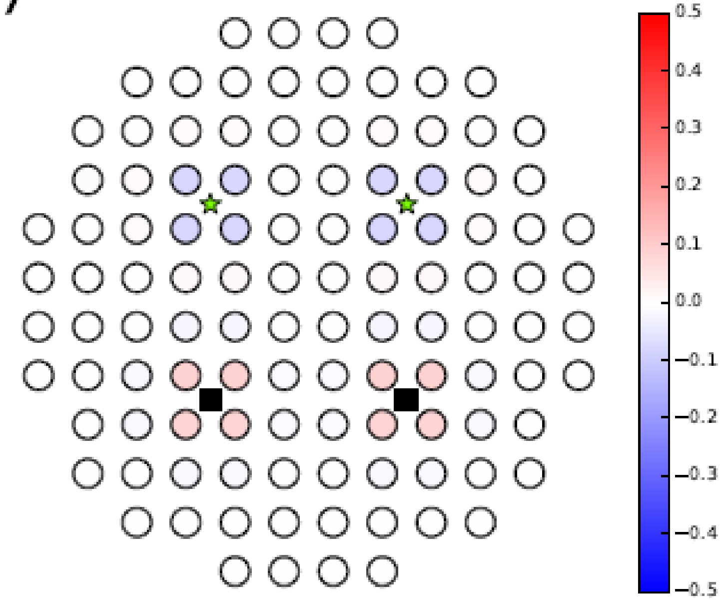


$$\Omega_{\vec{p}}(r) \equiv - \sum_{i \text{ inside circle}} \left(\langle n_i \rangle_{with} - \langle n_i \rangle_{without} \right)$$

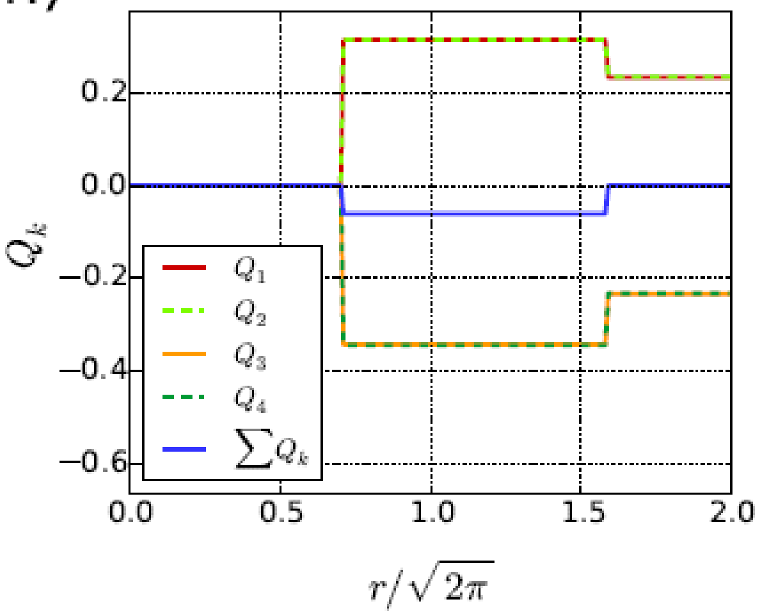
Nielsen, Glasser, Rodriguez, NJP **20**, 033029 (2018)

Quasiholes and quasielectrons in lattice Moore-Read states

g)



h)



Exchange properties are also as desired

Manna, Wildeboer, Nielsen, PRB **99**, 045147 (2019)

Examples of exact parent Hamiltonians

Bosonic Laughlin model from CFT

$$\phi_{s_j}(z_j) = e^{i\pi(j-1)(s_j+1)/2} : e^{is_j\varphi(z_j)/\sqrt{2}} : \qquad s_j \in \{-1,1\}$$

CFT: $SU(2)_1$ WZW

$$H = \frac{1}{2}\sum_{i\neq j}|w_{ij}|^2 - \frac{2i}{3}\sum_{i\neq j\neq k}\bar{w}_{ij}w_{ik}\boldsymbol{S}_i\cdot(\boldsymbol{S}_j\times\boldsymbol{S}_k)$$
$$+\frac{2}{3}\sum_{i\neq j}|w_{ij}|^2\boldsymbol{S}_i\cdot\boldsymbol{S}_j + \frac{2}{3}\sum_{i\neq j\neq k}\bar{w}_{ij}w_{ik}\boldsymbol{S}_j\cdot\boldsymbol{S}_k$$

$$\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z), \quad [S_i^a, S_j^b] = i\delta_{ij}\varepsilon_{abc}S_i^c$$

$$w_{ij} = \frac{g(z_i)}{z_i - z_j} + h(z_i)$$

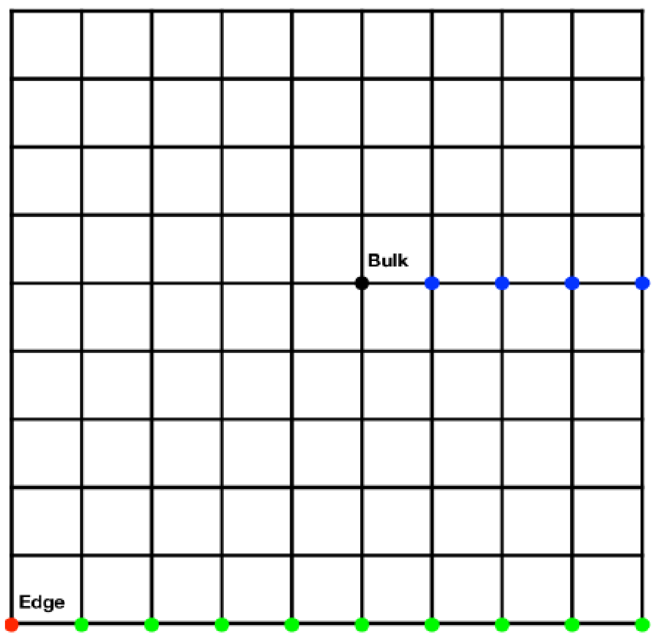
(g and h are arbitrary functions)

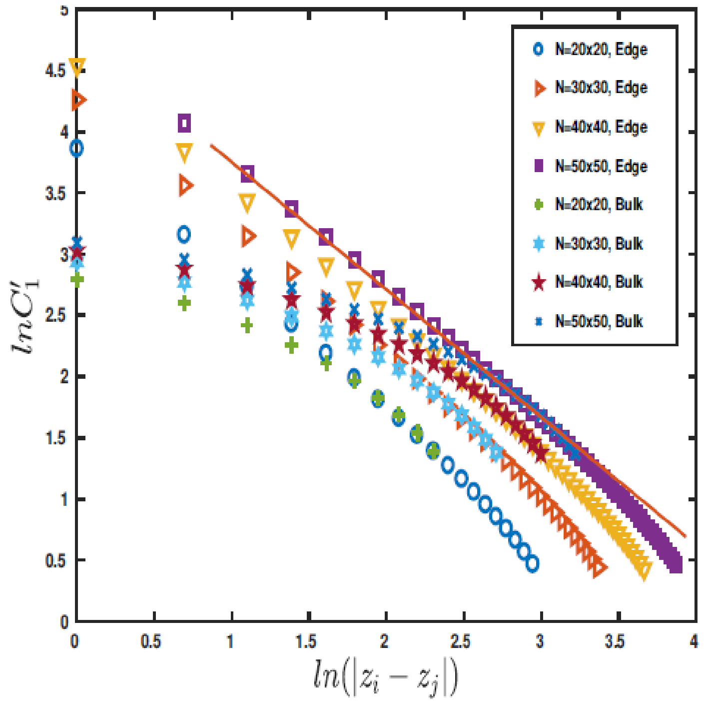
Generalization of the Kalmeyer-Laughlin state to arbitrary lattices.

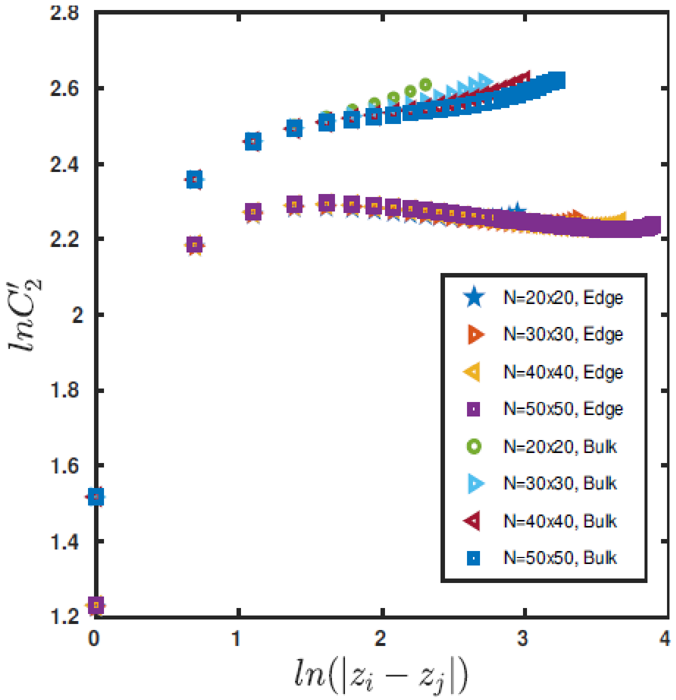
Hamiltonian for Laughlin states

$$\phi_{n_j}(z_j) = e^{i\pi(j-1)n_j} : e^{i(qn_j-1)\varphi(z_j)/\sqrt{q}} :$$

$$H = \sum_i \Lambda_i^\dagger \Lambda_i \qquad \Lambda_i = \sum_{j(\neq i)} \frac{1}{z_i - z_j}\left(d_j - d_i(qn_j - 1)\right)$$



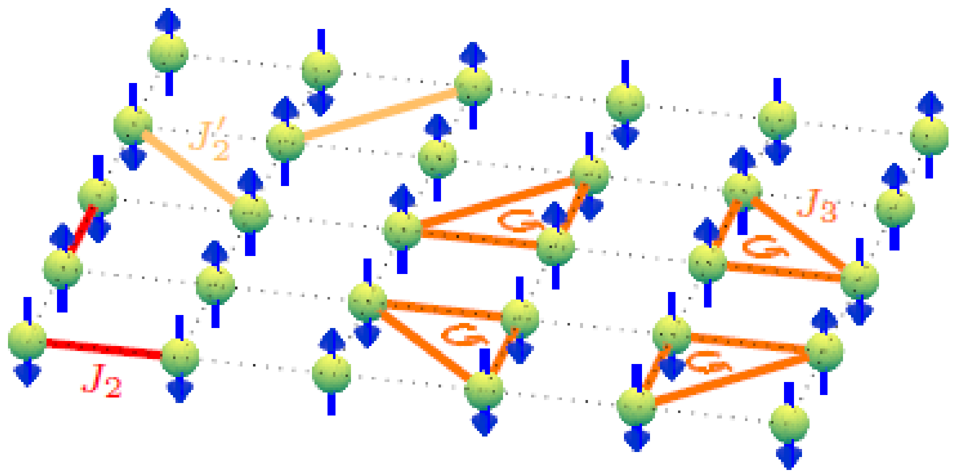




Can we truncate the Hamiltonians to obtain local models?

Local Hamiltonian on a square lattice

Approach: Truncate Hamiltonian, adjust, optimize.



Works also on kagome lattice, but not on triangular lattice.

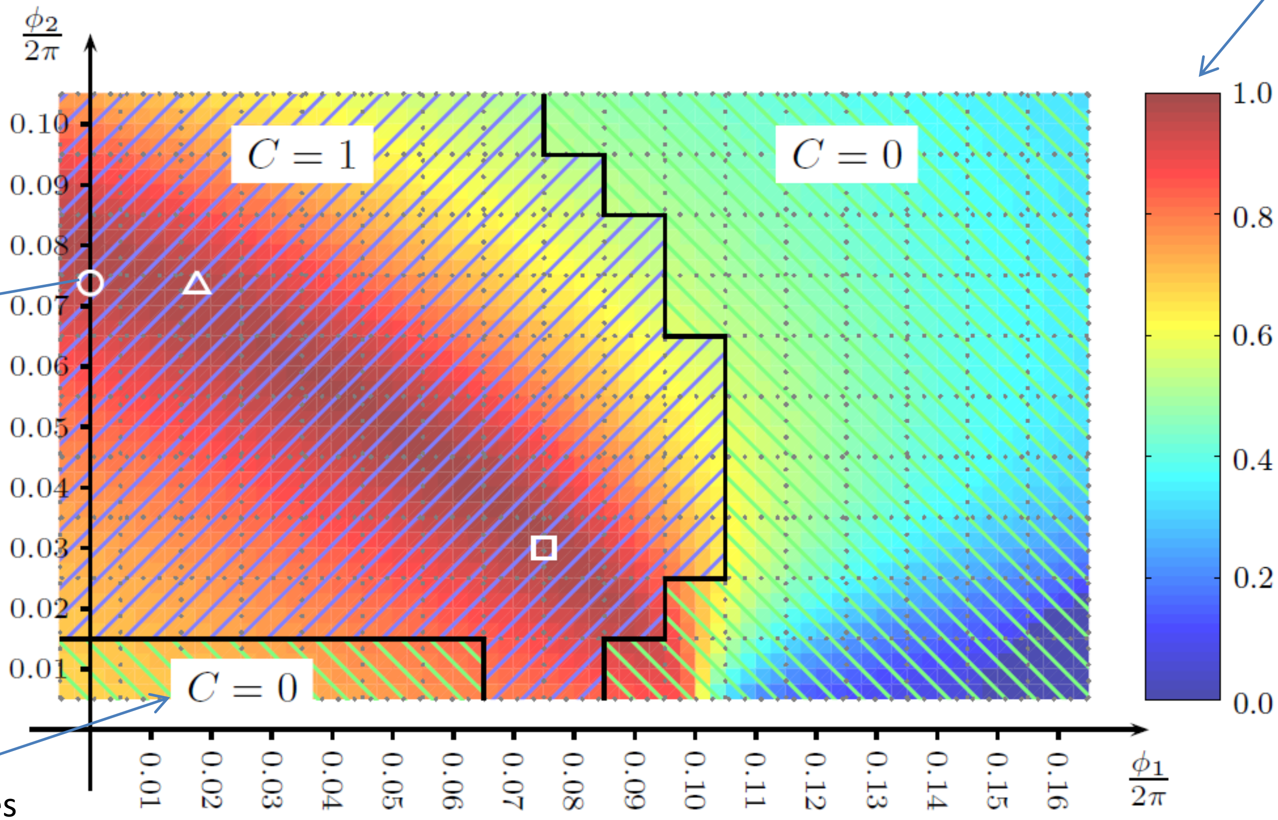
$$H = J_2 \sum_{<n,m>} 2\vec{S}_n \cdot \vec{S}_m + J'_2 \sum_{\ll n,m \gg} 2\vec{S}_n \cdot \vec{S}_m - J_3 \sum_{<n,m,p>_{\circlearrowleft}} 4\vec{S}_n \cdot (\vec{S}_m \times \vec{S}_p)$$

Nielsen, Sierra, Cirac, Nat. Commun. **4**, 2864 (2013).

Local Hamiltonian

Overlap with CFT state on 4×5 lattice

$J_2 = 1,$
$J'_2 = 0,$
$J_3 = 1/2$



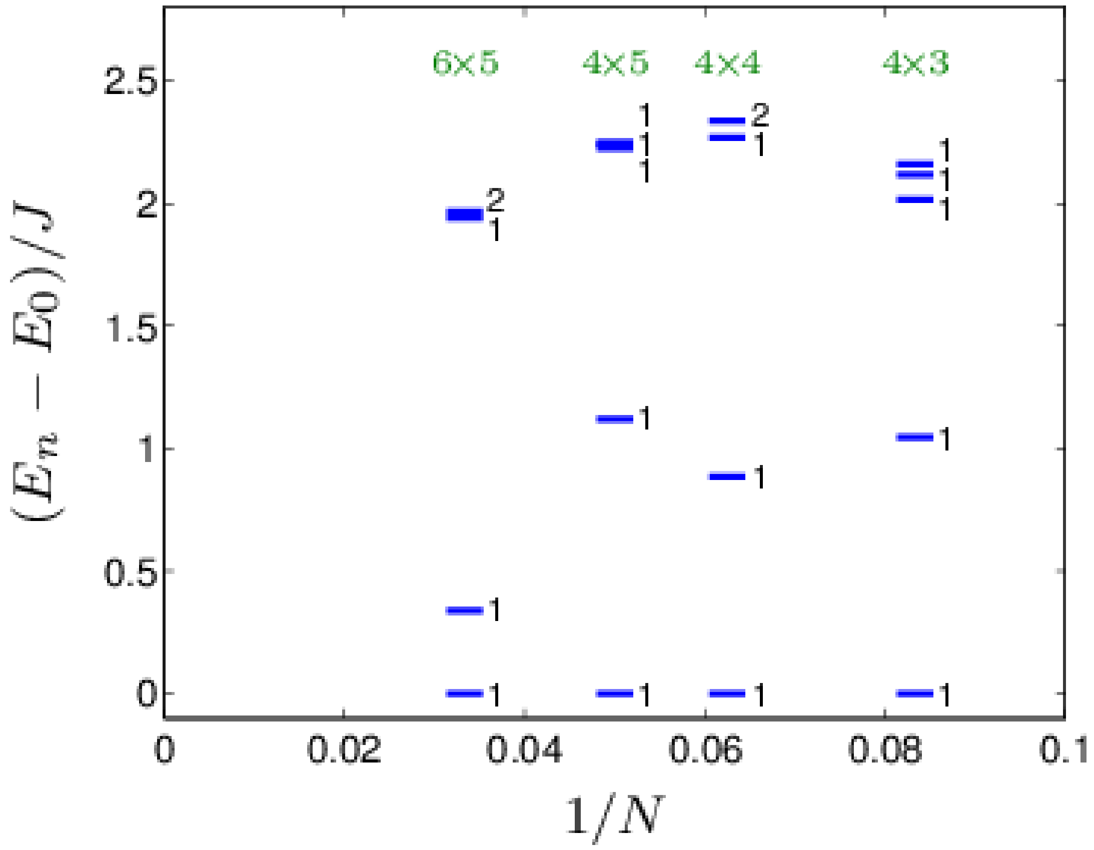
Total Chern number of ground states on the torus

$$H = J_2 \sum_{<n,m>} 2\vec{S}_n \cdot \vec{S}_m + J'_2 \sum_{\ll n,m \gg} 2\vec{S}_n \cdot \vec{S}_m - J_3 \sum_{<n,m,p>_\circlearrowleft} 4\vec{S}_n \cdot (\vec{S}_m \times \vec{S}_p)$$

$$J_2 = \cos(\phi_1)\cos(\phi_2), \quad J'_2 = \sin(\phi_1)\cos(\phi_2), \quad J_3 = \sin(\phi_2)$$

Nielsen, Sierra, Cirac, Nat. Commun. **4**, 2864 (2013).

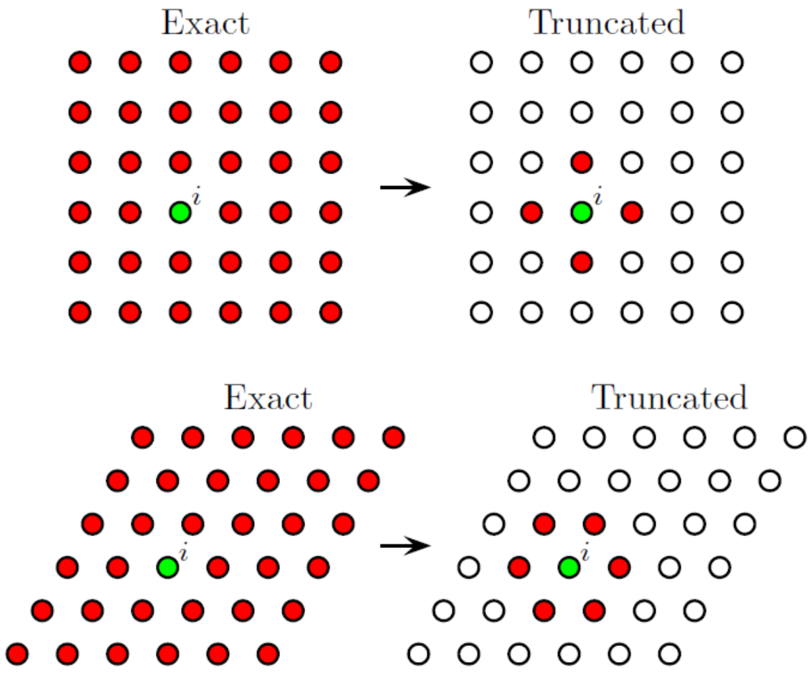
Is the ground state degeneracy correct?



Nielsen, Sierra, Cirac, Nat. Commun. **4**, 2864 (2013).

Can we do something more well-defined?

$$H = \sum_i \Lambda_i^\dagger \Lambda_i \qquad \Lambda_i = \sum_{j(\neq i)} \frac{1}{z_i - z_j}\left(d_j - d_i(qn_j - 1)\right)$$



Approach: Truncate Λ_i operator directly.

Advantages:

- The Λ_i operator is simpler to truncate than the Hamiltonian.
- The result of the truncation does not depend on the number of sites in the lattice.
- It is clear how to obtain models with periodic boundary conditions.
- No optimization needed.

Nandy, Srivatsa, Nielsen, arXiv:1902.09017

Results for $q = 2$ and $q = 4$

Square lattice

TABLE I. Overlap Δ and overlap per site $\Delta^{1/N}$ between the exact state on the torus and the lowest energy eigenstate of H^{Local} for the square lattice with $L_x \times L_y$ unit cells.

$L_x \times L_y$	$q=2$		$q=4$	
	Δ	$\Delta^{1/N}$	Δ	$\Delta^{1/N}$
3×4	0.8679	0.9883	0.8317	0.9848
4×4	0.9692	0.9980	0.9431	0.9963
4×5	0.9239	0.9961	0.9122	0.9954
4×6	0.9226	0.9966	0.7657	0.9889
5×6	0.9164	0.9971		

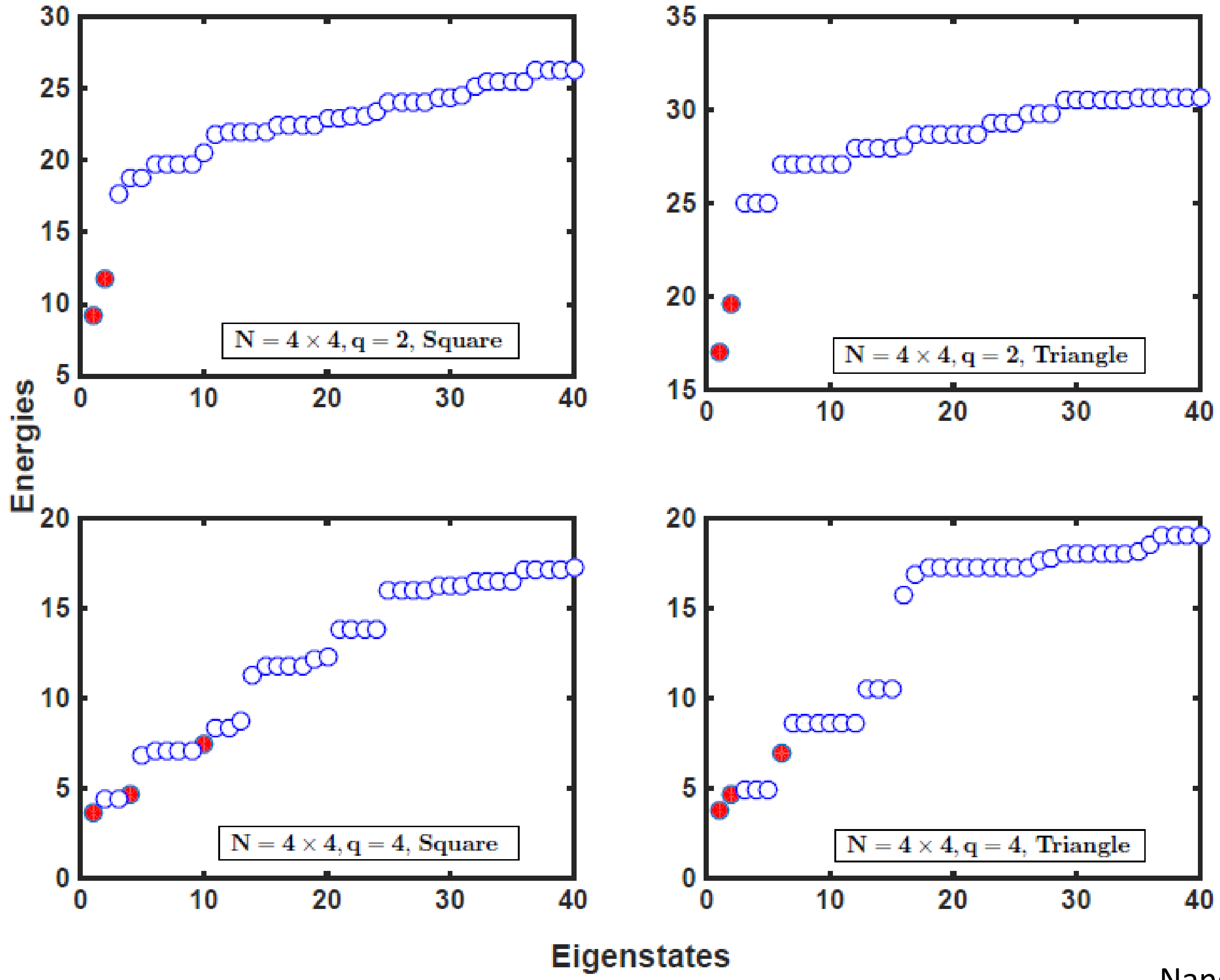
Triangular lattice

TABLE II. Overlap Δ and overlap per site $\Delta^{1/N}$ between the exact state on the torus and the lowest energy eigenstate of H^{Local} for the triangular lattice with $L_x \times L_y$ unit cells.

$L_x \times L_y$	$q=2$		$q=4$	
	Δ	$\Delta^{1/N}$	Δ	$\Delta^{1/N}$
3×4	0.8400	0.9856	0.9317	0.9941
4×4	0.9507	0.9968	0.8710	0.9913
4×5	0.9098	0.9953	0.7512	0.9857
4×6	0.8913	0.9952	0.6827	0.9842
5×6	0.8210	0.9934		

Nandy, Srivatsa, Nielsen, arXiv:1902.09017

Is the ground state degeneracy correct?



Nandy, Srivatsa, Nielsen, arXiv:1902.09017

Conclusion

Conclusion

- We have constructed families of lattice models with analytical ground states and few-body Hamiltonians and investigated their properties.
- We have shown how to construct trial wavefunctions and parent Hamiltonians for Laughlin and Moore-Read quasiholes and quasielectrons in lattices.
- We have investigated different ways to use the exact Hamiltonians as a starting point to find local, few-body Hamiltonians with almost the same ground states.

Thank you!

Sourav Manna

Dillip K. Nandy

Julia Wildeboer

Srivatsa N.S

MPQ collaborators:

Ignacio Cirac
Ivan Glasser
Hong-Hao Tu
Benedikt Herwerth
Ivan D. Rodriguez

UAM collaborators:

German Sierra