Exclusion bounds for extended anyons

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based on joint work with

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Outline of Talk

1. The extended anyon model
2. Ground state energy in the thermodynamic limit
3. Statistical repulsion and Lieb–Thirring inequalities
4. A local approach to Lieb–Thirring inequalities
5. Magnetic Hardy inequalities
6. Local effective 1D and 2D problems

The free anyon gas

Particle exchange in $\mathbb{R}^2$: statistics parameter $\alpha \in \mathbb{R}$

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) \text{ ''$=$'' } e^{i\alpha\pi}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N).$$

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- **Fermions**: $\alpha \in 1 + 2\mathbb{Z}$
- **Anyons**: any $\alpha \in \mathbb{R}$

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Free anyon gas: kinetic energy $T_0 := \sum_{j=1}^{N}(-i\nabla_{\mathbf{x}_j})^2$
(units chosen so that $\hbar^2/(2m) = 1$).

The magnetic gauge

Singular gauge transformation: Ψ_α section of anyon bundle E_α

$$\Psi_\alpha = U^\alpha \Psi_0, \qquad U := \prod_{j<k} e^{i\phi_{jk}} = \prod_{j<k} \frac{z_j - z_k}{|z_j - z_k|},$$

where Ψ_0 is **bosonic** and z_j coordinates in $\mathbb{R}^2 \sim \mathbb{C}$.

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Modified free energy in $L^2_{sym}(\mathbb{R}^{2N})$

$$T_\alpha := \sum_{j=1}^{N} (-i\nabla_{\mathbf{x}_j} + \alpha \boldsymbol{A}_j^0)^2, \qquad \boldsymbol{A}_j^0 := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}.$$

Aharonov–Bohm type magnetic interaction.

R-extended anyons

For $R > 0$ let

$$\boldsymbol{A}_j^{R} := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|_{R}^{2}}, \qquad |\mathbf{x}|_R := \max\{|\mathbf{x}|, R\}.$$

Constant magnetic field on disk of radius R around each $\mathbf{x}_k$, $k \neq j$

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Consider the **kinetic energy** in $L^2_{sym}(\mathbb{R}^{2N})$

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$$T_\alpha^R := \sum_{j=1}^N (-i\nabla_{\mathbf{x}_j} + \alpha \boldsymbol{A}_j^R)^2.$$

- Periodicity in α destroyed $\Rightarrow$ consider all $\alpha \geq 0$.
- Emergence of anyons as bosons with attached magnetic flux.

[Choi–Lee–Lee '92, Trudenberger '92, Mashkevich '96, Lundholm–Rougerie 15']

Ground state energy — Thermodynamic limit

Goal: Understand the N-anyon **ground state energy**

$$E(N) := \inf \sigma(H_N), \qquad H_N := T_\alpha^R + \sum_{j=1}^{N} V(\mathbf{x}_j).$$

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Thermodynamic limit confined anyon gas:

Theorem (L.–Lundholm 2018)

$Q = [0,L]^2, H_N = T_\alpha^R$ *on* Q^N *with* **Dirichlet** *boundary conditions.*

There exists $C > 0$ *s.t.*

$$\liminf_{\substack{N,L\to\infty \\ N/L^2=\bar\varrho}} \frac{E^{\mathcal{D}}(N)}{N} \geq C e(\alpha, \bar\gamma := R\sqrt{\bar\varrho})\,\bar\varrho$$

where

$$e(\alpha,\gamma) \sim \begin{cases} \frac{2\pi}{|\ln\gamma|} + \pi(j'_{\alpha_*})^2 \geq 2\pi\alpha_*, & \gamma \to 0, \\ 2\pi|\alpha|, & \gamma \gtrsim 1. \end{cases}$$

Ground state energy — Thermodynamic limit

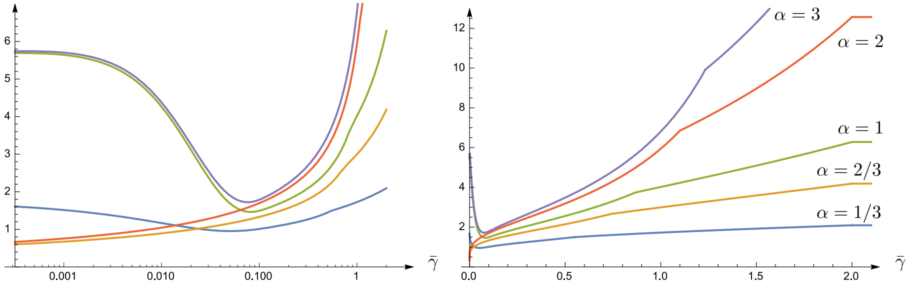

Fig 1. The function $\bar{\gamma} \mapsto e(\alpha, \bar{\gamma})$ for different α.

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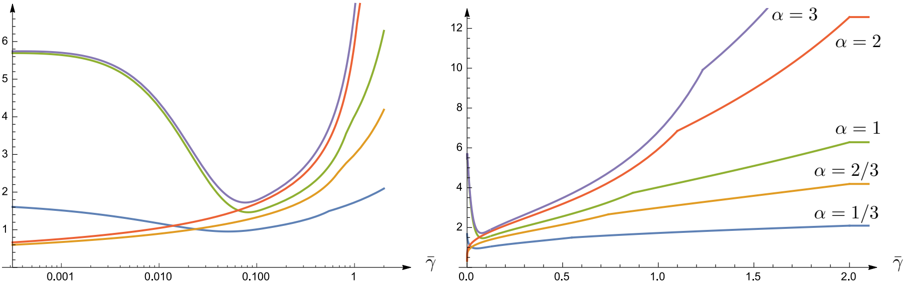

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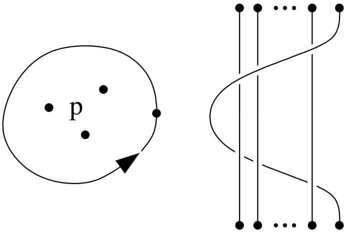
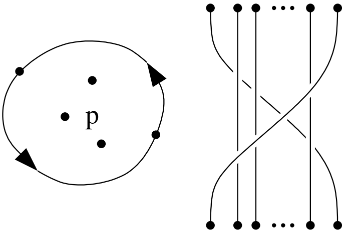
$$\alpha_N := \min_{\substack{p \in \{0,1,\ldots,N-2\} \\ k \in \mathbb{Z}}} |(2p+1)\alpha - 2k|$$ [Lundholm–Solovej '13]

$$\alpha_* := \lim_{N \to \infty} \alpha_N = \begin{cases} \frac{1}{\nu} & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Statistical repulsion — ideal anyons



$e^{i2p\alpha\pi}$



$e^{i(2p+1)\alpha\pi}$

Recall: 2-particle exchange **ideal anyons** $\Rightarrow$ phase $(2p+1)\alpha$ times π.
Cancel phase by pairwise relative angular momenta $\pm 2k$.

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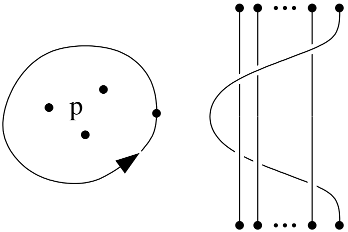

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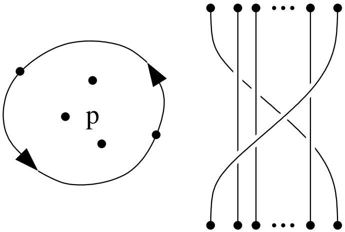

$e^{i(2p+1)\alpha\pi}$

Recall: 2-particle exchange **ideal anyons** $\Rightarrow$ phase $(2p+1)\alpha$ times π.
Cancel phase by pairwise relative angular momenta $\pm 2k$.
$\Rightarrow$ effective **statistical repulsion** [Lundholm, Solovej, '13]

$$V_{\text{stat}}(r) = |(2p+1)\alpha - 2k|^2 \frac{1}{r^2} \;\geq\; \frac{\alpha_N^2}{r^2}$$

Thermodynamic limit — confined Fermi gas in 2D

Fermi gas in a square Q: $H_N = \sum_j (-\Delta_j)$ on $L^2_{asym}(Q^N)$ **Dirichlet b.c.**

$$\inf \sigma(H_N) = \sum_{j=1}^{N} \lambda_j(H_1) \sim 2\pi(\underbrace{N/|Q|}_{\bar{\varrho}})^2 |Q|, \qquad \Psi_0 = \bigwedge_{j=1}^{N} \varphi_j$$

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$\rightsquigarrow$ Thomas–Fermi approximation: [Thomas, Fermi '27 – precursor to modern DFT]

$$\big\langle \Psi, H_N \Psi \big\rangle \approx 2\pi \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 d\mathbf{x},$$

one-particle density

$$\varrho_\Psi(\mathbf{x}) = \sum_{j=1}^{N} \int_{\mathbb{R}^{2(N-1)}} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_N)|^2 \prod_{k \neq j}^{N} d\mathbf{x}_k.$$

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The **Lieb–Thirring inequality**: [Lieb–Thirring '75]

$$\langle \Psi, H_N \Psi \rangle \geq C \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 d\mathbf{x}.$$

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The **Lieb–Thirring inequality**: [Lieb–Thirring '75]

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- Uses crucially the **antisymmetry** of Ψ (Pauli exclusion principle)

Lieb–Thirring inequalities for ideal anyons

[Lundholm–Solovej '13]

Theorem (LT inequality for ideal anyons)

There exists a $C > 0$*. Such that for any* $\Psi \in L^2_{\mathsf{sym}}(\mathbb{R}^{2N})$ *and* $\alpha \in \mathbb{R}$

$$\left\langle \Psi, T^0_\alpha \Psi \right\rangle \geq C\alpha_N^2 \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x}.$$

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- Trivial in the limit $N \to \infty$ if $\alpha_* = 0$.
- Non-trivial thermodynamic limit for $\alpha = \frac{\mu}{\nu}$ with **odd** μ and $\nu \geq 1$.
- Not expected behaviour with respect to α in dilute limit as $\alpha \to 0$.

[Lundholm–Rougerie '15, Correggi–Lundholm–Rougerie '17]

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[Lundholm–Solovej '13, L.–Lundholm '18]

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where $j'_\nu \geq \sqrt{2\nu}$ *is the first positive zero of* J'_ν*.*

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- **Has** expected behaviour with respect to α in dilute limit as $\alpha \to 0$.

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Lieb–Thirring inequalities for ideal anyons

[Lundholm–Solovej '13, L.–Lundholm '18, Lundholm–Seiringer '18, L.–Lundholm–Nam '19]

Theorem (LT inequality for ideal anyons)

There exists a $C > 0$. Such that for any $\Psi \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ and $\alpha \in \mathbb{R}$

$$\langle \Psi, T^0_\alpha \Psi \rangle \geq C \alpha_2 \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x}.$$

Recall $\alpha_2 = \min_{k\in\mathbb{Z}} |\alpha - 2k|$.

- Trivial **only** for $\alpha \in 2\mathbb{Z}$ (**bosons**).
- Non-trivial thermodynamic limit for **all** $\alpha \notin 2\mathbb{Z}$.
- **Has** expected behaviour with respect to α in dilute limit as $\alpha \to 0$.

[Lundholm–Rougerie '15, Correggi–Lundholm–Rougerie '17]

A local approach to Lieb–Thirring inequalities

Let $N \geq 1$, $\Psi \in H^s(\mathbb{R}^{dN})$ s.t. $\|\Psi\|_{L^2(\mathbb{R}^{dN})} = 1$, Q a d-cube with volume $|Q|$, and $\mathcal{E}_Q[\Psi] := \big\langle \Psi, \sum_{j=1}^N (-\Delta_{\mathbf{x}_j})^s_{|Q} \Psi \big\rangle = \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} \|\Psi\|^2_{\dot{H}^s_{\mathbf{x}_j}(Q)} d\mathrm{x}'$

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Lemma (Local uncertainty principle ∼ Poincaré–Sobolev)

$$\mathcal{E}_Q[\Psi] \;\geq\; \frac{1}{C_1} \frac{\int_Q \varrho_\Psi^{1+2s/d}}{(\int_Q \varrho_\Psi)^{2s/d}} - C_1 \frac{\int_Q \varrho_\Psi}{|Q|^{2s/d}}.$$

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Lemma (Local exclusion principle)

$$\mathcal{E}_Q[\Psi] \geq C_2 |Q|^{-2s/d} \left[\int_Q \varrho_\Psi(\mathbf{x})\, d\mathbf{x} - q \right]_+ ,$$

where $q := \#\{\text{multi-indices } \alpha \in \mathbb{N}_0^d : 0 \leq |\alpha| < s\}$.

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- Don't need Pauli's exlcusion principle, only effective local exclusion.

A local approach to Lieb–Thirring inequalities

Lemma (Covering lemma)

Let $0 \leq f \in L^1(\mathbb{R}^d)$ *be such that* $\int_{\mathbb{R}^d} f \geq \Lambda > 0$. *Then* $\operatorname{supp} f$ *can be covered by disjoint cubes* $\{Q\}$ *in* $\mathbb{R}^d$ *such that* $\int_Q f \leq \Lambda$ $\forall Q$ *and*

$$\sum_Q \frac{1}{|Q|^\alpha}\left(\left[\int_Q f - q\right]_+ - b\int_Q f\right) \geq 0$$

for all $\alpha > 0$ *and* $0 \leq q < \Lambda 2^{-d}$, *where*

$$b := \left(1 - \frac{2^d q}{\Lambda}\right)\frac{2^{d\alpha} - 1}{2^{d\alpha} + 2^d - 2} > 0.$$

[Lundholm–Solovej '13, Lundholm–Nam–Portmann '15]

A local approach to Lieb–Thirring inequalities

Proof of LT:
Let $\Lambda = 2^d q + 1$. If $N \leq \Lambda$ use Gagliardo–Nirenberg–Sobolev, while if $N > \Lambda$ use the covering lemma with $f = \varrho_\Psi$:

$$
\begin{aligned}
(\varepsilon+1)\mathcal{E}_{\mathbb{R}^d}[\Psi] &\geq \varepsilon \sum_Q \left[\frac{1}{C_1} \frac{\int_Q \varrho_\Psi^{1+2s/d}}{\left(\int_Q \varrho_\Psi\right)^{2s/d}} - \frac{C_1}{|Q|^{2s/d}} \int_Q \varrho_\Psi \right] \\
&\quad + \sum_Q \frac{C_2}{|Q|^{2s/d}} \left[\int_Q \varrho_\Psi(x)\,dx - q \right]_+ \\
&\geq \frac{\varepsilon}{C_1} \frac{\int_{\mathbb{R}^d} \varrho_\Psi^{1+2s/d}}{\Lambda^{2s/d}}
\end{aligned}
$$

for any fixed constant $\varepsilon > 0$ satisfying $\varepsilon C_1 \leq C_2 b$.

Reduction of local exclusion — general formulation

Lemma (Covariant energy bound, L.–Lundholm–Nam '19)

Assume that to any $n \in \mathbb{N}_0$ *and any* d*-cube* $Q \subset \mathbb{R}^d$ *there is an* **"energy"** $E_n(Q) \geq 0$ *satisfying, for some* $s > 0$*:*

- **(scale-covariance)** $E_n(\lambda Q) = \lambda^{-2s} E_n(Q)$ *for all* $\lambda > 0$*;*
- **(translation-invariance)** $E_n(Q + \mathbf{x}) = E_n(Q)$ *for all* $\mathbf{x} \in \mathbb{R}^d$*;*
- **(superadditivity)** *For any collection of disjoint cubes* $\{Q_j\}_{j=1}^J$ *such that their union is a cube,*

$$E_n\Big(\bigcup_{j=1}^J Q_j\Big) \geq \min_{\{n_j\} \in \mathbb{N}_0^J \text{ s.t. } \sum_j n_j = n} \sum_{j=1}^J E_{n_j}(Q_j);$$

- **(a priori positivity)** *There exists* $q \geq 0$ *such that* $E_n(Q) > 0$ *for all* $n \geq q$.

Then there exists a constant $C > 0$ *independent of* n *and* Q *s.t.*

$$E_n(Q) \geq C|Q|^{-2s/d} n^{1+2s/d}, \quad \forall n \geq q.$$

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- Applicable for **ideal** anyons but not extended

Relative magnetic Hardy inequality

Lemma (Magnetic Hardy inequality with symmetry)

Let $\Omega = B_{R_2}(0) \setminus \bar{B}_{R_1}(0)$. *Let* $\boldsymbol{a}$ *be a* **vector potential** *corresponding to a magnetic field* $b = \operatorname{curl} \boldsymbol{a}$, *defined on* $B_{R_2}(0)$.
Assume that

- $\Phi(r) := \frac{1}{2\pi} \int_{\partial B_r} \boldsymbol{a} \cdot d\boldsymbol{r}' < \infty \ \forall r \in (R_1, R_2)$,
- $\boldsymbol{a}$ *is antipodal-***antisymmetric** $\boldsymbol{a}(-\boldsymbol{r}) = -\boldsymbol{a}(\boldsymbol{r})$
- b *is antipodal-***symmetric** $b(-\boldsymbol{r}) = b(\boldsymbol{r})$.

*Then, for any antipodal-***symmetric** $u \in C^\infty(\Omega)$, *i.e. with* $u(-\boldsymbol{r}) = u(\boldsymbol{r})$ *for all* $\boldsymbol{r} \in \Omega$ *and* $r = |\boldsymbol{r}|$

$$\int_\Omega |(-i\nabla + \boldsymbol{a})u|^2 \, d\boldsymbol{r} \ \geq \ \int_\Omega \left(\big|\partial_r |u|\big|^2 + \inf_{k \in \mathbb{Z}} \big|\Phi(r) - 2k\big|^2 \frac{|u|^2}{r^2} \right) d\boldsymbol{r}.$$

[Laptev–Weidl '98, Lundholm–Solovej '13, L.–Lundholm '18]

- Keeping the $\partial_r |u|$-term is crucial!

Relative magnetic Hardy inequality

Lemma (Magnetic Hardy inequality with **anti**symmetry)

Let $\Omega = B_{R_2}(0) \setminus \bar{B}_{R_1}(0)$. *be a* **vector potential** *corresponding to a magnetic field* $b = \operatorname{curl} \boldsymbol{a}$, *defined on* $B_{R_2}(0)$.
Assume that

- $\Phi(r) := \frac{1}{2\pi} \int_{\partial B_r} \boldsymbol{a} \cdot d\boldsymbol{r}' < \infty \ \forall r \in (R_1, R_2)$,
- $\boldsymbol{a}$ *is antipodal-***antisymmetric** $\boldsymbol{a}(-\boldsymbol{r}) = -\boldsymbol{a}(\boldsymbol{r})$
- b *is antipodal-***symmetric** $b(-\boldsymbol{r}) = b(\boldsymbol{r})$.

*Then, for any antipodal-***antisymmetric** $u \in C^\infty(\Omega)$, *i.e. with* $u(-\boldsymbol{r}) = -u(\boldsymbol{r})$ *for all* $\boldsymbol{r} \in \Omega$ *and* $r = |\boldsymbol{r}|$

$$\int_\Omega |(-i\nabla + \boldsymbol{a})u|^2 \, d\boldsymbol{r} \ \geq \ \int_\Omega \left(\left|\partial_r |u|\right|^2 + \inf_{k \in \mathbb{Z}} \left|\Phi(r) - (2k+1)\right|^2 \frac{|u|^2}{r^2} \right) d\boldsymbol{r}.$$

[Laptev–Weidl '98, Lundholm–Solovej '13, L.–Lundholm '18]

- Keeping the $\partial_r |u|$-term is crucial!

Sketch of proof — symmetric case

In **polar coordinates** (r,φ) with $a_r = r^{-1}\boldsymbol{r}\cdot\boldsymbol{a}$ and $a_\varphi = r^{-1}\boldsymbol{r}^\perp\cdot\boldsymbol{a}$

$$\int_\Omega |(-i\nabla+\boldsymbol{a})u|^2 d\boldsymbol{r} = \int_{R_1}^{R_2}\int_0^{2\pi}\bigl(|(-i\partial_r + a_r)u|^2 + r^{-2}|(-i\partial_\varphi + ra_\varphi)u|^2\bigr) r\,d\varphi\,dr$$

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Eigenfunctions and eigenvalues of $K(r)$ are: For $k \in \mathbb{Z}$

$$\lambda_k(r) = -k + \Phi(r), \qquad \psi_k(r, \varphi) = (2\pi)^{-1/2} e^{i[\varphi\lambda_k(r) - r\int_0^\varphi a_\varphi(r,\eta)d\eta]}$$

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By antipodal-**antisymmetry** of $\boldsymbol{a}$ and antipodal-**symmetry** of u

$$\psi_k(r,\varphi+\pi) = (-1)^k\psi(r,\varphi) \quad\Longrightarrow\quad u(r,\varphi) = \sum_{k\in 2\mathbb{Z}} u_k(r)\psi_k(r,\varphi).$$

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Parseval's identity $\Rightarrow$

$$\int_\Omega |(-i\nabla + \boldsymbol{a})u|^2 d\boldsymbol{r} = \int_{R_1}^{R_2}\int_0^{2\pi} \Big(\left|\partial_r |u|\right|^2 + \inf_{k \in \mathbb{Z}} \left|\Phi(r) - 2k\right|^2 \frac{|u|^2}{r^2} \Big) r\, d\varphi\, dr$$

Pairwise relative coordinates

Local contributions to the energy: $Q \subset \mathbb{R}^2$, $1 \leq n \leq N$

$$\sum_{j=1}^{n} \int_{Q^n} |D_j \Psi|^2 \, d\mathrm{x} \qquad D_j = -i\nabla_{\mathbf{x}_j} + \alpha \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2}.$$

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Since $\sum_{j=1}^n |\mathbf{z}_j|^2 = \frac{1}{n} \sum_{1 \leq j < k \leq n} |\mathbf{z}_j - \mathbf{z}_k|^2 + \frac{1}{n} \Big| \sum_{j=1}^n \mathbf{z}_j \Big|^2, \quad \forall \mathbf{z}_j \in \mathbb{C}^2,$

$$\sum_{j=1}^{n} \int_{Q^n} |D_j \Psi|^2 \, d\mathrm{x} = \frac{1}{n} \sum_{1 \leq j < k \leq n} \int_{Q^{n-2}} \Big(\int_{Q^2} |(D_j - D_k)\Psi|^2 d\mathbf{x}_j d\mathbf{x}_k \Big) d\mathrm{x}' + \underbrace{\frac{1}{n} \int_{Q^n} \Big| \sum_{j=1}^{n} D_j \Psi \Big|^2 d\mathrm{x}}_{\geq 0 \text{ discard}}$$

Pairwise relative coordinates — local 4D problem

Relative coordinates $\boldsymbol{r}_{jk} = \dfrac{\mathbf{x}_j - \mathbf{x}_k}{2}$ and $\boldsymbol{X}_{jk} = \dfrac{\mathbf{x}_j + \mathbf{x}_k}{2}$

$$\int_{Q^2} |(D_j - D_k)\Psi|^2 d\mathbf{x}_j d\mathbf{x}_k = \int_{Q^2} \Big| \Big(-i\nabla_{\boldsymbol{r}_{jk}} + \alpha \widetilde{\boldsymbol{A}}(\boldsymbol{X}_{jk}, \boldsymbol{r}_{jk}) \Big) \Psi \Big|^2 d\mathbf{x}_j d\mathbf{x}_k$$

where

$$\widetilde{\boldsymbol{A}}_{jk}(\boldsymbol{X}, \boldsymbol{r}) = \boldsymbol{a}_0(\boldsymbol{r}) + \sum_{\substack{l=1 \\ l \notin \{j,k\}}}^{N} \big[\boldsymbol{a}_l(\boldsymbol{X}, \boldsymbol{r}) - \boldsymbol{a}_l(\boldsymbol{X}, -\boldsymbol{r}) \big].$$

and $\boldsymbol{a}_0(\boldsymbol{r}) = \frac{\boldsymbol{r}^\perp}{|\boldsymbol{r}|^2_{R/2}}$, $\boldsymbol{a}_l(\boldsymbol{X}, \boldsymbol{r}) = \frac{(\boldsymbol{X} + \boldsymbol{r} - \mathbf{x}_l)^\perp}{|\boldsymbol{X} + \boldsymbol{r} - \mathbf{x}_l|^2_R}$

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For fixed $\mathbf{x}_l$, $l \notin \{j, k\}$ and $\boldsymbol{X}_{jk}$

- $\widetilde{\boldsymbol{A}}$ is anitpodal-**antisymmetric** w.r.t. $\boldsymbol{r}$ on $Q^\circ_{\boldsymbol{X}_{jk}} = B_{\mathrm{dist}(\boldsymbol{X}_{jk}, \partial Q)}(0)$.
- Ψ is anitpodal-**symmetric** w.r.t. $\boldsymbol{r}$ on $Q^\circ_{\boldsymbol{X}_{jk}} = B_{\mathrm{dist}(\boldsymbol{X}_{jk}, \partial Q)}(0)$.

Local 2D problem

Fix $\boldsymbol{X}$ and $\mathbf{x}_l$, $l \in \{1, \ldots, N\} \setminus \{j, k\}$.

$$\int_{Q^\circ_{\boldsymbol{X}}} \Big| \Big(-i\nabla_{\boldsymbol{r}} + \alpha \widetilde{\boldsymbol{A}}(\boldsymbol{X}, \boldsymbol{r}) \Big) u \Big|^2 d\boldsymbol{r} \geq \int_{Q^\circ_{\boldsymbol{X}}} \Big(\big| \partial_r |u| \big|^2 + \frac{\rho(r)}{r^2} |u|^2 \Big) d\boldsymbol{r}$$

with

$$\rho(r) := \inf_{k \in \mathbb{Z}} \Big| \Phi(r) - 2k \Big|^2$$

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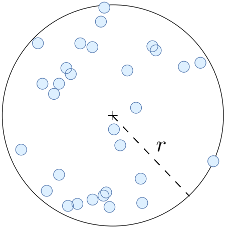
and where

$$R > 0, \;\; \Phi(r) = \alpha \Bigg(\underbrace{\int_{B_r(0)} \frac{\mathbb{1}_{B_{R/2}(0)}(\boldsymbol{r})}{\pi (R/2)^2} d\boldsymbol{r}}_{\text{interaction } \mathbf{x}_j \leftrightarrow \mathbf{x}_k} + 2 \sum_{\substack{l=1 \\ l \notin \{j,k\}}}^{N} \underbrace{\int_{B_r(0)} \frac{\mathbb{1}_{B_R(\mathbf{x}_l - \boldsymbol{X})}(\boldsymbol{r})}{\pi R^2} d\boldsymbol{r}}_{\text{interaction } \substack{\mathbf{x}_j \leftrightarrow \mathbf{x}_l \\ \mathbf{x}_k \leftrightarrow \mathbf{x}_l}} \Bigg)$$

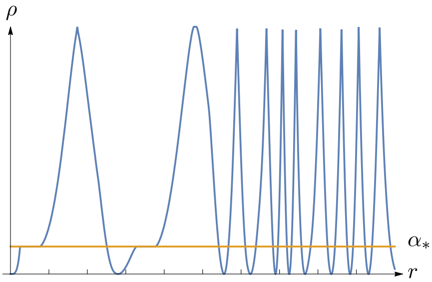
$$R = 0, \;\; \Phi(r) = \alpha \Big(1 + 2\# \big\{ l \in \{1, \ldots, N\} \setminus \{j, k\} : \mathbf{x}_l \in B_r(\boldsymbol{X}) \big\} \Big)$$

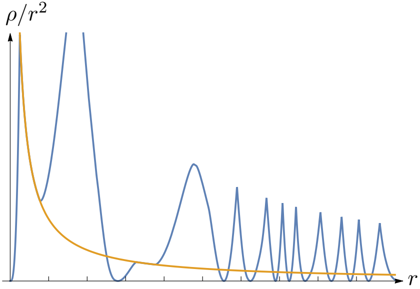
Local 2D problem — effective potential



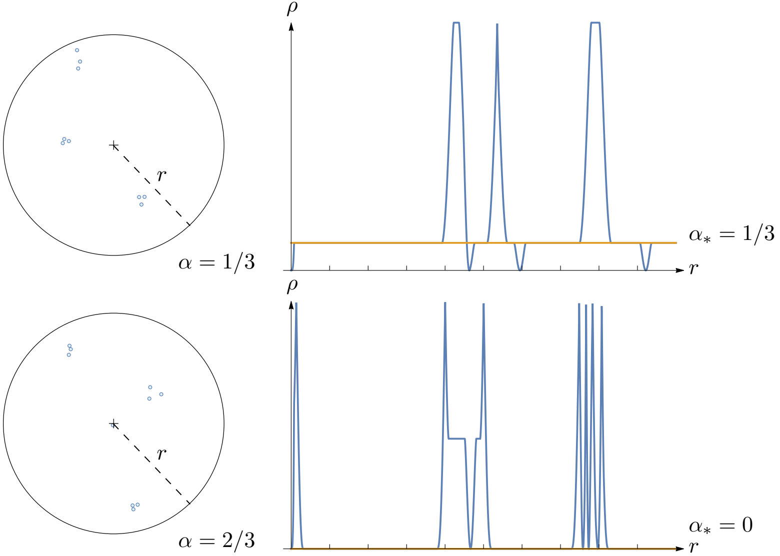
$$\rho(r) = \inf_{k\in\mathbb{Z}} \left|\Phi(r) - 2k\right|^2$$

$$\alpha = \frac{1}{3} \;\Rightarrow\; \alpha_* = \frac{1}{3}$$





Local 2D problem — effective potential with clustering



Local radial (1D) problem

Lemma (Long-range interaction, L.–Lundholm '18)

For $0 \le R \le \delta/6$, $\kappa \in [0,1]$, $u \in W^{1,2}([R,\delta], r dr)$, *and* ρ *as above with arbitrary* $(\mathbf{x}_l)_l \in \mathbb{R}^{2(N-2)}$ *it holds that*

$$\int_R^\delta \Big(|u'|^2 + \frac{\rho(r)}{r^2}|u|^2\Big) r dr \ge \int_R^\delta \Big((1-\kappa)|u'|^2 + c(\kappa)^2 \frac{\alpha_N^2}{r^2} \mathbb{1}_{[3R,\delta-3R]}(r)|u|^2\Big) r dr,$$

with $c(\kappa) = c_0 \frac{\kappa}{1+2\kappa}$. *If* $R = 0$ *one can take* $c(\kappa) \equiv 1$.

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- Removes oscillations of ρ by paying κ times radial derivative.
- Proof uses (repeated) Neumann bracketing on appropriately sized intervals, analysis of precise oscillations of ρ.

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- Removes oscillations of ρ by paying κ times radial derivative.
- Proof uses (repeated) Neumann bracketing on appropriately sized intervals, analysis of precise oscillations of ρ.
- Looks **quadratic** in α_N!

Local radial (1D) problem

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For $0 \le R < \delta/6$, $\kappa \in [0,1]$, $u \in W^{1,2}([R,L], rdr)$, *and* ρ *as above with arbitrary* $(\mathbf{x}_l)_l \in \mathbb{R}^{2(N-2)}$ *it holds that*

$$\int_R^\delta \Big(|u'|^2 + \frac{\rho(r)}{r^2}|u|^2\Big) rdr \ge (1-\kappa)\frac{g\Big(\frac{c(\kappa)\alpha_N}{\sqrt{1-\kappa}}, \gamma\Big)^2}{\delta^2(1-3R/\delta)^2}\int_{3R}^{\delta-3R}|u|^2 rdr,$$

with $\gamma = \frac{3R/\delta}{1-3R/\delta}$ *and* $c(\kappa)$ *as before.*

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- For $\nu \in [0,\infty)$ and $0 \leq \gamma < 1$, $g(\nu,\gamma) := \sqrt{\lambda}$ with λ smallest positive solution associated to **Bessel equation** on $(\gamma, 1)$ with **Neumann b.c.**

$$-u''(r) - u'(r)/r + \nu^2 u(r)/r^2 = \lambda u(r).$$

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- For ν fixed, g is **decreasing in** γ. For γ fixed, g is **increasing in** ν.

  Moreover, $$g(\nu,\gamma) \sim \begin{cases} j'_\nu \geq \sqrt{2\nu}, & \gamma \to 0, \\ \nu, & \gamma \to 1. \end{cases}$$

Returning to 2N dimensions

Theorem (Long-range magnetic interaction, L.–Lundholm '18)

Let $\alpha \in \mathbb{R}$, $R \geq 0$, $1 \leq n \leq N$ *and* $\Omega \subset \mathbb{R}^2$ *open and convex. Then, for* $\kappa \in [0, 1)$,

$$\sum_{j=1}^{n} \int_{\Omega^n} |D_j \Psi|^2 d\mathrm{x} \geq \frac{1-\kappa}{n} \sum_{j<k} \int_{\Omega^n} g\Big(\tfrac{c(\kappa)\alpha_N}{\sqrt{1-\kappa}}, \tfrac{3R/\delta(\boldsymbol{X}_{jk})}{1-3R/\delta(\boldsymbol{X}_{jk})} \Big)^2 \frac{\mathbb{1}_A(\mathbf{x}_j, \mathbf{x}_k)}{\delta(\boldsymbol{X}_{jk})^2} |\Psi|^2 d\mathrm{x},$$

where $\delta(\boldsymbol{X}_{jk}) := \mathrm{dist}(\boldsymbol{X}_{jk}, \partial\Omega)$ *and*

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- For $R = 0$ and $n = N = 2$ yields **correct energy** in limit $\alpha \to 0$. [Lundholm–Seiringer '18]
- However, **not good** if magnetic filling ratio $\bar{\gamma} = R\bar{\varrho}^{1/2}$ is large.

Returning to 2N dimensions

Lemma 1 (Short-range magnetic interaction, L.–Lundholm '18)

Let $\alpha \in \mathbb{R}$, $R \geq 0$, $1 \leq N$. *Then*

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |D_j \Psi|^2 dx \geq 2\pi|\alpha| \sum_{j \neq k} \int_{\mathbb{R}^{2N}} \frac{\mathbb{1}_{B_R(0)}(\mathbf{x}_j - \mathbf{x}_k)}{\pi R^2} |\Psi|^2 dx.$$

- **Good** when $\bar{\gamma}$ is large $\rightsquigarrow$ many flux disks overlapping.
- $\Psi \in L^2(\mathbb{R}^{2N})$ or **local Dirichlet boundary condition** is **necessary**.

Returning to 2N dimensions

Lemma 1 (Short-range magnetic interaction, L.–Lundholm '18)

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- **Good** when $\bar{\gamma}$ is large $\rightsquigarrow$ many flux disks overlapping.
- $\Psi \in L^2(\mathbb{R}^{2N})$ or **local Dirichlet boundary condition** is **necessary**. Start with **global** Ψ and apply inequality before reduction to local energy.

Application to the homogeneous gas

Analysis in the thermodynamic limit:

- Split kinetic energy into three parts: Use **diamagnetic inequality** and the **interaction bounds** $\Rightarrow$ Many-body Schrödinger operator with **scalar** interaction potentials.

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- Partition $Q = [0, L]^2$ into smaller squares of a **equal** size $\sim$ local Lieb–Thirring strategy.
- Depending on $\bar{\gamma}$ use **long-range and/or short-range** bounds.
- For the short-range bound analysis corresponds to a **soft-core dilute bose gas** with effective interaction strength $\sim \alpha\bar{\gamma}^{-2}$.

Thank you for your attention!