Zero modes of Pauli hamiltonian in 2d

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up to identifying $\psi \leftrightarrow \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ or $\psi \leftrightarrow \begin{pmatrix} 0 \\ \psi \end{pmatrix}$ we get

$$\left((P - A)_x \pm i(P - A)_y\right) \psi = 0$$

Using complex notation $z = x + iy$

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y),\ \bar{\partial}_z = \tfrac{1}{2}(\partial_x + i\partial_y)$$

$$A = \frac{1}{2}(A_x - iA_y),\ \bar{A} = \tfrac{1}{2}(A_x + iA_y)$$

we get

$$(\bar{\partial}_z - i\bar{A})\psi = 0$$

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$$\psi_k(z) = z^k\psi_0(z) \in L^2(\mathbb{R}^2) \quad \Leftrightarrow \quad k - \phi < -1$$

$\bar{\partial}_z \log \psi = i\bar{A}$ implies

- Superposition principle: $A = \sum A_a \Rightarrow \psi = \prod \psi_a$.

- ψ solution $\Rightarrow$ $P(z)\psi$ solution $\forall$ holomorphic $P(z)$. (Provided $P\psi \in L^2$.)

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Hence in our problem

$$\psi = P(z,t) \prod \psi_a(z - \zeta_a(t))$$

We know $\{\zeta_a(t)\}$. Need to find $P(z,t)$.

Main example: pointlike fluxons $\frac{1}{2\pi}B = \sum \phi_a \delta^{(2)}(z-\zeta_a)$

$$\psi = P(z,t)\prod(z-\zeta_a(t))^{-\phi_a} = \sum p_k(t) z^k \prod(z-\zeta_a(t))^{-\phi_a}$$

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Phase or cut of $(z - \zeta_a)^{-\phi_a}$ depends on gauge.

(E.g in Coulomb gauge just replace it by absolute value.)

Adiabatic evolution:

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In complex notations $A_0 = -\sum(v_a A_a + \bar{v}_a \bar{A}_a)$

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Explicitly: $\forall Q$

$$\langle Q(z) \prod \psi_a | \frac{\partial}{\partial t} + i(v_a A_a + \bar{v}_a \bar{A}_a) | P(z,t) \prod \psi_a \rangle = 0$$

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and $\bar{v}_a \bar{A}_a \psi = \bar{v}_a(-i\bar{\partial}_z \psi_a) = i\bar{v}_a \bar{\partial}_{\zeta_a}\psi_a$.

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We simplify the equation into:

$$\langle Q\prod\psi_a|dP\prod\psi_a\rangle+\underbrace{d\zeta_a\partial_a}_{\text{Dolbeault operator}}\langle Q\prod\psi_a|P\prod\psi_a\rangle=0$$

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Only holomorphic derivatives ! (no $\bar{\partial} \equiv d\bar{\zeta}_a \bar{\partial}_a$ part).
Connection 1-form ω is a (1,0)-form.

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Connection 1-form ω is a (1,0)-form.

Suggests that the zero-modes have a holomorphic structure preserved by the connection. (As a vector bundle over $\{(\zeta_1,...,\zeta_N)\in\mathbb{C}^N|\forall a\neq b\ \zeta_a\neq\zeta_b\}$)

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Indeed preserving the hermitian metric g requires

$$dg - \underbrace{g\omega}_{\partial g} - \underbrace{\omega^\dagger g}_{\bar{\partial} g} = 0$$

hence $D = d + \omega, \quad \omega = g^{-1}\partial g.$

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The holonomy along a path is given by how $\Upsilon(\zeta)$ changes along it.

holonomy $= \Upsilon^{-1}(\zeta_f)\Upsilon(\zeta_i)$.

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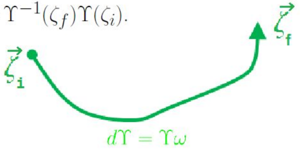


Pointlike fluxons: $\psi_k = z^k \prod(z-\zeta_a)^{-\phi_a}$

$$g_{ij} = \langle\psi_i|\psi_j\rangle = \frac{i}{2}\int \bar{\psi}_i(\bar{z})\psi_j(z)dz \wedge d\bar{z} = -\frac{i}{2}\int d\bar{\Psi}_i \wedge d\Psi_j$$

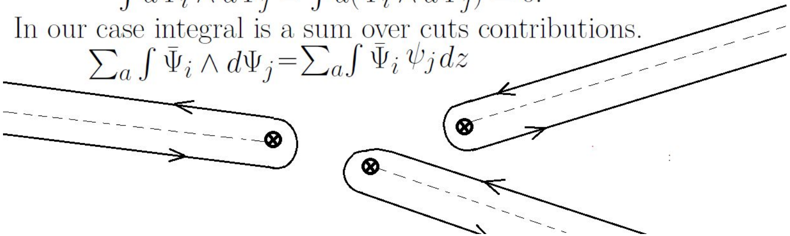
where Ψ_i is the primitive integral of ψ_i.

For single valued ψ this would vanish by Stokes:

$$\int d\bar{\Psi}_i \wedge d\Psi_j = \int d(\bar{\Psi}_i \wedge d\Psi_j) = 0.$$

In our case integral is a sum over cuts contributions.

$$\sum_a \int \bar{\Psi}_i \wedge d\Psi_j = \sum_a \int \bar{\Psi}_i \psi_j dz$$



$$\int \left((\overline{\Psi}_i \psi_j)_- - (\overline{\Psi}_i \psi_j)_+ \right) dz = \int \overline{\Psi_i(\zeta_a)} \left(\psi_j(z_-) - \psi_j(z_+) \right) dz =$$
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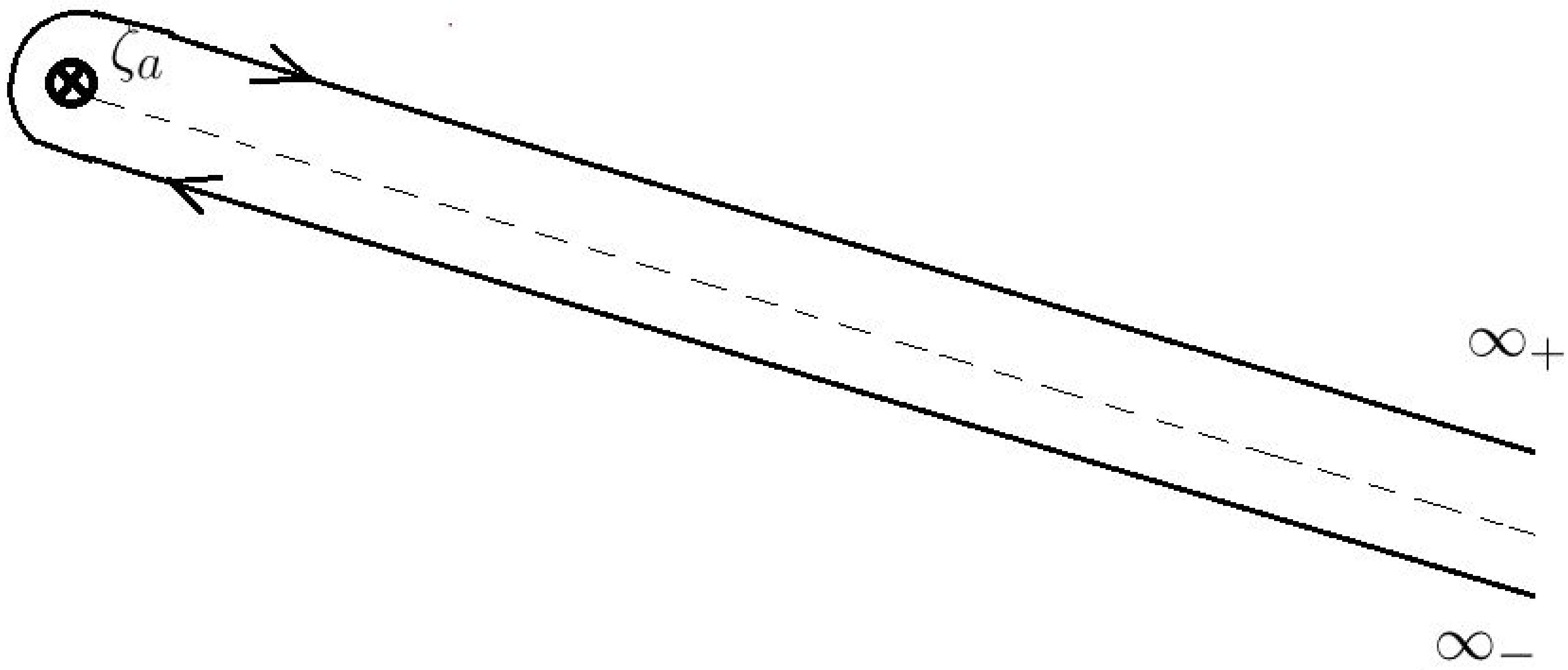

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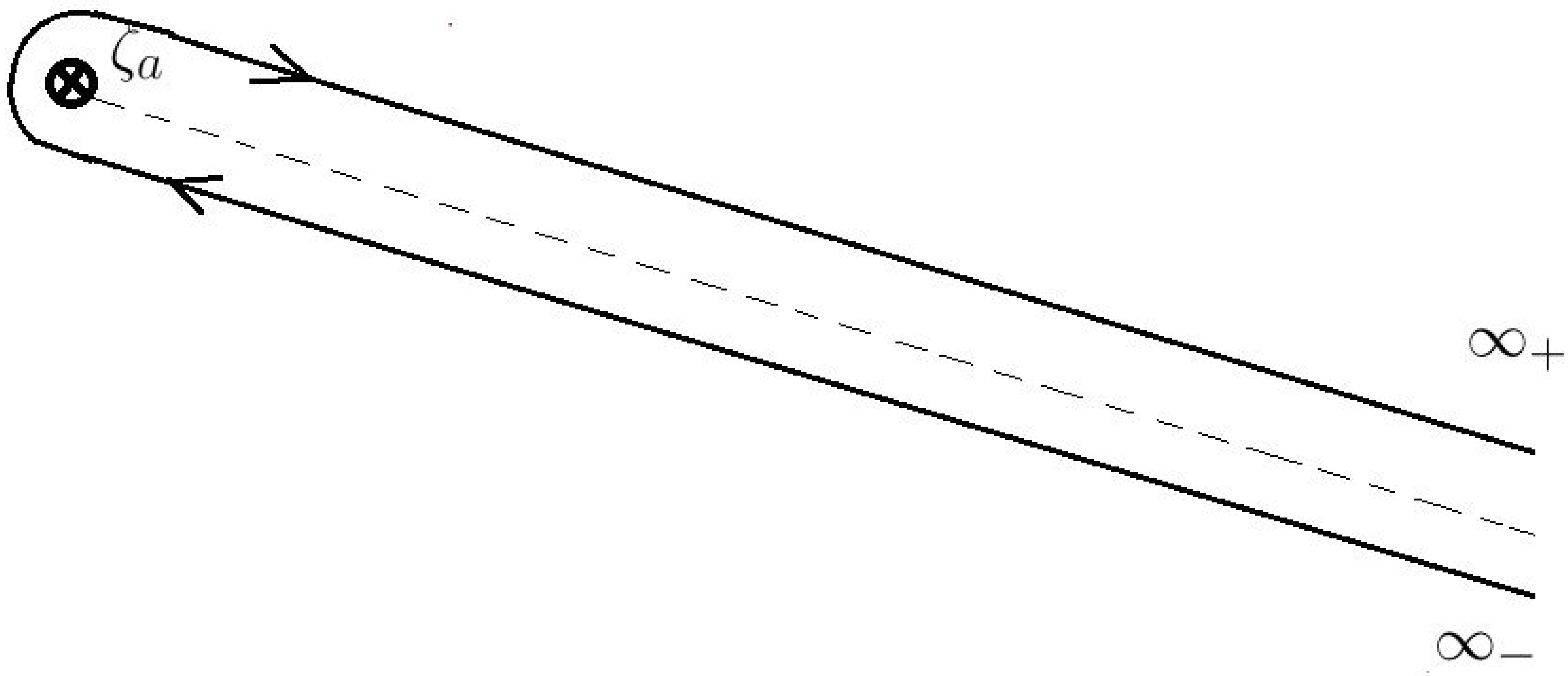
$$\overline{\Psi_i(\zeta_a; \zeta_1, ...\zeta_N)} \left(\Psi_j(\infty_{a-}; \zeta_1, ...\zeta_N) - \Psi_j(\infty_{a+}; \zeta_1, ...\zeta_N) \right)$$



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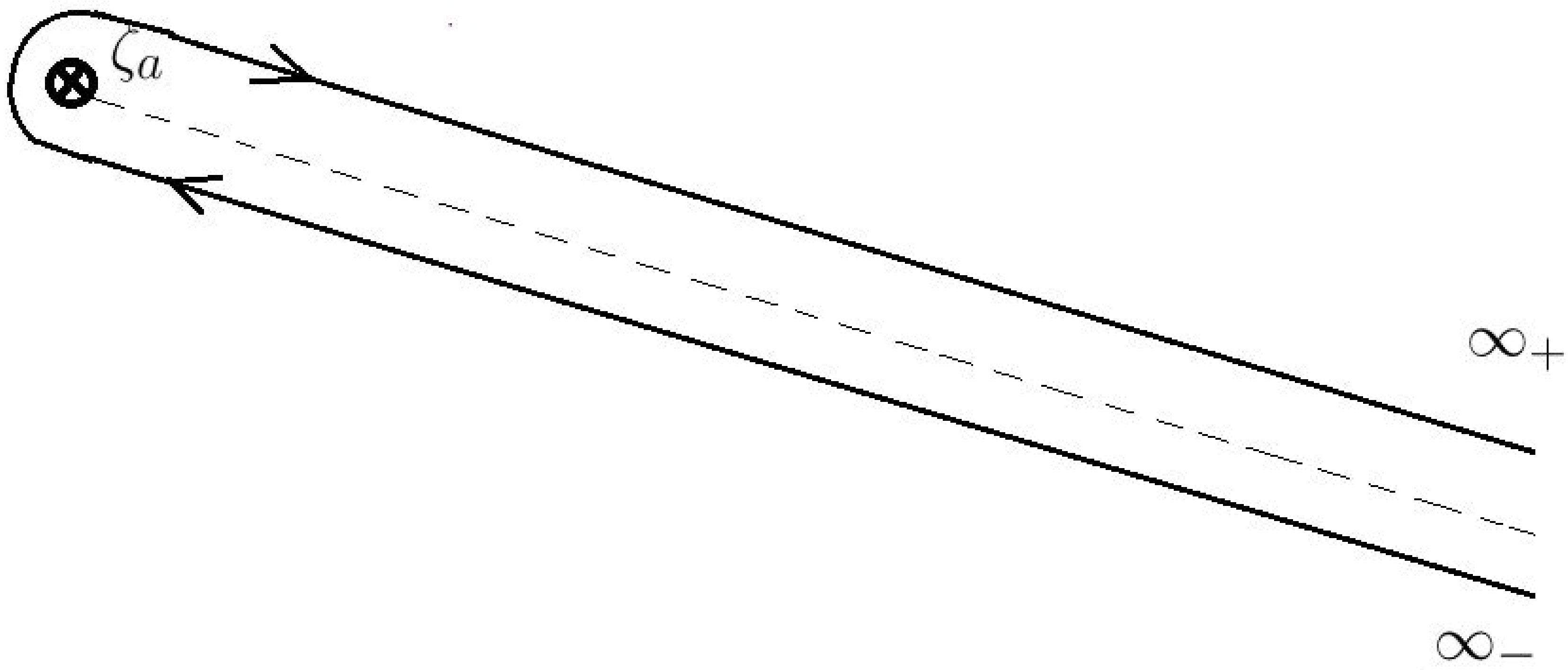
$$\underbrace{\overline{\Psi_i(\zeta_a; \zeta_1, ...\zeta_N)}}_{antiholomorphic} \underbrace{\left(\Psi_j(\infty_{a-}; \zeta_1, ...\zeta_N) - \Psi_j(\infty_{a+}; \zeta_1, ...\zeta_N) \right)}_{holomorphic}$$



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More symmetric form $\sim \overline{\Upsilon}_{ai} \Upsilon_{aj}$.

Summing $\sum_a$ leads to $g = \Upsilon^\dagger \Upsilon$ with $\Upsilon(\zeta)$ holomorphic.

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Holds $\Leftrightarrow$#(Zero-modes)=#(Cuts)-1

Both determined by the fluxes $\{\phi_a\}$.

$\Rightarrow$Condition $\sum(1-\phi_a) < 1$.

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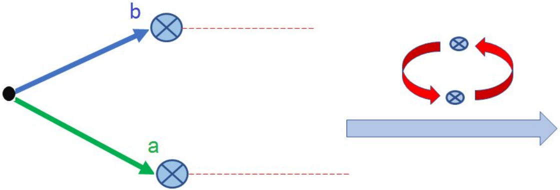
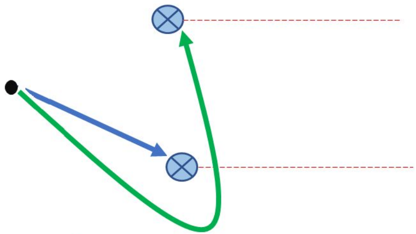
$\Rightarrow$Condition $\sum(1-\phi_a) < 1$.

$\equiv$ Condition for unitarity of the Gassner representation (with parameters $t_a = e^{i\phi_a}$) of the braid group.
(Identical flxons $\Rightarrow$ Gassner$\equiv$ Burau representation.)
One may check that the holonomy(=multivaluedness of $\Upsilon_{aj}(\zeta) \sim \Psi_i(\zeta_a;\zeta_1,...,\zeta_N)$) satisfies this representation.

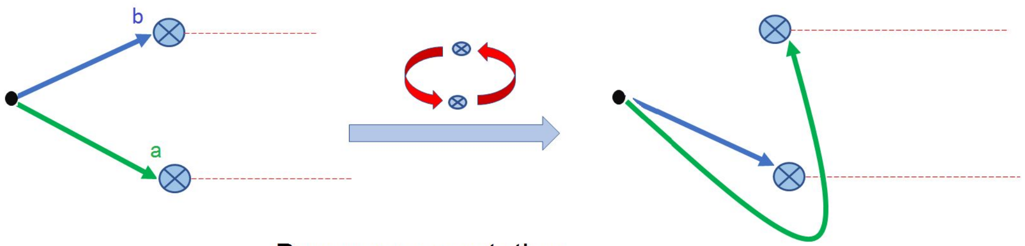




$$\Upsilon_{bi} \sim \Psi_i(\zeta_b) \rightarrow \Psi_i(\zeta_a)$$

$$\Upsilon_{ai} \sim \Psi_i(\zeta_a) \rightarrow \Psi_i(\zeta_a) + e^{i\phi_b}\left(\Psi_i(\zeta_b) - \Psi_i(\zeta_a)\right)$$

$$\begin{pmatrix} \Upsilon'_{bi} \\ \Upsilon'_{ai} \end{pmatrix} = \begin{pmatrix} 1-e^{i\phi_b} & e^{i\phi_b} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Upsilon_{bi} \\ \Upsilon_{ai} \end{pmatrix}$$



Burau representation

$$\Upsilon' = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1-e^{i\phi_b} & e^{i\phi_b} & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \Upsilon$$