Disorder and topology. The cases of Floquet and of chiral systems

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Mathematical physics of anyons and topological states of matter
Nordita
11-16 March 2019
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based on joint work with J. Shapiro, C. Tauber
Outline

Topological insulators

Chiral systems
  An experiment
  A chiral Hamiltonian and its indices

Time periodic systems
  Definitions and results
  Some numerics
Topological insulators

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Topological insulators: definition stated

- **Insulator in the Bulk**: Excitation gap
  For independent electrons: spectral gap at Fermi energy $\mu$

![Diagram showing energy levels and gap at Fermi energy $\mu$]
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- **Insulator** in the Bulk: Excitation gap
  For independent electrons: spectral gap at Fermi energy $\mu$

- **Topology**: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open

$\mu$ $\rightarrow \mu'$ $\rightarrow \mu E$
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  Ordinary insulator: Can be deformed to the limit of well-separated atoms (or void).
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- Classification by suitable indices (e.g. homotopy equivalence)
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- Termination of **bulk** of a topological insulator implies **edge states:** Bulk index vs. edge index
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The role of disorder

The spectrum of a single-particle Hamiltonian

- extended states (continuous spectrum)
- localized states (pure point spectrum: Anderson localization)

For a periodic (crystalline) medium:
- Method of choice: Bloch theory and vector bundles (Thouless et al.)
- Gap is spectral

For a disordered medium:
- Method of choice: Non-commutative geometry (Bellissard; Avron et al.)
- Fermi energy may lie in a spectral gap or (better, and more generally) in a mobility gap.

μ: Fermi energy (Pauli principle)
Spectral vs. Mobility gap, technically speaking

- Hamiltonian $H$ on $\ell^2(\mathbb{Z}^d)$
- Fermi energy $\mu$ in gap
- $P_\mu = I_{(-\infty, \mu)}(H)$: Fermi projection with matrix elements $P_\mu(x, x')$, $(x, x' \in \mathbb{Z}^d)$
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- Spectral gap

Strong off-diagonal decay:

$$P_\mu(x, x') \lesssim e^{-\nu|x-x'|}$$
Spectral vs. Mobility gap, technically speaking

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  ![Spectral gap diagram]

  Strong off-diagonal decay:

  \[ P_\mu(x, x') \lesssim e^{-\nu|x-x'|} \]

- **Mobility Gap:** Localization holds at Fermi energy

  ![Mobility Gap diagram]

  \[
  \sup_{x' \in \mathbb{Z}^d} \sum_{x \in \mathbb{Z}^d} e^{-\varepsilon|x'|} e^{\nu|x-x'|} |P_\mu(x, x')| < \infty
  \]

  (some \( \nu > 0 \), all \( \varepsilon > 0 \))
Spectral vs. Mobility gap, technically speaking

- **Mobility Gap**: Localization holds at Fermi energy

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\sup_{x'\in\mathbb{Z}^d} e^{-\varepsilon|x'|} \sum_{x\in\mathbb{Z}^d} e^{\nu|x-x'|} |P_\mu(x, x')| < \infty
\]

(some \(\nu > 0\), all \(\varepsilon > 0\))

- Proven in (virtually) all cases where localization is known.
- Trivially false for extended states at \(E = \mu\).
Periodic vs. non-periodic case

Difference illustrated for the conductance $\sigma_H$ of (integer) quantum Hall effect (Kubo formula)
Periodic vs. non-periodic case

Difference illustrated for the conductance $\sigma_H$ of (integer) quantum Hall effect (Kubo formula)

- **Periodic case.** (Thouless et al., Avron)

\[
\sigma_H = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}} d^2k \text{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])
\]

where $\mathbb{T}$: Brillouin zone (torus); $P(k)$ Fermi projection on the space of states of quasi-momentum $k = (k_1, k_2)$; $\partial_i = \partial/\partial k_i$
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where $\mathbb{T}$: Brillouin zone (torus); $P(k)$ Fermi projection on the space of states of quasi-momentum $k = (k_1, k_2)$; $\partial_i = \partial/\partial k_i$

**Remark.**

\[
2\pi \sigma_H = \text{ch}(P)
\]

is the Chern number (index) of the vector bundle over $\mathbb{T}$ and fiber range $P(k)$
Periodic vs. non-periodic case

- **Periodic case.** (Thouless et al., Avron)

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\sigma_H = -\frac{i}{(2\pi)^2} \int_T d^2k \text{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])
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- **Non-periodic case.** (Bellissard et al., Avron et al.)

\[
\sigma_H = i \text{tr} P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]
\]

where \(\Lambda_i = \Lambda(x_i), \ (i = 1, 2)\) are switch functions

\[
\Lambda(x)
\]

\[
\Lambda(x) \uparrow 1
\]

\[
\chi
\]
Periodic vs. non-periodic case

- **Periodic case.** (Thouless et al., Avron)

  \[ \sigma_H = -\frac{i}{(2\pi)^2} \int_T d^2k \text{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)]) \]

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- Alternative treatment of disorder (Thouless): Large, but finite system (square); \((k_1, k_2) \sim (\varphi_1, \varphi_2)\) phase slips in boundary conditions
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An experiment: Amo et al.

Figure: Zigzag chain of coupled micropillars and lasing modes (polaritons)
An experiment: Amo et al.

**Figure:** Lasing modes: bulk and edge
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The Su-Schrieffer-Heeger model (1 dimensional)

Alternating chain with nearest neighbor hopping

\[ \psi_{n-1}^+ \xrightarrow{A_n} \psi_n^- \xleftarrow{B_n} \psi_n^+ \xrightarrow{A_n} \psi_{n+1}^- \xleftarrow{B_n} \psi_{n+1}^+ \]
The Su-Schrieffer-Heeger model (1 dimensional)

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\[ \psi_{n-1}^+ \xleftrightarrow{A_n} \psi_n^- \xleftrightarrow{B_n} \psi_n^+ \xleftrightarrow{A_n} \psi_{n+1}^- \]

Hilbert space: sites arranged in dimers

\[ \mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix} \quad n \in \mathbb{Z} \]

Hamiltonian

\[ H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \]

with \( S, \ S^* \) acting on \( \ell^2(\mathbb{Z}, \mathbb{C}^N) \) as

\[ (S\psi^+)_n = A_n\psi_{n-1}^+ + B_n\psi_n^+ , \quad (S^*\psi^-)_n = A^*_{n+1}\psi_{n+1}^- + B^*\psi_n^- \]

\( (A_n, B_n \in \text{GL}(N) \text{ almost surely}) \)
Chiral symmetry

\[ \Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \{H, \Pi\} \equiv H\Pi + \Pi H = 0 \]

hence

\[ H\psi = \lambda \psi \implies H(\Pi \psi) = -\lambda (\Pi \psi) \]
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Energy \( \lambda = 0 \) is special:

- Eigenspace of \( \lambda = 0 \) invariant under \( \Pi \)
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Energy \( \lambda = 0 \) is special:

- Eigenspace of \( \lambda = 0 \) invariant under \( \Pi \)

Eigenvalue equation \( H\psi = \lambda\psi \) is

\[ S\psi^+ = \lambda\psi^-, \quad S^*\psi^- = \lambda\psi^+, \quad \text{i.e.} \]

\[ A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n, \quad A^*_n+1\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n \]

is one 2nd order difference equation, but two 1st order for \( \lambda = 0 \)
Bulk index

Let

\[ \Sigma = \text{sgn } H \]

**Definition.** The Bulk index is

\[ \mathcal{N} = \frac{1}{2} \text{tr}(\Pi \Sigma [\Lambda, \Sigma]) \]

with \( \Lambda = \Lambda(n) \) a switch function (cf. Prodan et al.)
Edge Hamiltonian and index

\[ \psi_{a-1}^+ \quad \psi_a^- \quad \psi_a^+ \quad \psi_{a+1}^- = 0 \]

Edge Hamiltonian \( H_a \) defined by restriction to \( n \leq a \) (Dirichlet boundary condition \( \psi_{a+1}^- = 0 \)). Chiral symmetry preserved.

\[ \text{Definition. The Edge index is } N_a = N_a^+ - N_a^- \text{ and can be shown to be independent of } a. \text{ Call it } N^\#_a. \]
Edge Hamiltonian and index

Edge Hamiltonian $H_a$ defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi_{a+1}^- = 0$). Chiral symmetry preserved.

Eigenspace of $\lambda = 0$ still invariant under $\Pi$. 
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Eigenspace of $\lambda = 0$ still invariant under $\Pi$.

\[ N_a^\pm := \dim \{ \psi \mid H_a\psi = 0, \Pi\psi = \pm\psi \} \]
Edge Hamiltonian and index

\[
\begin{align*}
\psi_{a-1}^+ & \quad \psi_a^- \\
\psi_a^+ & \quad \psi_{a+1}^- = 0
\end{align*}
\]

Edge Hamiltonian \( H_a \) defined by restriction to \( n \leq a \) (Dirichlet boundary condition \( \psi_{a+1}^- = 0 \)). Chiral symmetry preserved.

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**Definition.** The Edge index is

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\mathcal{N}_a = \mathcal{N}_a^+ - \mathcal{N}_a^-
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and can be shown to be independent of \( a \). Call it \( \mathcal{N}^\# \).
Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\#$$
Bulk-edge duality: Remarks

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Remarks.

- Spectral gap case ($0 \notin \sigma_{ess}(H) \supset \sigma_{ess}(H_a)$)

$$H_a = \begin{pmatrix} 0 & S_a^* \\ S_a & 0 \end{pmatrix} \quad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{N}_a^\# := \dim \ker S_a - \dim \ker S_a^* = \text{ind} S_a \quad \text{(Fredholm index)}$$

Bulk-edge duality by Schulz-Baldes. In mobility gap case, $S_a$ is not Fredholm.
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- **Supersymmetry:** Is realized as $(H_a, \Pi) = (\text{supercharge}, \text{grading})$. Then $\mathcal{N}_a^\#$ is Witten index.
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- Periodic case

$$S = \int_{S^1}^{\oplus} S(k)$$

Toeplitz index theorem:

$$\mathcal{N}^\# = -\text{Wind}(k \mapsto \det S(k))$$
Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

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Bulk-edge duality: Lyaponov exponents

Theorem (G., Shapiro). Assume \( \lambda = 0 \) lies in a mobility gap. Then

\[ \mathcal{N} = \mathcal{N}^\# \]

Remark. Consider the dynamical system \( A_n \psi_{n-1}^+ + B_n \psi_n^+ = 0 \) with Lyaponov exponents

\[ \gamma_1 \geq \ldots \geq \gamma_N \]

The assumption is satisfied if \( \gamma_i \neq 0 \); then \( \mathcal{N}^\# = \# \{ i \mid \gamma_i > 0 \} \).
Bulk-edge duality: Lyapunov exponents

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The assumption is satisfied if $\gamma_i \neq 0$; then $\mathcal{N}^\# = \#\{i \mid \gamma_i > 0\}$. Phase boundaries correspond to $\gamma_i = 0$ (cf. Prodan et al.)
**Bulk-edge duality: Lyapunov exponents**

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Lyapunov spectrum of the full chain has \( 2N \) exponents, spectrum is even (Example: \( N = 4 \))

- at energy \( \lambda \neq 0 \) (simple spectrum)
  
  ![Diagram](https://via.placeholder.com/150)

- Spectrum is simple because measure on transfer matrices is irreducible
- so \( \gamma = 0 \) is not in the spectrum; localization follows
**Bulk-edge duality: Lyaponov exponents**

**Theorem (G., Shapiro).** Assume $\lambda = 0$ lies in a mobility gap. Then

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**Remark.** Consider the dynamical system $A_n\psi_{n-1}^+ + B_n\psi_n^+ = 0$ with Lyaponov exponents

$$\gamma_1 \geq \ldots \geq \gamma_N$$

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Lyapunov spectrum of the full chain has $2N$ exponents, spectrum is even (Example: $N = 4$)

- at energy $\lambda \neq 0$ (simple spectrum)

  ![Diagram](image)

- At $\lambda = 0$ chains decouple: $\mathbb{C}^N \oplus 0$ and $0 \oplus \mathbb{C}^N$ are invariant subspaces
**Bulk-edge duality: Lyapunov exponents**

**Theorem** (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\#$$

**Remark.** Consider the dynamical system

$$A_n\psi_{n-1}^+ + B_n\psi_n^+ = 0$$

with Lyapunov exponents

$$\gamma_1 \geq \ldots \geq \gamma_N$$

The assumption is satisfied if $\gamma_i \neq 0$; then $\mathcal{N}^\# = \#\{i \mid \gamma_i > 0\}$. Phase boundaries correspond to $\gamma_i = 0$ (cf. Prodan et al.)

Lyapunov spectrum of the full chain has $2N$ exponents, spectrum is even (Example: $N = 4$)

- at energy $\lambda \neq 0$ (simple spectrum)
  
  ![Diagram 1](image1)

- of the upper ($+$) and lower ($-$) chains, at energy $\lambda = 0$
  
  ![Diagram 2](image2)

- at energy $\lambda = 0$ (phase boundary)
  
  ![Diagram 3](image3)
Some numerics

Left/right column: two parameterized chiral models \((N = 1)\)
upper/lower row: index and Lyapunov exponent (from Prodan et al.)
Proof

Recall $\mathcal{N}_a = \text{tr}(\Pi P_{0,a})$, where

$$1 = P_{-,a} + P_{0,a} + P_{-,a}$$

is decomposition into states of energies $< 0, = 0, > 0$
Proof

Recall $\mathcal{N}_a = \text{tr}(\prod P_{0,a})$, where

$$1 = P_{-,a} + P_{0,a} + P_{-,a}$$

is decomposition into states of energies $< 0, = 0, > 0$

Lemma. The common value of $\mathcal{N}_a$ is

$$\mathcal{N}^\# = \lim_{a \to +\infty} \text{tr}(\prod \Lambda P_{0,a})$$
Proof

Lemma. The common value of \( N_a \) is

\[
N^\# = \lim_{a \to +\infty} \text{tr}(\Pi \Lambda P_{0,a})
\]

Proof of Theorem. On the Hilbert space \( \mathcal{H}_a \) corresponding to \( n \leq a \)

\[
\text{tr}(\Pi \Lambda) = N \left( \sum_{n \leq a} \Lambda(n) \right) \text{tr}_{C^2} \Pi = 0
\]

though \( \|\Pi \Lambda\|_1 = \|\Lambda\|_1 \to \infty, (a \to +\infty) \)
Lemma. The common value of $N_a$ is

$$N^{\#} = \lim_{a \to +\infty} \text{tr}(\Pi \Lambda P_{0,a})$$

Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\text{tr}(\Pi \Lambda) = 0$$

$$\text{tr}(\Pi \Lambda) = \text{tr}(\Pi \Lambda P_{0,a}) + \text{tr}(\Pi \Lambda P_{+,a}) + \text{tr}(\Pi \Lambda P_{-,a})$$
Proof

Lemma. The common value of $N_a$ is

$$N^\# = \lim_{a \to +\infty} \text{tr}(\Pi \Lambda P_{0,a})$$

Proof of Theorem. On the Hilbert space $H_a$ corresponding to $n \leq a$

$$\text{tr}(\Pi \Lambda) = 0$$

$$\text{tr}(\Pi \Lambda) = \text{tr}(\Pi \Lambda P_{0,a}) + \text{tr}(\Pi \Lambda P_{+,a}) + \text{tr}(\Pi \Lambda P_{-,a})$$

$$\text{tr}(\Pi \Lambda P_{+,a}) = \text{tr}(P_{+,a} \Pi \Lambda P_{+,a}) = \text{tr}(\Pi P_{-,a} \Lambda P_{+,a})$$

$$= \text{tr}(\Pi P_{-,a} [\Lambda, P_{+,a}])$$
Proof

Lemma. The common value of $\mathcal{N}_a$ is

$$\mathcal{N}^\# = \lim_{a \to +\infty} \text{tr}(\Pi\Lambda P_0, a)$$

Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\text{tr}(\Pi\Lambda) = 0$$

$$\text{tr}(\Pi\Lambda) = \text{tr}(\Pi\Lambda P_0, a) + \text{tr}(\Pi\Lambda P_+, a) + \text{tr}(\Pi\Lambda P_-, a)$$

$$\text{tr}(\Pi\Lambda P_+, a) = \text{tr}(P_+, a \Pi\Lambda P_+, a) = \text{tr}(\Pi P_-, a \Lambda P_+, a)$$

$$= \text{tr}(\Pi P_-, a [\Lambda, P_+, a]) \rightarrow \text{tr}(\Pi P_- [\Lambda, P_+]) \quad (a \to +\infty)$$
Proof

Lemma. The common value of $\mathcal{N}_a$ is

$$\mathcal{N}^\# = \lim_{a \to +\infty} \text{tr}(\Pi \Lambda P_0, a)$$

Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\text{tr}(\Pi \Lambda) = 0$$

So,

$$\text{tr}(\Pi \Lambda) = \text{tr}(\Pi \Lambda P_0, a) + \text{tr}(\Pi \Lambda P_+, a) + \text{tr}(\Pi \Lambda P_-, a)$$

In fact by $\Sigma = P_+ - P_-$ the last expression is

$$-(1/2) \text{tr}(\Pi \Sigma [\Lambda, \Sigma]) = -\mathcal{N}$$

q.e.d.
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Floquet topological insulators

\[ H = H(t) \text{ (bulk) Hamiltonian in the plane with period } T \]

\[ H(t + T) = H(t) \]

(disorder allowed, no adiabatic setting)
Floquet topological insulators

\[ H = H(t) \] (bulk) Hamiltonian in the plane with period \( T \)

\[ H(t + T) = H(t) \]

(disorder allowed, no adiabatic setting)

\( U(t) \) propagator for the interval \((0, t)\)

\( \hat{U} = U(T) \) fundamental propagator
Floquet topological insulators

\[ H = H(t) \text{ (bulk) Hamiltonian in the plane with period } T \]

\[ H(t + T) = H(t) \]

(disorder allowed, no adiabatic setting)

\( U(t) \) propagator for the interval \((0, t)\)
\( \hat{U} = U(T) \) fundamental propagator

Assumption: Spectrum of \( \hat{U} \) has gaps:

\[ \text{spec } \hat{U} \subset S^1 \]
Bulk index

Special case first: $U(t)$ periodic, i.e.

$$\hat{U} = 1$$
Bulk index

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$$\hat{U} = 1$$

Bulk index

$$\mathcal{N}_B = \frac{1}{2} \int_0^T dt \, \text{tr}(U^* \partial_t U [U^* [\Lambda_1, U], U^* [\Lambda_2, U]])$$

with $U = U(t)$ and switches $\Lambda_i = \Lambda(x_i), (i = 1, 2)$
Bulk index

Special case first: $U(t)$ periodic, i.e.

$$\hat{U} = 1$$

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with $U = U(t)$ and switches $\Lambda_i = \Lambda(x_i)$, ($i = 1, 2$)

**Remark.** Extends the formula for the periodic case (Rudner et al.)

$$\mathcal{N}_B = \frac{1}{8\pi^2} \int_0^T dt \int_{\mathbb{T}} d^2k \, \text{tr}(U^* \partial_t U[U^* \partial_1 U, U^* \partial_2 U])$$

with $U = U(t, k)$ acting on the space of states of quasi-momentum $k = (k_1, k_2)$.
Bulk index

Special case first: $U(t)$ periodic, i.e.

$$\hat{U} = 1$$

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Edge index

$H_E(t)$ restriction of $H(t)$ to right half-space $x_1 > 0$

$\hat{U}_E$ corresponding fundamental propagator
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Remarks.

- The trace is well-defined
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**Remarks.**

- The trace is well-defined

  $\mathcal{N}_E$ is charge that crossed the line $x_2 = 0$ during a period.

  $\mathcal{N}_E$ is independent of $\Lambda_2$ and an integer.
General case: Pair of Hamiltonians

\[ \hat{U} \neq 1 \]
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Pair of periodic Hamiltonians \( H_i(t), (i = 1, 2) \) with

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Define Hamiltonian \( H(t) \) with period \( 2T \) by

\[
H(t) = \begin{cases} 
H_1(t) & (0 < t < T) \\
-H_2(-t) & (-T < t < 0)
\end{cases}
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Then

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U(t) = \begin{cases} 
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**Theorem** (G., Tauber) $\mathcal{N} = \mathcal{N}_E$
Duality in time and space

Let the interface Hamiltonian $H_1(t)$ be a bulk Hamiltonian with

$$H_1(t) = \begin{cases} H_1(t) \\ H_2(t) \end{cases}$$

on states supported on large $\pm x_1$

(still assuming $\hat{U}_1 = \hat{U}_2 =: \hat{U}_\bullet$)
Duality in time and space

Let the interface Hamiltonian \( H_I(t) \) be a bulk Hamiltonian with

\[
H_I(t) = \begin{cases} 
H_1(t) & \text{on states supported on large } \pm x_1 \\
H_2(t) & 
\end{cases}
\]

(still assuming \( \hat{U}_1 = \hat{U}_2 =: \hat{U}_\bullet \))

Interface index

\[
\mathcal{N}_I = \text{tr}(\hat{U}_\bullet^* \hat{U}_I[\Lambda_2, \hat{U}_\bullet^* \hat{U}_I])
\]
Duality in time and space

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Interface index

\[
N_I = \text{tr}(\hat{U}_\bullet \hat{U}_I [\Lambda_2, \hat{U}_\bullet \hat{U}_I])
\]

Theorem (G., Tauber) The indices for the two diagrams agree:

\[
(\mathcal{N} =) N_E = N_I
\]
Back to single Hamiltonian

\[ \hat{U} \neq 1 \]

\[ \text{spec } \hat{U} \subset S^1 \]
Back to single Hamiltonian

\[ \hat{U} \neq 1 \]

Let \( \alpha \in \mathbb{R} \) and \( \omega = e^{i\alpha} \). For \( z \notin \omega \mathbb{R}_+ \) (ray) define the branch

\[ \log_\alpha z = \log |z| + i \arg_\alpha z \]

by \( \alpha - 2\pi < \arg_\alpha z < \alpha \).
Back to single Hamiltonian

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Let \( \alpha \in \mathbb{R} \) and \( \omega = e^{i\alpha} \). For \( z \notin \omega \mathbb{R}_+ \) (ray) define the branch

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Comparison Hamiltonian \( H_\alpha \): For \( \omega = e^{i\alpha} \notin \text{spec} \hat{U} \) set

\[ -i H_\alpha T := \log_{\alpha} \hat{U} \]

So,

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- \( U_{\alpha+2\pi}(t) = U_{\alpha}(t)e^{2\pi it/T} \)
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So,

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- \( U_{\alpha+2\pi}(t) = U_\alpha(t)e^{2\pi it/T} \)
- \( N_{\text{B},\alpha+2\pi} = N_{\text{B},\alpha} =: N_\omega \) by

\[ N_{\text{B}} = \frac{1}{2} \int_0^T dt \tr (U^* \partial_t U [U^* [\Lambda_1, U], U^* [\Lambda_2, U]]) \]
Back to single Hamiltonian

\[ \hat{U} \neq 1 \]

Comparison Hamiltonian \( H_\alpha \): For \( \omega = e^{i\alpha} \notin \text{spec} \hat{U} \) set

\[-iH_\alpha T := \log_\alpha \hat{U}\]

**Theorem** (Rudner et al.; G., Tauber) For \( \omega, \omega' \) in gaps

\[ N_{\omega'} - N_\omega = i \text{ tr } P[[P, \Lambda_1], [P, \Lambda_2]] \]

where \( P = P_{\omega, \omega'} \) is the spectral projection associated with \( \text{spec} \hat{U} \) between \( \omega, \omega' \) (counter-clockwise)
Topological insulators

Chiral systems
   An experiment
   A chiral Hamiltonian and its indices

Time periodic systems
   Definitions and results
   Some numerics
Bulk and Edge spectrum

Bulk spectrum
$J = 5.30, \delta = 6.28, \gamma = 7.85, N=M=40$

Edge spectrum
$J = 5.30, \delta = 6.28, \gamma = 7.85, N=M=40$
Computing the edge index

Edge index $N_{E,\alpha}$ based on the pair $(H, H_{\alpha})$ (with $\alpha = \pi$)

$$N_{E,\alpha} = \text{tr} A \quad A = \hat{U}_{E}^{\ast} \Lambda_{2} \hat{U}_{E} - \hat{U}_{\alpha,E}^{\ast} \Lambda_{2} \hat{U}_{\alpha,E}$$

The diagonal integral kernel $A(x, x)$ as $\log |A(x, x)|$

Boundary conditions:
- Vertical edges: Dirichlet
- Horizontal edges: Periodic
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The diagonal integral kernel $A(x, x)$ as $\log |A(x, x)|$

Boundary conditions:
- Vertical edges: Dirichlet
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The transition

**Invariant wrt J for d = 3.14, \(dr=3.14\)**

- **L=24**
- **L=32**
- **L=40**
Summary

- Chiral symmetry
- Floquet topological insulator
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Thank you for your attention!