Group-Theoretical Approach to Anyons and Topological Quantum Configurations

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An overview of topics in the talk

1. Diffeomorphism groups, local current algebra, and quantum theory: general approach
2. Inequivalent representations describing different quantum systems
3. Unitary representations of $Diff_0(M)$
4. Induced representations, particle statistics, and topology
5. Anyons and nonabelian anyons in two-space
6. Intertwining creation and annihilation fields, q-commutation relations
7. An approach to relativistic theory
8. Infinite systems, additional topological possibilities, and general configuration spaces

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1. Diffeomorphism groups, local current algebra, and quantum theory: general approach

Some well-known ideas:

Symmetry groups describe transformations under which kinematics or dynamics may be invariant, or partially invariant.

Locality makes use of points or regions in a manifold (of space or of spacetime) to express that "action at a distance" does not occur in quantum physics, and that "what we measure" is always in the "here and now."

These ideas are joined to describe *local symmetry* by means of local current groups or gauge groups, diffeomorphism groups, their semidirect products, and (sometimes) their extensions.

Local current groups and diffeomorphism groups

A local current group S associates a Lie group L with points in a manifold M (of space or spacetime). Group elements are compactly-supported C^∞ functions $f : M \to L$, with the group operation in S defined pointwise by the operation in L. Restricting the support of the functions f to a compact region $B \subset M$ defines a subgroup for each (local) region.

The diffeomorphsim group $G = Diff_0(M)$ is the group of compactly-supported, invertible C^∞ maps $\phi : M \to M$ (with C^∞ inverse), under composition. A subgroup of G is defined for a compact region $B \subset M$ by restricting the support of the diffeomorpisms in the subgroup to B.

Both S and G are endowed with the topology of uniform convergence in all derivatives.

Let M be the manifold of physical space, and consider the natural *semidirect product* of S with G. We thus have local (spatial) commutativity.

Quantum mechanics from the group representations

We shall focus on the easiest case $L = \mathbf{R}$ (under addition), so the map group is just the function-space $\mathcal{D}$ of C^∞, compactly-supported functions on M. So we consider $\mathcal{D} \rtimes G$.

For $f_1, f_2 \in \mathcal{D}$, and $\phi_1, \phi_2 \in G$, we have the semidirect product group law,

$$(f_1, \phi_1)\,(f_2, \phi_2) = (f_1 + \phi_1 f_2, \phi_1 \phi_2), \tag{1.1}$$

where $\phi_1 f_2 = f_2 \circ \phi_1$, and $\phi_1 \phi_2 = \phi_2 \circ \phi_1$.

A continuous, irreducible unitary representation $U(f)V(\phi)$ of $\mathcal{D} \rtimes G$ in Hilbert space $\mathcal{H}$ describes a quantum system in the physical space M. The diffeomorphism group describes its *local kinematical symmetries*.

The distinct (inequivalent) representations of $\mathcal{D} \rtimes G$ describe different quantum systems. This leads to a unification describing a wide variety of possible quantum configurations and statistics.

The self-adjoint generators ...

The corresponding infinite-dimensional Lie algebra is the semidirect sum of the Lie algebras of generators of $\mathcal{D}$ and of $Diff_0(M)$.

Unitary representations of the group give (under appropriate technical conditions) self-adjoint representations of this Lie algebra (on a common dense invariant domain of essential self-adjointness).

Such a representation is by operators $\hat{\rho}(f)$ and $\hat{J}(\mathbf{g})$; where f is a compactly-supported, real-valued C^∞ function on M, and $\mathbf{g}$ is a compactly-supported (tangent) vector field.

They generate the continuous 1-parameter unitary subgroups

$$U(sf) = \exp i(s/m)\hat{\rho}(f), \quad V(\phi_s^{\mathbf{g}}) = \exp i(s/\hbar)\hat{J}(\mathbf{g}) \quad (s \in \mathbf{R}), \tag{1.2}$$

where m is a unit mass and $\phi_s^{\mathbf{g}}$ is the flow on M generated by $\mathbf{g}$.

The local current algebra ...

The *current algebra for quantum mechanics* is,

$$[\hat{\rho}(f_1), \hat{\rho}(f_2)] = 0\,, \quad [\hat{\rho}(f), \hat{J}(\mathbf{g})] = i\hbar\hat{\rho}(\mathbf{g}\cdot\nabla f), \tag{1.3}$$

$$[\hat{J}(\mathbf{g}_1), \hat{J}(\mathbf{g}_2)] \;=\; -i\hbar\hat{J}([\mathbf{g}_1, \mathbf{g}_2])\,, \tag{1.4}$$

where $\mathbf{g}\cdot\nabla f$ is the Lie derivative of f in the direction of $\mathbf{g}$, and

$$[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_1\cdot\nabla\mathbf{g}_2 - \mathbf{g}_2\cdot\nabla\mathbf{g}_1 \tag{1.5}$$

is the Lie bracket of the vector fields.

The local current algebra is thus very natural and geometrical. In a representation, $\hat{\rho}(f)$ is interpreted physically as the mass density (spatially averaged with the scalar function f), and $\hat{J}(\mathbf{g})$ as the momentum density (spatially averaged with the vector field $\mathbf{g}$).

They are operator-valued distributions over test-function spaces of scalar functions and vector fields (respectively).

... and the 1-particle representation

For example, with $M = \mathbf{R}^d$, the quantum mechanics of a single point particle is described in the usual Hilbert space $\mathcal{L}^2_{dx}(\mathbf{R}^d)$ by the unitary representation,

$$U(f)\psi(x) = \exp[if(x)]\psi(x), \quad V(\phi)\psi(x) = V(\phi(x))\sqrt{\mathcal{J}_\phi(x)} \tag{1.6}$$

where $\mathcal{J}_\phi$ is the Jacobian of the diffeomorphism ϕ. Equivalently, the current algebra is represented by the self-adjoint operators,

$$[\hat{\rho}(f)\psi](x) = mf(x)\psi(x), \quad [\hat{J}(\mathbf{g})\psi](x) = (\hbar/2i)\{\mathbf{g}\cdot\nabla\psi(x) + \nabla\cdot[\mathbf{g}\psi(x)]\}\,, \tag{1.7}$$

where m is the particle mass.

Compare with conventional quantization: Taking $f(x) \sim \delta(x-y)$, then the expectation value $(\psi, \hat{\rho}(f)\psi) \sim m|\psi(y)|^2$. And taking $\mathbf{g}$ to approximate a constant vector field in, say, the x^j direction, then $\hat{J}(\mathbf{g}) \sim -i\hbar\partial/\partial x^j$, the total momentum in the x^j direction. This is consistent with the usual physical interpretations of $\hat{\rho}(f)$ and $\hat{J}(\mathbf{g})$.

2. Inequivalent representations describing different quantum systems

The (unitarily) inequivalent representations of the local current algebra correspond to distinct quantum systems. A wide variety of quantum-mechanical possibilities are unified in their description by this method – and some new ones were predicted!

(a) N-particle quantum mechanics, with particles distinguished by their masses

(b) Systems of indistinguishable particles obeying Bose or Fermi exchange statistics (in two or more space dimensions)

(c) Systems of indistinguishable particles (or excitations) obeying intermediate, anyon statistics in two space dimensions (for given anyonic phase shift under counterclockwise exchange)

(d) Systems of distinguishable particles in two-space with distinct relative phase shifts under counterclockwise exchange

(e) Systems of particles obeying parastatistics (in two or more space dimensions)

(f) Systems of nonabelian anyons in two space dimensions

Inequivalent representations ... (continued)

(g) Systems of tightly bound charged particles – point dipoles, quadrupoles, etc.

(h) Particles with spin, arranged in spin towers according to representations of the general linear group

(i) Particles with fractional spin, in two space dimensions

(j) Systems of infinitely many particles, in locally finite configurations, corresponding to a free or interacting Bose gas (Poisson measures, Gibbs measures)

(k) Systems of infinitely many particles obeying Fermi statistics, or exotic statistics

(l) Systems of infinitely many particles with accumulation points

(m) Quantized vortex systems, with filaments of vorticity in two space dimensions, or ribbons of vorticity in three dimensions, obtained from representations of area- (resp., volume-) preserving diffeomorphisms

Inequivalent representations ... (continued)

(n) Configurations of extended objects, including loops and strings, knotted configurations, and configurations of objects with nontrivial topology and/or nontrivial internal symmetry

(o) Quantum mechanics on physical spaces that themselves are manifolds with boundary, with singularities, or with nontrivial topology

(p) Quantum particles having nonlinear time-evolutions, possibly equivalent to linear theories via nonlinear gauge transformations, but otherwise violating the "no signal" property

(q) Other ... ?

The mathematical theory behind many of the above descriptions is incomplete or only partially developed. There are many unanswered questions, opportunities for new constructions, and (probably) new predictions to be made of a fundamental nature.

3. Unitary representations of $\mathit{Diff}_0(M)$: *A quick look*

Under very general conditions, a unitary representation of $G = \mathit{Diff}_0(M)$ may be written,

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}\,, \quad \text{where :} \tag{3.1}$$

γ belongs to a *configuration space* Δ, which carries an action of G inherited somehow from M,

μ is a measure on Δ which is quasi-invariant under the action of G,

Ψ is a function on Δ taking values in an inner product space $\mathcal{W}$ (a complex Hilbert space), with $\langle\Psi(\gamma),\Psi(\gamma)\rangle_{\mathcal{W}}$ integrable with respect to μ; so that $\Psi \in \mathcal{H} = \mathcal{L}^2_{d\mu}(\Delta,\mathcal{W})$. It may be that Ψ is just complex-valued, so that $\mathcal{W} = \mathbf{C}$,

χ is a *unitary 1-cocycle* acting in $\mathcal{W}$.

Inner product in $\mathcal{H}$, and semidirect product of $\mathcal{D}$ with $Diff_0(M)$

The inner product in $\mathcal{H}$ is given by

$$(\Phi, \Psi) = \int_\Delta \langle \Phi(\gamma), \Psi(\gamma) \rangle_{\mathcal{W}} \, d\mu(\gamma) \,, \tag{3.2}$$

where $\langle \ , \ \rangle_{\mathcal{W}}$ is the inner product in $\mathcal{W}$.

To represent the full local current algebra of quantum mechanics, we also need to associate configurations γ with (continuous) linear functionals on $\mathcal{D}$. There are a number of systematic ways to do so. Then for $f \in \mathcal{D}$, we have $\langle \gamma, f \rangle \in \mathbf{R}$. We write,

$$[U(f)\Psi](\gamma) = \exp i \langle \gamma, f \rangle \Psi(\gamma) \tag{3.3}$$

which, combined with the general formula (**??**) for $V(\phi)$, is the desired representation of the semidirect product group in $\mathcal{H}$.

Cocycles

The cocycle equation for χ is,

$$\chi_{\phi_1\phi_2}(\gamma) = \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma)\,. \tag{3.4}$$

The equations (3.1)-(3.4) hold outside sets of μ-measure zero in Δ.

Given Δ and a quasiinvariant measure μ on Δ, one may always choose $\mathcal{W} = \mathbf{C}$ and $\chi_\phi(\gamma) \equiv 1$, to obtain a unitary group representation on complex-valued wave functions.

But inequivalent (non-cohomologous) unitary 1-cocycles describe unitarily inequivalent representations! This is the reason for the exchange statistics of indistinguishable particles, and for anyon statistics in two-space.

We remark that the system of Radon-Nikodym derivatives $\alpha_\phi(\gamma) = [d\mu_\phi/d\mu](\gamma)$ is also a real 1-cocycle, and consequently so is $\sqrt{\alpha_\phi(\gamma)}$.

4. Induced representations, particle statistics, and topology

Define the configuration space $\Delta^{(N)}$, $N = 1, 2, 3, \ldots$, to consist of N-point subsets of $\mathbf{R}^d$. Let diffeomorphims of $\mathbf{R}^d$ act on $\Delta^{(N)}$ in the natural way.

The measure μ is here the (local) Lebesgue measure on $\Delta^{(N)}$.

The corresponding diffeomorphism group and local current algebra representations describe N indistinguishable quantum particles in d-dimensional space. The wave function Ψ may be complex-valued or, more generally, vector-valued.

Inequivalent cocycles on $\Delta^{(N)}$ are obtained (for $d \geq 2$) by inducing (generalizing Mackey's method) from inequivalent unitary representations of the fundamental group (first homotopy group) $\pi_1[\Delta^{(N)}]$.

This is where the topology of configuration space enters in a fundamentally important way. Let us visualize how this occurs.

The stability subgroup of a configuration maps to the fundamental group

Unitary representations are induced by representations of the stability subgroup of a configuration (the "little group").

Consider the stability subgroup of an N-point configuration in 3-space (for example). A diffeomorphism that leaves $\gamma = \{x_1, ..., x_N\} \subset \mathbf{R}^3$ fixed can do so by leaving each point fixed: but also, by permuting the points. So this stability subgroup maps (via a natural surjective homomorphism) onto the symmetric group S_N.

Unitary representations of S_N immediately provide continuous unitary representations of the stability subgroup of the diffeomorphism group. These in turn induce representations of the full diffeomorphism group.

For $d \geq 3$, S_N is the fundamental group of the configuration space $\Delta^{(N)}$.

The fundamental group of configuration space (continued)

When Ψ is complex-valued and $\chi \equiv 1$, we have bosonic exchange symmetry. This corresponds to inducing by the identity representation of S_N.

The alternating representation of S_N, $N \geq 2$, gives fermionic exchange symmetry.

The higher-dimensional representations of S_N induce representations that describe particles satisfying the parastatistics of Greenberg and Messiah. Here we have wave functions taking vector values, according to the dimension of the representation of S_N.

But more generally, the fundamental group of a manifold (first homotopy group) is defined from closed paths (i.e., pointed loops). But diffeomorphisms of the physical space act on not only on objects, but on paths in configuration space.

Thus the stability subgroup maps naturally to the fundamental group for all configuration spaces, even when the fundamental group is different from S_N.

When the configuration space is a space of embeddings of a compact manifold K into physical space M, then the stability subgroup of $Diff_0(M)$ maps naturally onto the fundamental group of K.

5. Anyons and nonabelian anyons in two-space

When $d = 2$ and $N \geq 2$, the fundamental group of the N-particle configuration space $\Delta^{(N)}$ is no longer S_N, but the braid group B_N. The 1-dimensional unitary representations (characters) of B_N then induce representations of the local current algebra describing intermediate (anyon) exchange statistics: a single counterclockwise exchange introduces an arbitrary phase $\exp i\theta$.

This development was one of three independent discoveries (about 40 years ago), the earliest being by Leinaas and Myrheim, the second being ours at Los Alamos, and the third being that of Wilczek who coined the term "anyons".

The theoretical possibility of nonabelian anyons, also called plektons, was first discovered from the local current algebra. Representations describing nonabelian anyons are induced by the higher-dimensional representations of B_N (in analogy with parastatistics in three space dimensions).

As is now well-known, these ideas find application in condensed matter physics, in descriptions of the quantum Hall effect, and more recently in quantum computing.

6. Intertwining creation and annihilation fields, q-commutation relations

The irreducible, unitary diffeomorphism group representations fall naturally into hierarchies, whose intertwining operators (satisfying a natural commutator bracket with the densities and currents) *create* and *annihilate* configurations of the same kind.

These intertwining operators have an interpretation as "second-quantized" fields in the Galilean theory. They are general enough to describe not only point particles but extended objects.

We are entitled to call the family of representations a *hierarchy* if for each N there exists an operator-valued distribution $\psi^*_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$ (the creation field), such that for all $h \in \mathcal{D}$, $f \in \mathcal{D}$, and $\phi \in \mathcal{K}$,

$$U_{N+1}(f)\psi^*_N(h) = \psi^*_N(U_{N=1}(f)h)U_N(f)\,, \tag{6.1}$$

$$V_{N+1}(\phi)\psi^*_N(h) = \psi^*_N(V_{N=1}(\phi)h)V_N(\phi)\,. \tag{6.2}$$

Intertwining fields ...

For example, take a family of N-particle representations, where $N = 0, 1, 2, \ldots$. For each N, we have the Hilbert space $\mathcal{H}_N$, and the unitary representation $U_N(f)V_N(\phi)$. Here, the preceding equations state the following:

Beginning with an N-particle state, create a new particle in state h. Then transform the resulting $(N+1)$-particle state by the unitary group representation U_{N+1} (respectively, V_{N+1}).

The result is the same as if: Beginning with the same N-particle state, first transform it by U_N (resp. V_N). Then create a new particle in the state obtained by transforming the 1-particle state h by U_1 (resp. V_1).

The nontrivial part of the construction involves ensuring that creation and annihilation within each hierarchy respect the particle statistics (bosonic, fermionic, or anyonic).

Intertwining fields ...

In effect, $\psi^*(x)$ creates a particle at $x \in M$, while its adjoint $\psi(x)$ annihilates a particle at x. The smeared field $\psi^*(h) = \int \psi^*(x)h(x)dx$ is the intertwining field for the hierarchy.

As a consequence of (6.1)-(6.2), ψ^* and ψ also obey natural brackets with the local current generators of the hierarchy:

$$[\hat{\rho}(f), \psi^*(h)] = \psi^*(\hat{\rho}_{N=1}(f)h), \quad [\hat{J}(\mathbf{g}), \psi^*(h)] = \psi^*(\hat{J}_{N=1}(g)h); \tag{6.3}$$

the brackets with ψ are given by the adjoint equations. Note that these are still all *commutator* brackets.

In terms of ψ^* and ψ, we may now write the current algebra generators (formally) as operator-valued distributions:

$$\hat{\rho}(x) = \psi^*(x)\psi(x), \quad \hat{\mathbf{J}}(x) = (\hbar/2i)\{\psi^*(x)\nabla\psi(x) - [\nabla\psi^*(x)]\psi(x)\}\,. \tag{6.4}$$

Anyonic q-commutation relations

But from the above, ψ^* and ψ satisfy certain equal-time brackets with themselves and each other. These turn out to be canonical commutation relations when ψ, ψ^* intertwine the symmetric hierarchy, anticommutation relations when they intertwine the antisymmetric hierarchy, and q-commutation relations when they intertwine the N-anyon hierarchy in two-space, where $q = \exp i\theta$ specifies the anyonic phase shift:

More specifically, set $[A, B]_q = AB - qBA$ (here q is a complex number of modulus 1). Then in two-space with $x = (x^1, x^2)$ and $y = (y^1, y^2)$, in the half-space $x^1 < y^1$,

$$[\psi(x), \psi(y)]_q = [\psi^*(x), \psi^*(y)]_q = 0, \quad [\psi(y), \psi^*(x)]_q = \delta(x - y); \tag{6.5}$$

in the complementary half-space, $\bar{q}$ replaces q. The choice of half-space is arbitrary, without physical consequence.

These ideas were introduced with D. H. Sharp (1996) and developed further with S. Majid (2002).

7. An approach to relativistic theory

All the preceding discussion pertains directly to "nonrelativistic" (Galilean) quantum theory. Here diffeomorphsims of physical space respect the causal structure of spacetime. But this is no longer true in Minkowskian spacetime.

In recent work with D. Sharp (2019), we describe an approach to relativistic quantum field theory making use of the above development. We apply this successfully to reconstruct the free relativistic scalar (Bose) field in the Fock representation.

The key idea is that physical measurements are spatial, take place in a fixed inertial reference frame, and can be described via noncovariant densities and currents that are local in space but nonlocal in spacetime. Spacetime symmetry and relativistic covariance are to be introduced at the end, not at the outset.

Steps in constructing relativistic theories ...

We envision the following steps in the program.

1. Choose an inertial frame of reference F (the frame of the observer). We have not yet built in how observations in one inertial frame are related to those in another.

2. Call the spacetime as observed from F by the name M_F. Introduce a coordinate system for M_F in which the coordinate $\mathbf{x}$ refers to space, and t to time, to define a "spacelike" surface Σ_F coordinatized by x. This plays the role of M in the previous discussion.

It is natural to regard the different spacetimes M_F as fibers in a bundle over a base space of inertial frames. Each fiber carries a copy of the theory. The spacetime symmmetry should eventually establish the isomorphism of the theories in different fibers.

3. Introduce the group $\mathcal{D} \rtimes \mathit{Diff}_0(\Sigma_F)$ with respect to Σ_F, and consider its continuous unitary representations as discussed above. Alternatively, $\mathcal{D}$ may be replaced by a local current group describing internal symmetries or gauge symmetry.

Relativistic theories ...

4. Identify one or more *hierarchies* of representations, describing configurations consisting of entities of the same type (to be interpreted as quanta of the same field). Introduce creation and annihilation fields as operators intertwining the unitary representations in the hierarchy.

At this stage in the general development, we have a full (Fock-like) description of the field quanta, but we do not yet have the field – nor do we have a description of the dynamics, which is to be provided as usual by a Hamiltonian operator.

5. Introduce a (covariant) *relativistic* field, defined making use of the creation and annihilation fields intertwining the hierarchy of representations in inertial frame F. This is the first place where the particular choice of spacetime symmetry (relating different reference frames) is introduced.

The particles or entities in the configuration spaces of the hierarchy are interpreted as quanta of the relativistic field as observable in the reference frame F.

Relativistic theories ...

6. Introduce the Hamiltonian H describing the (relativistic) dynamics. While H may be expressed in terms of the relativistic quantum field, the preceding construction should mean it can also be expressed explicitly in terms of the local currents (the infinitesimal generators in the diffeomorphism group representations with which we started), together with the (nonrelativistic) intertwining fields.

7. At the end of the construction, the physics described by the relativistic field and Hamiltonian should not depend on the particular reference frame F with which we began.

8. Infinite systems, additional topological possibilities, and general configuration spaces

Many of the open questions and much of the yet-undeveloped mathematical foundations of this group-theoretic approach, pertain to infinite quantum systems (including anyons and non-abelian anyons), and systems of extended objects (including those with nontrivial topology).

For example, these include measure-theoretic questions (characterizing classes of measures on manifolds, particularly infinite-dimensional manifolds, quasi-invariant under diffeomorphisms).

To address such questions requires working within a category of quantum configuration spaces. This also raises the overarching question of selecting a category of configuration spaces as a "universal configuration space" for quantum mechanics.

Let us conclude with a rapid overview of some categories of configuration spaces – some extensively studied, some less so – that are pertinent to the present discussion.

General configuration spaces ...

Infinite but locally finite (point) configurations

When Δ is the configuration space of infinite but locally finite subsets of $\mathbf{R}^d$, or more generally of a locally compact, σ-compact manifold M, representations of the same current algebra describes the physics of infinite gases. This configuration space is standard in continuum classical or quantum statistical mechanics. It is probably pertinent to the anyonic case.

Two important kinds of quasi-invariant measures μ pertinent to the bosonic case are Poisson measures (associated with gases of noninteracting particles at fixed average density), or Gibbs measures (associated with translation-invariant two-body interactions).

Other classes of measures are needed for fermionic and anyonic statistics.

General configuration spaces ...

Distributions

The space of generalized functions (or distributions) $\mathcal{D}'(M)$ has the advantage that for $\gamma \in \mathcal{D}'$ and $f \in \mathcal{D}$, we immediately have $\langle \gamma, f \rangle$ (needed to represent the semidirect product group).

Here we not only have infinite, locally finite point configurations, but derivatives of δ-functions and other, extended-object possibilities.

Diffeomorphisms of M act on $\mathcal{D}$ as specified by the semidirect product law in G, and act on $\mathcal{D}'$ by the dual action.

Embeddings or immersions, parameterized or unparameterized

Let N be a manifold (typically of lower dimension than M). Configurations are (not typically infinitely differentiable) embeddings (or, more generally, immersions) of N in M – e.g., with $N = S^1$, loops in M, parameterized or unparameterized. For $N = S^1$ and M 3-dimensional, we have classes of orbits for different kinds of knots and links.

More generally, we have embedded manifolds with a variety of topological properties, and cocycles associated with their fundamental groups.

General configuration spaces ...

Closed subsets

General configuration spaces may be defined as spaces of closed subsets of M, as proposed and developed by Ismagilov.

Unparameterized embeddings or immersions of N in M are special cases of such closed subsets, while parameterized embeddings or immersions are not. These spaces also include infinite point configurations with accumulation points.

Countable subsets

Still more general configuration spaces contain countable subsets of M (with no condition of local finiteness). This generalizes locally finite configurations to allow accumulation points.

But it also generalizes closed subsets (and thus unparameterized embeddings), in that (for M separable) a closed subset can be recovered as the closure of many distinct countable subsets.

Parameterized configurations require consideration of *ordered* countable subsets.

General configuration spaces ...

Coadjoint orbits and their unions

Consideration of the *coadjoint representation* of G, suggests that one construct configuration spaces from the dual space to the corresponding (infinite-dimensional) Lie algebra – i.e., the dual space to the current algebra of compactly-supported scalar functions and vector fields on M.

Then one needs to introduce a "polarization" (in the spirit of geometric quantization) in the corresponding coadjoint orbit or class of orbits, which amounts to selecting certain coordinates as "position-like" and others as "momentum-like" — with the former defining the quantum configurations.

The symplectic structure on coadjoint orbits provides a systematic way to obtain cocycles in this context.

General configuration spaces ...

Bundles

Finite or countably infinite subsets of bundles over M provide another approach to configuration spaces. E.g., in $\mathcal{D}'$, derivatives of δ-functions (including higher derivatives) lead to quantum theories of point-like dipoles, quadrupoles, etc. Such configurations belong not to M itself, but to the *jet bundle* over M, to which the action of diffeomorphisms on M lifts naturally.

This suggests working within the more general category of bundles over M to which the action of diffeomorphisms of M lifts.

Marked configuration spaces

In the spirit of bundles over M, there is a physically important generalization to "marked configuration spaces." Here one identifies a compact manifold K describing the internal degrees of freedom of a particle, and a compact Lie group L that acts on K. One then associates to each point in an ordinary configuration a value or "mark" in K.

The local symmetry group itself can be correspondingly enlarged to include compactly supported C^∞ mappings from M to L, and/or to include bundle diffeomorphisms of $M \times K$.

Comments on general configuration spaces, anyons, and topological quantum theory

To describe systems of anyons, infinite systems, or extended quantum configurations with nontrivial topologies in M, or spaces M with nontrivial topological properties, in a unified, group-theoretic approach, some generalized configuration space over M is needed. An accompanying description of classes of measures quasi-invariant under diffeomorphisms of M is likewise necessary.

Each of the above methods of characterizing quantum configuration spaces has some significant literature, and certain methods are associated with a point of view about quantization or about quantum mechanics.

The diffeomorphism group approach helps us understand these distinct but overlapping methods as techniques for the construction of classes of unitary group representations, embodying the local symmetry of physical space in the quantum kinematics.

Each construction has its own set of open questions, sometimes of a technical nature. In addition, there is the overarching question of selecting a category of configuration spaces as a "universal configuration space" for quantum mechanics.

Thank you for your kind attention.

Comparison with conventional, canonical quantization

Conventionally, a quantum particle is described by a self-adjoint representations of the Heisenberg algebra $[Q, P] = C$, where C is the central element.

In a self-adjoint, irreducible representation, $C \to i\hbar I$, where $\hbar$ is a (positive) constant. Fixing the value of $\hbar$, the irreducible representation $\hat{Q}, \hat{P}$ is unique up to unitary equivalence, given [for ψ in the Hilbert space $\mathcal{L}^2_{dx}(\mathbf{R})$] by

$$\hat{Q}\psi(x) = x\psi(x) \quad \hat{P}\psi = -i\hbar\frac{\partial}{\partial x}\psi\,. \tag{8.1}$$

Normalizing so that $\int |\psi(x)|^2\, dx = 1$, the function $\rho(x) = |\psi(x)|^2$ defines the probability density for finding the quantum particle at x.

In d space dimensions, we similarly introduce Q_j and P_k for $j, k = 1, \ldots d$,

$$[Q_j, P_k] = C\delta_{jk}, \quad \text{and} \quad \hat{Q}_j = x_j\,, \quad \hat{P}_k = -i\hbar\partial/\partial x_k\,. \tag{8.2}$$

But in contrast, representations of the local current algebra of (1.1) are *not unique*.

Interpretation of the 1-particle representation

In fact, if we allow f to approximate a δ-function centered at the point y, then the expectation value of $\hat{\rho}(f)$ approximates $m|\psi(y)|^2$, the mass times the usual probability density $\rho(y)$. So $\hat{\rho}(f)$ is a mass density operator.

And if we allow the components of $\mathbf{g}$ to approximate δ-functions centered at y, then the expectation value of $\hat{J}(\mathbf{g})$ approximates $(\hbar/2i)\{\overline{\psi(y)}\nabla\psi(y) - [\nabla\overline{\psi(y)}]\psi(y)\}$, the mass times the usual probability flux $\mathbf{j}(y)$.

So we recover the usual quantum probability density and flux as expectation values of $\hat{\rho}$ and $\hat{J}$ in 1-particle quantum mechanics. This feature is particular to the single point case, where the configuration space is naturally identified with physical space.

For multiparticle systems, the operators $\hat{\rho}$ and $\hat{J}$ are interpreted (respectively) as mass and momentum density operators on *physical space*. To obtain the probability density and flux on *configuration space*, one must consider expectation values of *products* of density and current operators.