On the Average-Field Functional for Anyons

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Mathematical Physics of Anyons and Topological States of Matter

NORDITA

joint work with R. Duboscq (Toulouse),
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Outline

1. Introduction:
   - Almost-bosonic limit and the Average-Field (AF) functional [LR];
   - Minimization of the AF functional.
2. Main results [CLR,CLR*]:
   - Existence of the Thermodynamic Limit (TL) for homogeneous anyons;
   - Local density approximation of the AF functional for trapped anyons in terms of a Thomas-Fermi (TF) effective energy;
3. Vortex structure (numerical simulations) [CDLR].

Main References

- [LR] D. Lundholm, N. Rougerie, *J. Stat. Phys.* **161** (2015);
- [CLR] MC, D. Lundholm, N. Rougerie, *Anal. PDE* **10** (2017), 1169–1200;
- [CLR*] MC, D. Lundholm, N. Rougerie, in *Contemp. Math.*, **717** (2018);
- [CDLR] MC, R. Duboscq, D. Lundholm, N. Rougerie, preprint *arXiv: 1901.10739 [cond-mat.quant-gas]*.

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Anyons

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- Any anyonic wave function Ψ can be mapped onto a bosonic one $\tilde{\Psi} \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ via the bosonization map

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{j<k} e^{i\alpha\phi_{jk}} \tilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N), \qquad \phi_{jk} = \arg \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|}.$$

Magnetic Gauge

On $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ the Schrödinger operator $\sum \left(-\Delta_j + V(\mathbf{x}_j) \right)$ is mapped to

$$\boxed{H_N = \sum_{j=1}^{N} \left[\left(-i\nabla_j + \alpha \mathbf{A}_j \right)^2 + V(\mathbf{x}_j) \right]}$$

with Aharonov-Bohm magnetic potentials $\mathbf{A}_j = \mathbf{A}(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2}$.

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AF Approximation

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- If the number of anyons is larger, i.e., $N \to \infty$ but at the same time $\alpha \simeq \beta N^{-1}$, then one expects a mean-field behavior, i.e.,

$$\alpha \mathbf{A}_j = (N\alpha) \frac{1}{N} \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2} \simeq \beta \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} \, \frac{(\mathbf{x} - \mathbf{y})^{\perp}}{|\mathbf{x} - \mathbf{y}|^2} \rho(\mathbf{y}),$$

with ρ the one-particle density associated to $\Phi \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$;

- We should then expect that

$$\frac{1}{N} \langle \Phi | \, H_N \, | \Phi \rangle \simeq \mathcal{E}^{\mathrm{af}}_{\beta}[u],$$

for some $u \in L^2(\mathbb{R}^2)$ such that $|u|^2(\mathbf{x}) = \rho(\mathbf{x})$ (self-consistency).

AF Functional

$$\boxed{\mathcal{E}^{\mathrm{af}}_{\beta}[u] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left\{ \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right\}}$$

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Minimization of $\mathcal{E}^{\mathrm{af}}_\beta$ (I)

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- Thanks to the symmetry $u, \beta \longrightarrow u^*, -\beta$, we can assume $\beta \geqslant 0$;
- The domain of $\mathcal{E}^{\mathrm{af}}_\beta$ is $\mathscr{D}[\mathcal{E}^{\mathrm{af}}] = H^1(\mathbb{R}^2)$, since by 3-body Hardy inequality

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left| \mathbf{A}[|u|^2] \right|^2 |u|^2 \leqslant C \left\| u \right\|^4_{L^2(\mathbb{R}^2)} \left\| \nabla |u| \right\|^2_{L^2(\mathbb{R}^2)} .$$

Proposition (Minimization [Lundholm, Rougerie '15])

For any $\beta \geqslant 0$, there exists a minimizer $u^{\mathrm{af}}_\beta \in \mathscr{D}[\mathcal{E}^{\mathrm{af}}_\beta]$ of the functional $\mathcal{E}^{\mathrm{af}}_\beta$:

$$E^{\mathrm{af}}_\beta := \inf_{\|u\|_2 = 1} \mathcal{E}^{\mathrm{af}}_\beta[u] = \mathcal{E}^{\mathrm{af}}_\beta[u^{\mathrm{af}}_\beta].$$

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MINIMIZATION OF $\mathcal{E}_\beta^{\mathrm{af}}$ (II)

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PROPOSITION (MINIMIZATION [LUNDHOLM, ROUGERIE '15])

For any $\beta \geqslant 0$, u_β^{af} solves the variational equation (semilinear with quintic nonlinearity in u and cubic nonlinearity in u^2 and ∇u)

$$\left[\left(-i\nabla + \beta \mathbf{A}_\beta^{\mathrm{af}} \right)^2 + V - 2\beta \nabla^\perp w_0 * \left(\beta \mathbf{A}_\beta^{\mathrm{af}} \left| u_\beta^{\mathrm{af}} \right|^2 + \mathbf{j}_\beta^{\mathrm{af}} \right) \right] u_\beta^{\mathrm{af}} = \lambda u_\beta^{\mathrm{af}},$$

where $\mathbf{A}_\beta^{\mathrm{af}} = \mathbf{A}\left[\left| u_\beta^{\mathrm{af}} \right|^2 \right]$, $\mathbf{j}_\beta^{\mathrm{af}} = \frac{i}{2} u_\beta^{\mathrm{af}} \nabla u_\beta^{\mathrm{af}\,*} + \mathrm{c.c.}$ is the current and

$$\lambda = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \left\{ 1 \left(\left| \nabla u_\beta^{\mathrm{af}} \right|^2 + V \left| u_\beta^{\mathrm{af}} \right|^2 \right) + 2 \cdot 2\beta \mathbf{A}_\beta^{\mathrm{af}} \cdot \mathbf{j}_\beta^{\mathrm{af}} + 3\beta^2 \left| \mathbf{A}_\beta^{\mathrm{af}} \right|^2 \left| u_\beta^{\mathrm{af}} \right|^2 \right\}.$$

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Almost-bosonic Limit

- Consider $N \to \infty$ non-interacting anyons with statistics parameter $\alpha = \dfrac{\beta}{N-1}$ for some $\beta \in \mathbb{R}$, i.e., in the almost-bosonic limit;
- Assume that the anyons are extended, i.e., the fluxes are smeared over a disc of radius $R = N^{-\gamma}$.

Theorem (AF Approximation [Lundholm, Rougerie '15])

Under the above hypothesis and assuming that V is trapping and $\gamma \leqslant \gamma_0$,

$$\boxed{\lim_{N\to\infty} \frac{\inf \sigma(H_{N,R})}{N} = \inf_{\|u\|_2=1} \mathcal{E}^{\mathrm{af}}_{\beta}[u]}$$

and the one-particle reduced density matrix of any sequence of ground states of $H_{N,R}$ converges to a convex combination of projectors onto AF minimizers.

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Motivations

- The AF approximation is used heavily in physics literature, but typically the nonlinearity is resolved by picking a given ρ (usually the constant density);

- As expected, when $\beta \to 0$, the anyonic gas behaves like a Bose gas.
- More interesting is the regime $\beta \to \infty$, i.e., "less-bosonic" anyons:
  - what is the energy asymptotics of E_β^{af}?
  - is $|u_\beta^{\mathrm{af}}|^2$ almost constant in the homogeneous case, i.e., for $V = 0$ and confinement to a bounded region?
  - how does the inhomogeneity of V modify the density $|u_\beta^{\mathrm{af}}|^2$?
  - what is u_β^{af} like? in particular how does its phase behave?

- The AF functional is not the usual mean-field-type energy (e.g., Hartree or Gross-Pitaevskii), since the nonlinearity depends on the density but acts on the phase of u via a magnetic field.

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Homogeneous Gas

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- $\Omega \subset \mathbb{R}^2$ bounded and simply connected with Lipschitz boundary;
- We consider the following two minimization problems

$$E_{\mathrm{N}/\mathrm{D}}(\Omega, \beta, M) := \inf_{u \in H_0^1(\Omega), \|u\|_2 = M} \int_\Omega \mathrm{d}\mathbf{x} \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 .$$

- We want to study the limit $\beta \to \infty$ of $E_{\mathrm{N}/\mathrm{D}}(\Omega, \beta, M)/\beta$;
- The above limit is equivalent to the TD limit ($\beta, \rho \in \mathbb{R}^+$ fixed)

$$\lim_{L \to \infty} \frac{E_{\mathrm{N}/\mathrm{D}}(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|} .$$

Lemma (Scaling Laws)

For any $\lambda, \mu \in \mathbb{R}^+$, $E_{\mathrm{N}/\mathrm{D}}(\Omega, \beta, M) = \frac{1}{\lambda^2} E_{\mathrm{N}/\mathrm{D}} \left(\mu\Omega, \frac{\beta}{\lambda^2 \mu^2}, \lambda^2 \mu^2 M \right)$.

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Heuristics ($\beta \gg 1$)

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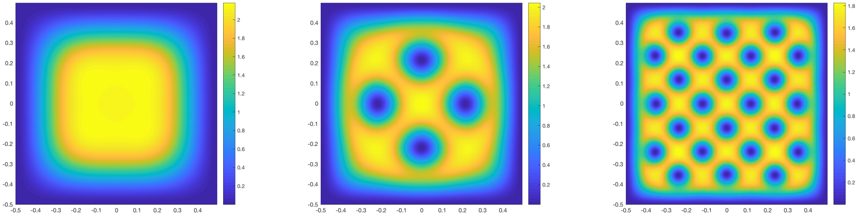

Numerical simulations: $|u_\beta^{\mathrm{af}}|^2$ (square trap with Dirichlet b.c.) for $\beta = 10, 30, 120$.

- In the homogeneous case, $|u_\beta^{\mathrm{af}}|^2$ can be constant only in a very weak sense (say in L^p, $p < \infty$ not too large);
- The phase of u_β^{af} contains vortices (with $\# \sim \beta$) almost uniformly distributed with average distance $\sim \frac{1}{\sqrt{\beta}}$ (Abrikosov lattice?), in order to compensate the huge magnetic field.

TD Limit

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Theorem (∃ TD Limit [MC, Lundholm, Rougerie '16])

Under the above hypothesis on Ω *and for any* $\beta, \rho \in \mathbb{R}^+$, *the limits*

$$\boxed{e(\beta,\rho) := \lim_{L\to\infty} \frac{E_{\mathrm{N/D}}(L\Omega, \beta, \rho L^2|\Omega|)}{L^2|\Omega|} = \beta|\Omega| \lim_{\tilde{\beta}\to\infty} \frac{E_{\mathrm{N/D}}(\Omega, \tilde{\beta}, \rho)}{\tilde{\beta}}}$$

exist, coincide and are independent of Ω. *Moreover*

$$\boxed{e(\beta,\rho) = \beta\rho^2 e(1,1)}$$

- The scaling property of $e(\beta,\rho)$ is a direct consequence of the scaling law mentioned before. Moreover $e(1,1)$ is a finite quantity such that

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Sketch of the Proof (I)

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1. $\exists$ of TD limit when Ω is a unit square with Dirichlet b.c.;
2. $E_{\rm D}(LQ, \beta, \rho L^2) - E_{\rm N}(LQ, \beta, \rho L^2) = o(L^2)$ for any domain Ω (IMS);
3. $\exists$ of TD for general domains Ω with Dirichlet b.c..

① $\exists$ TD limit for Dirichlet b.c.

- Key observation: the magnetic field generated by a bounded region can be gauged away outside (Newton's theorem);
- Pick a smooth and radial f with $\mathrm{supp}(f) \subset \mathcal{B}_\delta(0)$ and $N \sim L^2$ points so that $|\mathbf{x}_j - \mathbf{x}_k| > 2\delta$: consider then the trial state

$$u(\mathbf{x}) = \sum_{j=1}^{N} f(\mathbf{x} - \mathbf{x}_j) e^{-i\phi_j}, \qquad \|u\|_2^2 = N \|f\|_2^2;$$

- In $\{|\mathbf{x} - \mathbf{x}_j| \leqslant \delta\}$ the magnetic field generated by the other discs is

$$\sum_{k \neq j} \nabla^{\perp} \left(w_0 * |f(\mathbf{x} - \mathbf{x}_k)|^2 \right) \simeq \|f\|_2^2 \nabla \sum_{k \neq j} \arg(z - z_k) =: \nabla \phi_j.$$

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Sketch of the Proof (II)

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$$\mathcal{E}^{\mathrm{af}}_{\beta}[u] = \int_{L\Omega} \mathrm{d}\mathbf{x}\, \left|\left(-i\nabla + \beta \mathbf{A}[|u|^2]\right) u\right|^2, \quad \mathbf{A}[\rho] = \nabla^{\perp}\left(w_0 * \rho\right), \quad \|u\|_2^2 = \rho L^2$$

① $\exists$ TD limit for squares with Dirichlet b.c.

- By testing $\mathcal{E}^{\mathrm{af}}_{\beta}$ on u_{trial}, one gets the sum of the energies in each $\mathcal{B}_{\delta}(\mathbf{x}_j)$, which yields the upper bound $E_{\mathrm{N/D}} \leqslant CL^2$.
- A trivial lower bound is given via the inequality (for any $u \in H^1_0(\Omega)$)

$$\int_{\Omega} \mathrm{d}\mathbf{x}\, \left|\left(-i\nabla + \beta \mathbf{A}\right) u\right|^2 \geqslant \beta \int_{\Omega} \mathrm{d}\mathbf{x}\, \mathrm{curl}\mathbf{A}\, |u|^2$$

which leads to the lower bound $E_{\mathrm{D}} \geqslant 2\pi\beta\rho^2 L^2$, since

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② Dirichlet vs. Neumann in squares

- Use a refined version of the IMS formula to estimate the mass well inside the square and the contribution of the boundary layer, to get a bound matching the obvious $E_{\mathrm{N}} \leqslant E_{\mathrm{D}}$.
- Error of order $L^{12/7+\varepsilon} \ll L^2$ but $\gg L$ (not optimal).

③ General domains

- Tile the domain with squares and use the Dirichlet minimizers in the upper bound (gauging away the magnetic field as before) and the Neumann minimizer in the lower bound.

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$e(1,1)$

Bounds on $e(1,1)$

- It is easy to see that $e(1,1)$ is bounded.
- The inequality $\left\| \left(-i\nabla + \mathbf{A}\left[|u|^2 \right] \right) u \right\|_2^2 \geqslant 2\pi$ implies that $e(1,1) \geqslant 2\pi$.

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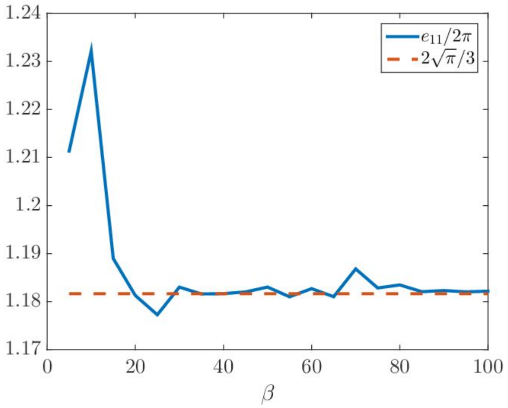

Numerical computation of $e(1,1)/(2\pi)$ using harmonically trapped anyons for increasing values of β.

Trapped Anyons

$$\mathcal{E}^{\mathrm{af}}_{\beta}[u] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left\{ \left| \left(-i\nabla + \beta \mathbf{A}[|u|^2] \right) u \right|^2 + V|u|^2 \right\}, \qquad \mathbf{A}[\rho] = \nabla^{\perp} \left(w_0 * \rho \right)$$

- Let $V(\mathbf{x})$ be a smooth homogenous potential of degree $s \geqslant 1$, i.e.,

$$V(\lambda \mathbf{x}) = \lambda^s V(\mathbf{x}), \qquad V \in C^{\infty}(\mathbb{R}^2),$$

and such that $\min_{|\mathbf{x}| \geqslant R} V(\mathbf{x}) \xrightarrow[R \to \infty]{} +\infty$ (trapping potential).

- We consider the minimization problem for $\beta \gg 1$

$$E^{\mathrm{af}}_{\beta} = \inf_{u \in \mathscr{D}[\mathcal{E}^{\mathrm{af}}], \|u\|_2 = 1} \mathcal{E}^{\mathrm{af}}_{\beta}[u],$$

with $\mathscr{D}[\mathcal{E}^{\mathrm{af}}] = H^1(\mathbb{R}^2) \cap \left\{ V|u|^2 \in L^1(\mathbb{R}^2) \right\}$ and u^{af}_{β} any minimizer.

- Since $B(\mathbf{x}) = \beta \mathrm{curl} \mathbf{A}[\rho] = 2\pi\beta\rho(\mathbf{x})$, if one could minimize the magnetic energy alone, the effective functional for $\beta \gg 1$ should be

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left[B(\mathbf{x}) + V(\mathbf{x}) \right] \rho = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left[2\pi\beta\rho^2 + V(\mathbf{x})\rho \right].$$

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TF Approximation

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TF Functional

The limiting functional for $\mathcal{E}_\beta^{\mathrm{af}}$ is

$$\boxed{\mathcal{E}_\beta^{\mathrm{TF}}[\rho] := \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \, \left[e(1,1)\beta\rho^2(\mathbf{x}) + V(\mathbf{x})\rho(\mathbf{x}) \right]}$$

with ground state energy $E_\beta^{\mathrm{TF}} := \inf_{\|\rho\|_1=1} \mathcal{E}_\beta^{\mathrm{TF}}[\rho]$ and minimizer $\rho_\beta^{\mathrm{TF}}(\mathbf{x})$.

- Under the hypothesis we made on V, we have
$$E_\beta^{\mathrm{TF}} = \beta^{\frac{s}{s+2}} E_1^{\mathrm{TF}}, \qquad \rho_\beta^{\mathrm{TF}}(\mathbf{x}) = \beta^{-\frac{2}{s+2}} \rho_1^{\mathrm{TF}} \left(\beta^{-\frac{1}{s+2}} \mathbf{x} \right).$$
- Given the chemical potential $\mu_1^{\mathrm{TF}} := E_1^{\mathrm{TF}} + e(1,1) \left\| \rho_1^{\mathrm{TF}} \right\|_2^2$, we have
$$\rho_1^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{2e(1,1)} \left[\mu_1^{\mathrm{TF}} - V(\mathbf{x}) \right]_+ .$$

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Local Density Approximation (I)

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Theorem (TF Approx. [MC, Lundholm, Rougerie '16])

Under the hypothesis on V and for any $R > 0$

$$\boxed{\lim_{\beta\to\infty} \frac{E^{\mathrm{af}}_{\beta}}{\beta^{\frac{s}{s+2}} E^{\mathrm{TF}}_{1}} = 1, \qquad \beta^{\frac{2}{s+2}} \left| u^{\mathrm{af}}_{\beta} \right|^2 \left(\beta^{\frac{1}{s+2}} \mathbf{x} \right) \xrightarrow[\beta\to\infty]{\left(C^{0,1}_0(\mathcal{B}_R)\right)^*} \rho^{\mathrm{TF}}_1(\mathbf{x})}$$

in the dual space of Lipschitz functions $C^{0,1}_0(\mathcal{B}_R)$ vanishing on $\partial\mathcal{B}_R$.

- The result applies to more general potentials, e.g., asymptotically homogeneous potentials;
- The homogeneous case (confinement to Ω, $V = 0$) is included: we recover the asymptotics $E_{\mathrm{N/D}}(\Omega, \beta, 1)/\beta \longrightarrow e(1,1)/|\Omega|$ and

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Average Density

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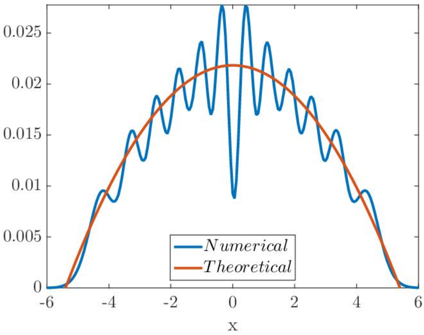

Theoretical (red) and numerical (blue) density profiles for $V(\mathbf{x}) = |\mathbf{x}|^2$, $\beta = 90$ (left) and $V(\mathbf{x}) = |\mathbf{x}|^4$, $\beta = 140$ (right).

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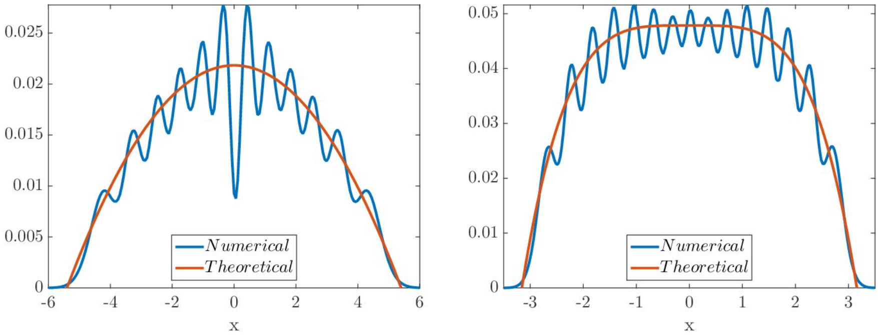

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In the homogeneous case, for any $\mathbf{x}_0 \in \Omega^{\circ}$ and any $R > 0$,

$$\left| u^{\mathrm{af}}_{\beta,\mathrm{N/D}} \left(\mathbf{x}_0 + \beta^{-\eta} \, \cdot \, \right) \right| \xrightarrow[\beta \to +\infty]{\left(C^{0,1}_0(\mathcal{B}_R) \right)^*} |\Omega|^{-1/2}, \qquad \textit{for any } 0 < \eta < \tfrac{1}{14}.$$

- We conjecture that $\rho^{\mathrm{af}}_{\beta}$ is well approximated in weak sense by a constant on any scale smaller than 1 up to $1/\sqrt{\beta}$;
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Vortex Structure of u_β^{af} (I)

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Why vortices?

- In presence of a large magnetic field of order $\beta \gg 1$, vortices can compensate for it: where $|u| \neq 0$, we can set $u = |u| e^{i\phi}$ and
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Vortex Structure of u_β^{af} (I)

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Simulations: Homogeneous Gas (I)

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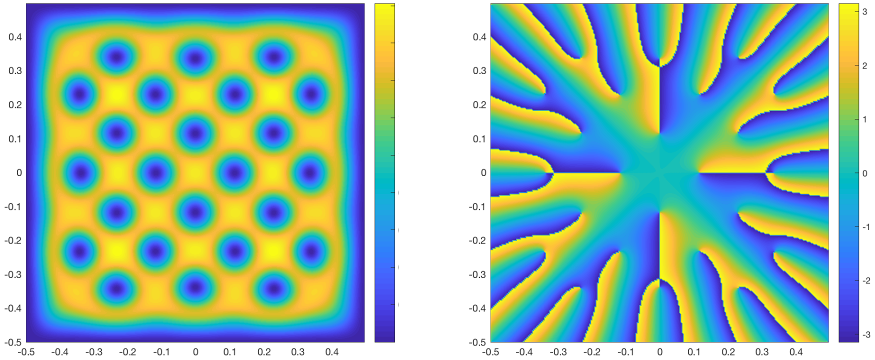

Modulus and phase of u_β^{af} in a square trap with Dirichlet boundary conditions for $\beta = 130$.

Simulations: Homogeneous Gas (II)

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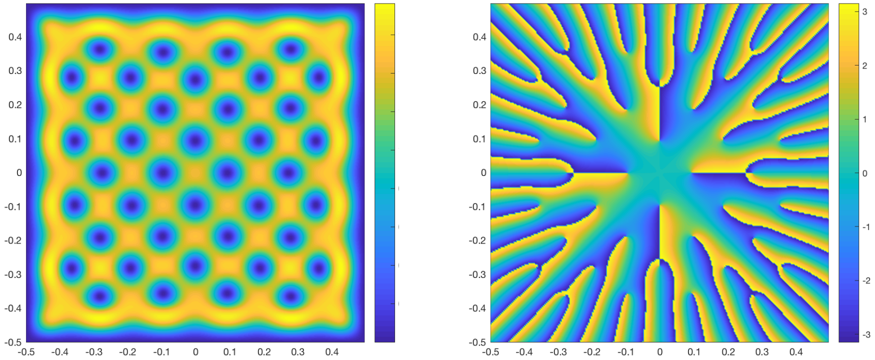
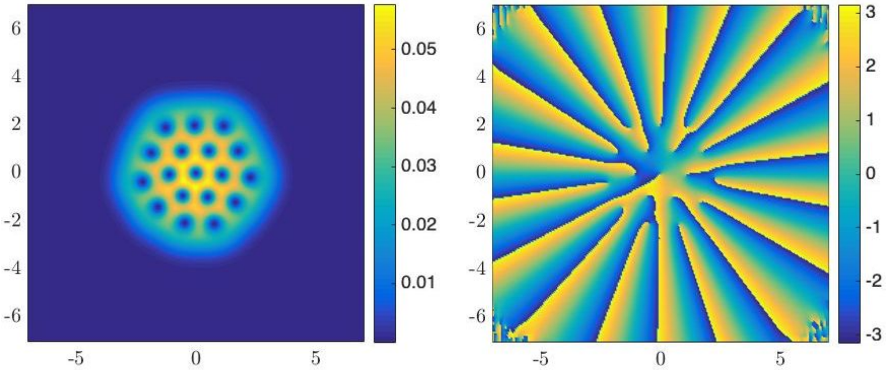


Modulus and phase of u_β^{af} in a square trap with Dirichlet boundary conditions for $\beta = 230$.

Simulations: Trapped Gas (I)

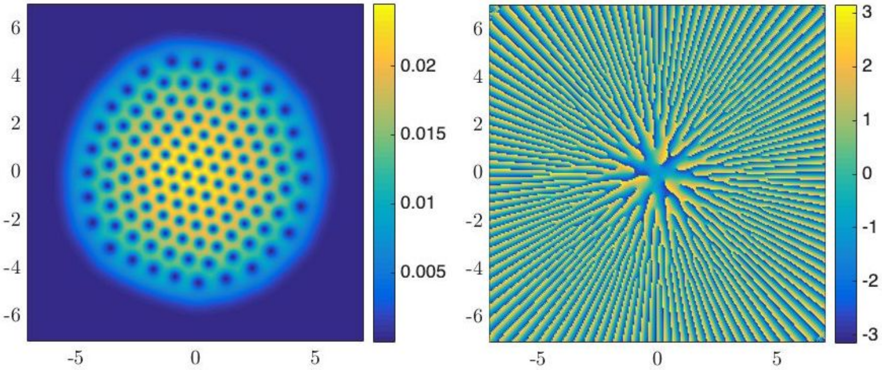
Sapienza Università di Roma



Modulus and phase of u_β^{af} in a harmonic trap for $\beta = 25$.

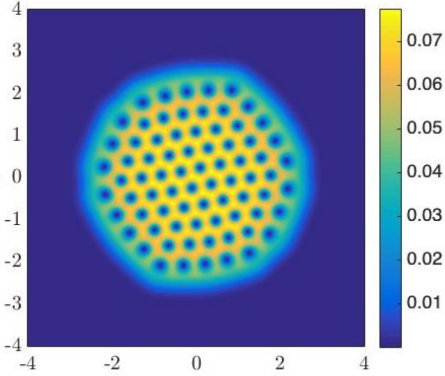
Simulations: Trapped Gas (II)

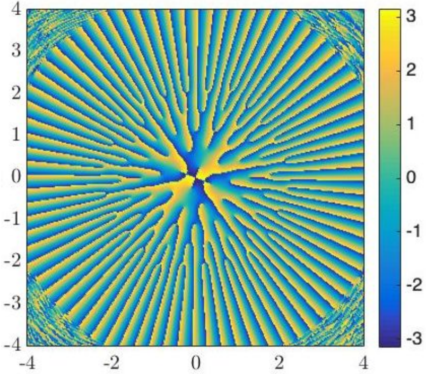
Sapienza Università di Roma



Modulus and phase of u_β^{af} in a harmonic trap for $\beta = 140$.

Simulations: Trapped Gas (III)

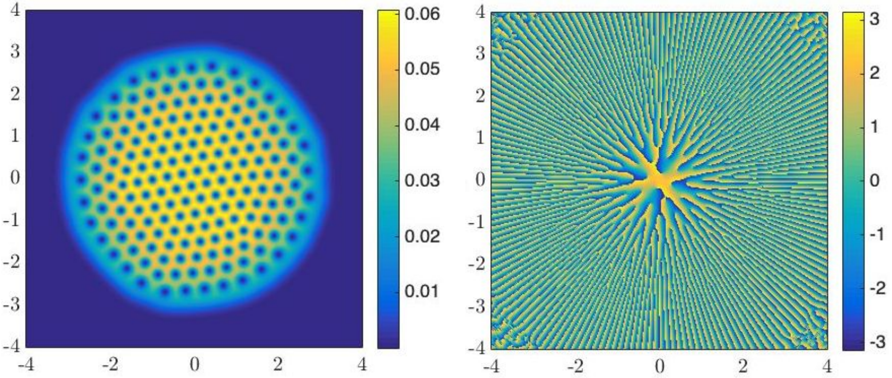




Modulus and phase of u_β^{af} in a quartic trap for $\beta = 90$.

Simulations: Trapped Gas (IV)

Sapienza Università di Roma



Modulus and phase of u_β^{af} in a quartic trap for $\beta = 195$.

Vortex Structure of u_β^{af} (I)

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- It is clear from these numerical experiments that u_β^{af} is expected to carry a large number of vortices.
- Vortices are expected to be distributed according to the density $\left|u_\beta^{\mathrm{af}}\right|^2$ and therefore their distribution is inhomogeneous in presence of a trapping potential.
- To check whether our expectations fit with the numerical data we can compare the number of vortices $N_v^{\mathrm{num}}(r)$ in a disc of radius r (counting the zeros of the wave function) with the expected value...
- Minimizing the kinetic term $\rho\left|\nabla\phi + \beta\mathbf{A}[\rho]\right|^2$ yields a vorticity density

$$\mu_v = \mathrm{curl}\nabla\phi \simeq -2\pi\beta\left|u_\beta^{\mathrm{af}}\right|^2,$$

which implies

$$N_v^{\mathrm{th}}(r) = 4\pi^2\beta \int_0^r \mathrm{d}r\, r\, \rho_\beta^{\mathrm{TF}}(r).$$

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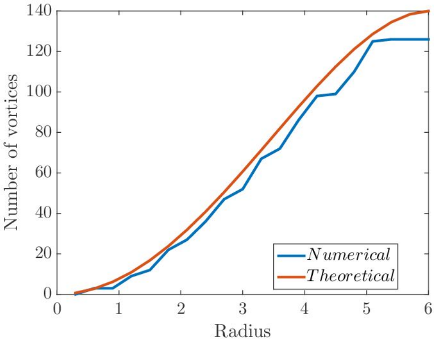

Theoretical (red) and numerical (blue) vortex density for $V(\mathbf{x}) = |\mathbf{x}|^2$, $\beta = 140$ (left) and $V(\mathbf{x}) = |\mathbf{x}|^4$, $\beta = 195$ (right).

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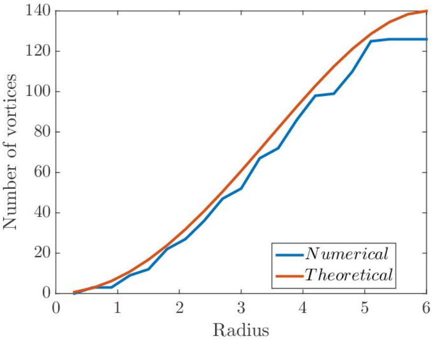

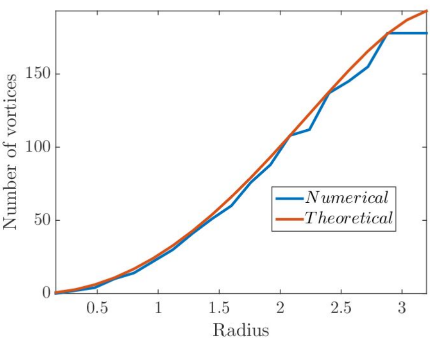

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Vortex Structure of u^{af}_{β} (III)

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Abrikosov-like problem?

- Typically in BECs or superconductors the vortex core is much smaller than the vortex mean distance $\Longrightarrow$ the density is approx. constant with isolated zeros (scale separation).
- There is a regime in BECs and superconductors where the core and mean distance are of the same order (no scale separation) $\Longrightarrow$ Abrikosov problem: minimize the energy of a periodic distribution of vortices $\Longrightarrow$ triangular lattice with energy e_{A} [Aftalion, Blanc '07; Fournais, Kachmar '11; Sandier, Serfaty '12].
- In $\mathcal{E}^{\mathrm{af}}_{\beta}$ there is only one asymptotic parameter $\Longrightarrow$ no scale separation and core $\sim$ mean distance.
- It seems natural to compare $e(1,1)$ with $e_{\mathrm{A}} \simeq 1.1596$, but not only $e(1,1) > 2\pi$, apparently also $e(1,1) > e_{\mathrm{A}}$...

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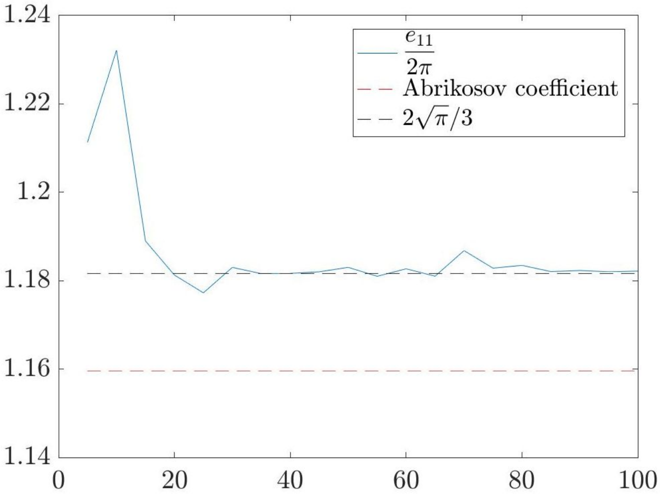
$e(1,1)$



1.24
1.22
1.2
1.18
1.16
1.14
0
20
40
60
80
100
$\frac{e_{11}}{2\pi}$
Abrikosov coefficient
$2\sqrt{\pi}/3$

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  - Recover the behavior $\beta \to \infty$ at the many-body level, in a limit $N \to \infty$, $\alpha = \alpha(N)$;
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