On the Average-Field Functional for Anyons

Michele Correggi

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Mathematical Physics of Anyons and Topological States of Matter

NORDITA

joint work with R. Duboscq (Toulouse), D. Lundholm (Stockholm) and N. Rougerie (Grenoble)
**Outline**

1. **Introduction:**
   - Almost-bosonic limit and the Average-Field (AF) functional [LR];
   - Minimization of the AF functional.

2. **Main results [CLR, CLR*]:**
   - Existence of the Thermodynamic Limit (TL) for homogeneous anyons;
   - Local density approximation of the AF functional for trapped anyons in terms of a Thomas-Fermi (TF) effective energy;

3. **Vortex structure (numerical simulations) [CDLR].**

**Main References**

- [CLR*] MC, D. Lundholm, N. Rougerie, in *Contemp. Math.*, **717** (2018);
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Any anyonic wave function $\Psi$ can be mapped onto a bosonic one $\tilde{\Psi} \in L^2_{\text{sym}}(\mathbb{R}^{2N})$ via the bosonization map

$$\Psi(x_1, \ldots, x_N) = \prod_{j<k} e^{i\alpha \phi_{jk}} \tilde{\Psi}(x_1, \ldots, x_N), \quad \phi_{jk} = \arg \frac{x_j - x_k}{|x_j - x_k|}.$$ 

**Magnetic Gauge**

On $L^2_{\text{sym}}(\mathbb{R}^{2N})$ the Schrödinger operator $\sum (-\nabla_j^2 + V(x_j))$ is mapped to

$$H_N = \sum_{j=1}^{N} \left[ (-i \nabla_j + \alpha A_j)^2 + V(x_j) \right]$$

with Aharonov-Bohm magnetic potentials $A_j = A(x_j) := \sum_{k \neq j}^{N} \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2}.$
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AF APPROXIMATION

- If the number of anyons is larger, i.e., $N \to \infty$ but at the same time $\alpha \approx \beta N^{-1}$, then one expects a mean-field behavior, i.e.,

$$\alpha A_j = (N \alpha) \frac{1}{N} \sum_{k \neq j} (x_j - x_k) \frac{1}{|x_j - x_k|^2} \approx \beta \int_{\mathbb{R}^2} dy \frac{(x - y) \perp}{|x - y|^2} \rho(y),$$

with $\rho$ the one-particle density associated to $\Phi \in L^2_{\text{sym}}(\mathbb{R}^{2N})$;

- We should then expect that

$$\frac{1}{N} \langle \Phi | H_N | \Phi \rangle \simeq \mathcal{E}_{\beta}^{\text{af}}[u],$$

for some $u \in L^2(\mathbb{R}^2)$ such that $|u|^2(x) = \rho(x)$ (self-consistency).

AF FUNCTIONAL

$$\mathcal{E}_{\beta}^{\text{af}}[u] = \int_{\mathbb{R}^2} dx \left\{ \left| (-i \nabla + \beta A[|u|^2]) u \right|^2 + V |u|^2 \right\}$$

with $A[\rho] = \nabla \perp (w_0 * \rho)$ and $w_0(x) := \log |x|$.
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The **AF Functional**

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with \( A[\rho] = \nabla^\perp (w_0 \ast \rho) \) and \( w_0(x) := \log |x| \).
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### AF Functional

$$E_{\beta}^{\text{af}}[u] = \int_{\mathbb{R}^2} dx \left\{ |(-i \nabla + \beta A[|u|^2]) u|^2 + V |u|^2 \right\}$$

with $A[\rho] = \nabla^{\perp} (w_0 * \rho)$ and $w_0(x) := \log |x|$.
Minimization of $E_{\beta}^{af}$ (I)

$E_{\beta}^{af}[u] = \int_{\mathbb{R}^2} dx \left\{ \left| (-i\nabla + \beta A[|u|^2]) u \right|^2 + V|u|^2 \right\}$, \quad $A[\rho] = \nabla \perp (w_0 * \rho)$

- Thanks to the symmetry $u, \beta \rightarrow u^*, -\beta$, we can assume $\beta \geq 0$;
- The domain of $E_{\beta}^{af}$ is $\mathcal{D}[E_{\beta}^{af}] = H^1(\mathbb{R}^2)$, since by 3-body Hardy inequality
  \[ \int_{\mathbb{R}^2} dx \ |A[|u|^2]|^2 |u|^2 \leq C \|u\|_{L^2(\mathbb{R}^2)}^4 \|\nabla |u||_{L^2(\mathbb{R}^2)}^2. \]

Proposition (Minimization [Lundholm, Rougerie ’15])

For any $\beta \geq 0$, there exists a minimizer $u_{\beta}^{af} \in \mathcal{D}[E_{\beta}^{af}]$ of the functional $E_{\beta}^{af}$:

$E_{\beta}^{af} := \inf_{\|u\|_2 = 1} E_{\beta}^{af}[u] = E_{\beta}^{af}[u_{\beta}^{af}]$. 
Introduction

Minimization of $\mathcal{E}^{\text{af}}_\beta$ (I)

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For any $\beta \geq 0$, there exists a minimizer $u^{\text{af}}_\beta \in \mathcal{D}[\mathcal{E}^{\text{af}}_\beta]$ of the functional $\mathcal{E}^{\text{af}}_\beta$:

$$E^{\text{af}}_\beta := \inf_{\|u\|_2=1} \mathcal{E}^{\text{af}}_\beta[u] = \mathcal{E}^{\text{af}}_\beta[u^{\text{af}}_\beta].$$
Minimization of $\mathcal{E}_{\beta}^{af}$ (II)

$$\mathcal{E}_{\beta}^{af}[u] = \int_{\mathbb{R}^2} dx \left\{ |(-i \nabla + \beta A[u^2]) u|^2 + V|u|^2 \right\}, \quad A[\rho] = \nabla^\perp (w_0 \ast \rho)$$

Proposition (Minimization [Lundholm, Rougerie ’15])

For any $\beta \geq 0$, $u_{\beta}^{af}$ solves the variational equation (semilinear with quintic nonlinearity in $u$ and cubic nonlinearity in $u^2$ and $\nabla u$)

$$\left[ (-i \nabla + \beta A_{\beta}^{af})^2 + V - 2\beta \nabla^\perp w_0 \ast \left( \beta A_{\beta}^{af} |u_{\beta}^{af}|^2 + j_{\beta}^{af} \right) \right] u_{\beta}^{af} = \lambda u_{\beta}^{af},$$

where $A_{\beta}^{af} = A[|u_{\beta}^{af}|^2]$, $j_{\beta}^{af} = \frac{i}{2} u_{\beta}^{af} \nabla u_{\beta}^{af^*} + c.c.$ is the current and

$$\lambda = \int_{\mathbb{R}^2} dr \left\{ 1(|\nabla u_{\beta}^{af}|^2 + V|u_{\beta}^{af}|^2) + 2 \cdot 2\beta A_{\beta}^{af} \cdot j_{\beta}^{af} + 3\beta^2 |A_{\beta}^{af}|^2 |u_{\beta}^{af}|^2 \right\}.$$
Minimization of $\mathcal{E}^\text{af}_\beta$ (II)

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Almost-bosonic Limit

- Consider $N \to \infty$ non-interacting anyons with statistics parameter $\alpha = \frac{\beta}{N-1}$ for some $\beta \in \mathbb{R}$, i.e., in the almost-bosonic limit;
- Assume that the anyons are extended, i.e., the fluxes are smeared over a disc of radius $R = N^{-\gamma}$.

Theorem (AF Approximation [Lundholm, Rougerie ’15])

Under the above hypothesis and assuming that $V$ is trapping and $\gamma \leq \gamma_0$,

$$\lim_{N \to \infty} \frac{\inf \sigma(H_{N,R})}{N} = \inf_{\|u\|_2 = 1} E^\text{af}_\beta [u]$$

and the one-particle reduced density matrix of any sequence of ground states of $H_{N,R}$ converges to a convex combination of projectors onto AF minimizers.
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and the one-particle reduced density matrix of any sequence of ground states of $H_{N,R}$ converges to a convex combination of projectors onto AF minimizers.
The AF approximation is used heavily in physics literature, but typically the nonlinearity is resolved by picking a given $\rho$ (usually the constant density);

As expected, when $\beta \to 0$, the anyonic gas behaves like a Bose gas.

More interesting is the regime $\beta \to \infty$, i.e., “less-bosonic” anyons:

- what is the energy asymptotics of $E^{af}_\beta$?
- is $|u^{af}_\beta|^2$ almost constant in the homogeneous case, i.e., for $V = 0$ and confinement to a bounded region?
- how does the inhomogeneity of $V$ modify the density $|u^{af}_\beta|^2$?
- what is $u^{af}_\beta$ like? in particular how does its phase behave?

The AF functional is not the usual mean-field-type energy (e.g., Hartree or Gross-Pitaevskii), since the nonlinearity depends on the density but acts on the phase of $u$ via a magnetic field.
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Homogeneous Gas

- \( \Omega \subset \mathbb{R}^2 \) bounded and simply connected with Lipschitz boundary;
- We consider the following two minimization problems

\[
E_{N/D}(\Omega, \beta, M) := \inf_{u \in H^1_0(\Omega), \|u\|_2 = M} \int_{\Omega} \left| (-i \nabla + \beta A |u|^2) u \right|^2.
\]

- We want to study the limit \( \beta \to \infty \) of \( E_{N/D}(\Omega, \beta, M) / \beta \);
- The above limit is equivalent to the TD limit (\( \beta, \rho \in \mathbb{R}^+ \) fixed)

\[
\lim_{L \to \infty} \frac{E_{N/D}(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|}.
\]

Lemma (Scaling Laws)

For any \( \lambda, \mu \in \mathbb{R}^+ \), \( E_{N/D}(\Omega, \beta, M) = \frac{1}{\lambda^2} E_{N/D} \left( \mu \Omega, \frac{\beta}{\lambda^2 \mu^2}, \lambda^2 \mu^2 M \right) \).
Main Results – Homogeneous Gas

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**Heuristics** ($\beta \gg 1$)

Numerical simulations: $|u^\text{af}_\beta|^2$ (square trap with Dirichlet b.c.) for $\beta = 10, 30, 120$.

- In the **homogeneous** case, $|u^\text{af}_\beta|^2$ can be constant only in a very weak sense (say in $L^p$, $p < \infty$ not too large);
- The phase of $u^\text{af}_\beta$ contains **vortices** (with $\# \sim \beta$) almost uniformly distributed with average distance $\sim \frac{1}{\sqrt{\beta}}$ (Abrikosov lattice?), in order to compensate the huge magnetic field.
Theorem (∃ TD Limit [MC, Lundholm, Rougerie ‘16])

Under the above hypothesis on $\Omega$ and for any $\beta, \rho \in \mathbb{R}^+$, the limits

$$e(\beta, \rho) := \lim_{L \to \infty} \frac{E_{N/D}(L\Omega, \beta, \rho L^2|\Omega|)}{L^2|\Omega|} = \beta|\Omega| \lim_{\beta \to \infty} \frac{E_{N/D}(\Omega, \tilde{\beta}, \rho)}{\tilde{\beta}}$$

exist, coincide and are independent of $\Omega$. Moreover

$$e(\beta, \rho) = \beta \rho^2 e(1, 1)$$

The scaling property of $e(\beta, \rho)$ is a direct consequence of the scaling law mentioned before. Moreover $e(1, 1)$ is a finite quantity such that $e(1, 1) \geq 2\pi$. 
Main Results – Homogeneous Gas

**TD Limit**

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M. Correggi (Roma 1)  Average-Field Functional  Nordita 12/3/2019
Main Results – Homogeneous Gas

Sketch of the Proof (I)

1. ∃ of TD limit when $\Omega$ is a unit square with Dirichlet b.c.;
2. $E_D(LQ, \beta, \rho L^2) - E_N(LQ, \beta, \rho L^2) = o(L^2)$ for any domain $\Omega$ (IMS);
3. ∃ of TD for general domains $\Omega$ with Dirichlet b.c..

1. TD limit for Dirichlet b.c.

- Key observation: the magnetic field generated by a bounded region can be gauged away outside (Newton’s theorem);
- Pick a smooth and radial $f$ with $\text{supp}(f) \subset B_\delta(0)$ and $N \sim L^2$ points so that $|x_j - x_k| > 2\delta$: consider then the trial state
  \[ u(x) = \sum_{j=1}^{N} f(x - x_j)e^{-i\phi_j}, \quad \|u\|^2_2 = N \|f\|^2_2; \]
- In $\{|x - x_j| \leq \delta\}$ the magnetic field generated by the other discs is
  \[ \sum_{k \neq j} \nabla_{\perp} (w_0 * |f(x - x_k)|^2) \simeq \|f\|^2_2 \nabla \sum_{k \neq j} \arg(z - z_k) =: \nabla \phi_j. \]
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1. TD LIMIT FOR DIRICHLET B.C.

- Key observation: the magnetic field generated by a bounded region can be gauged away outside (Newton’s theorem);
- Pick a smooth and radial $f$ with $\text{supp}(f) \subset B_\delta(0)$ and $N \sim L^2$ points so that $|x_j - x_k| > 2\delta$: consider then the trial state
  \[
  u(x) = \sum_{j=1}^{N} f(x - x_j)e^{-i\phi_j}, \quad \|u\|_2^2 = N \|f\|_2^2;
  \]
- In $\{|x - x_j| \leq \delta\}$ the magnetic field generated by the other discs is
  \[
  \sum_{k \neq j} \nabla \perp (w_0 * |f(x - x_k)|^2) \simeq \|f\|_2^2 \nabla \sum_{k \neq j} \text{arg}(z - z_k) =: \nabla \phi_j.
  \]
Sketch of the Proof (II)

\[ \mathcal{E}_\beta^{af}[u] = \int_{L\Omega} dx \left| (-i\nabla + \beta A[|u|^2]) u \right|^2, \quad A[\rho] = \nabla^\perp (w_0 \ast \rho), \quad \|u\|_2^2 = \rho L^2 \]

① TD limit for squares with Dirichlet b.c.

- By testing \( \mathcal{E}_\beta^{af} \) on \( u_{\text{trial}} \), one gets the sum of the energies in each \( B_\delta(x_j) \), which yields the upper bound \( E_{N/D} \leq CL^2 \).

- A trivial lower bound is given via the inequality (for any \( u \in H^1_0(\Omega) \))

\[
\int_\Omega dx \left| (-i\nabla + \beta A) u \right|^2 \geq \beta \int_\Omega dx \text{curl} A \left| u \right|^2
\]

which leads to the lower bound \( E_D \geq 2\pi\beta\rho^2L^2 \), since

\[
\text{curl} A \left[ |u|^2 \right] = 2\pi |u|^2
\]

- Decompose the square into small squares and use a trial state obtained by gluing together the Dirichlet minimizers in each smaller square.
\[ \mathcal{E}_\beta^{af}[u] = \int_{L\Omega} d\mathbf{x} \left| (-i\nabla + \beta \mathbf{A}[|u|^2]) u \right|^2, \quad \mathbf{A}[\rho] = \nabla^\perp (w_0 \ast \rho), \quad \|u\|^2_2 = \rho L^2 \]

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Main Results – Homogeneous Gas

Sketch of the Proof (III)

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2 Dirichlet vs. Neumann in squares

- Use a refined version of the IMS formula to estimate the mass well inside the square and the contribution of the boundary layer, to get a bound matching the obvious \( E_N \leq E_D \).
- Error of order \( L^{12/7 + \epsilon} \ll L^2 \) but \( \gg L \) (not optimal).

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- Tile the domain with squares and use the Dirichlet minimizers in the upper bound (gauging away the magnetic field as before) and the Neumann minimizer in the lower bound.
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- It is easy to see that $e(1, 1)$ is bounded.
- The inequality $\|(-i\nabla + A |u|^2) u\|_2^2 \geq 2\pi$ implies that $e(1, 1) \geq 2\pi$. 
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Numerical computation of \( e(1, 1)/(2\pi) \) using harmonically trapped anyons for increasing values of \( \beta \).
**Main Results – Trapped Gas**

**Trapped Anyons**

\[ \mathcal{E}^{af}_\beta[u] = \int_{\mathbb{R}^2} dx \left\{ \left| (-i \nabla + \beta A[|u|^2]) u \right|^2 + V|u|^2 \right\}, \quad A[\rho] = \nabla^\perp (w_0 \ast \rho) \]

- Let \( V(x) \) be a smooth **homogenous** potential of degree \( s \geq 1 \), i.e.,
  \[ V(\lambda x) = \lambda^s V(x), \quad V \in C^\infty(\mathbb{R}^2), \]
  and such that \( \min_{|x| \geq R} V(x) \xrightarrow{R \to \infty} +\infty \) (**trapping potential**).

- We consider the minimization problem for \( \beta \gg 1 \)
  \[ E^{af}_\beta = \inf_{u \in \mathcal{D}[\mathcal{E}^{af}], \|u\|_2=1} \mathcal{E}^{af}_\beta[u], \]
  with \( \mathcal{D}[\mathcal{E}^{af}] = H^1(\mathbb{R}^2) \cap \{ V|u|^2 \in L^1(\mathbb{R}^2) \} \) and \( u^{af}_\beta \) any minimizer.

- Since \( B(x) = \beta \text{curl} A[\rho] = 2\pi \beta \rho(x) \), if one could minimize the magnetic energy alone, the effective functional for \( \beta \gg 1 \) should be
  \[ \int_{\mathbb{R}^2} dx \{ B(x) + V(x) \} \rho = \int_{\mathbb{R}^2} dx \left[ 2\pi \beta \rho^2 + V(x) \rho \right]. \]
Trapped Anyons

\[ \mathcal{E}_{\beta}^{af}[u] = \int_{\mathbb{R}^2} \text{d}x \left\{ \left| \left( -i \nabla + \beta A[|u|^2] \right) u \right|^2 + V|u|^2 \right\}, \quad A[\rho] = \nabla^\perp (w_0 * \rho) \]

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TF APPROXIMATION

TF FUNCTIONAL

The limiting functional for $\mathcal{E}_\beta^{af}$ is

$$
\mathcal{E}_\beta^{TF}[\rho] := \int_{\mathbb{R}^2} dx \left[ e(1, 1) \beta \rho^2(x) + V(x) \rho(x) \right]
$$

with ground state energy $E^{TF}_\beta := \inf_{\|\rho\|_1=1} \mathcal{E}_\beta^{TF}[\rho]$ and minimizer $\rho_{\beta}^{TF}(x)$.

- Under the hypothesis we made on $V$, we have
  $$
  E^{TF}_\beta = \beta^{s+2} E^{TF}_1, \quad \rho^{TF}_\beta(x) = \beta^{-\frac{2}{s+2}} \rho^{TF}_1 \left( \beta^{-\frac{1}{s+2}} x \right).
  $$

- Given the chemical potential $\mu^{TF}_1 := E^{TF}_1 + e(1, 1) \|\rho^{TF}_1\|_2^2$, we have
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  \rho^{TF}_1(x) = \frac{1}{2e(1,1)} \left[ \mu^{TF}_1 - V(x) \right]_+.
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Local Density Approximation (I)

Theorem (TF Approx. [MC, Lundholm, Rougerie ‘16])

Under the hypothesis on $V$ and for any $R > 0$

\[
\lim_{\beta \to \infty} \frac{E_{\beta}^{af}}{\beta s + 2 E_{1}^{TF}} = 1, \quad \beta s + 2 |u_{\beta}^{af}|^2 \left( \beta \frac{1}{s+2} x \right) \xrightarrow{\beta \to \infty} \rho_{1}^{TF}(x)
\]

in the dual space of Lipschitz functions $C_{0,1}^{0,1}(\mathcal{B}_R)$ vanishing on $\partial \mathcal{B}_R$.

- The result applies to more general potentials, e.g., asymptotically homogeneous potentials;
- The homogeneous case (confinement to $\Omega$, $V = 0$) is included: we recover the asymptotics $E_{N/D}(\Omega, \beta, 1)/\beta \to e(1, 1)/|\Omega|$ and

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|u_{\beta}^{af}|^2 (x) \xrightarrow{\beta \to \infty} \rho_{1}^{TF}(x) \equiv |\Omega|^{-1/2}.
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Theoretical (red) and numerical (blue) density profiles for $V(x) = |x|^2$, $\beta = 90$ (left) and $V(x) = |x|^4$, $\beta = 140$ (right).
Average Density

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Main Results – Trapped Gas

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**Theorem (LDA [MC, Lundholm, Rougerie '16])**

In the homogeneous case, for any \( x_0 \in \Omega^\circ \) and any \( R > 0 \),

\[ \left| u_{\beta, N/D}^{af} (x_0 + \beta^{-\eta} \cdot) \left( C_{0,1}^0(\mathcal{B}_R)^* \right) \right|_{\beta \to +\infty} \xrightarrow{\beta \to +\infty} \left| \Omega \right|^{-1/2}, \quad \text{for any } 0 < \eta < \frac{1}{14}. \]

- We conjecture that \( \rho_{\beta}^{af} \) is well approximated in weak sense by a constant on any scale smaller than 1 up to \( 1/\sqrt{\beta} \);
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- Analogous result for trapped anyons...
In presence of a large magnetic field of order $\beta \gg 1$, vortices can compensate for it: where $|u| \neq 0$, we can set $u = |u|e^{i\phi}$ and

$$|(-i\nabla + \beta A[|u|^2])u|^2 = |\nabla|u|^2| + |u|^2|\nabla\phi + \beta A|^2.$$ 

Hence several point singularities of $\phi$ can “reconstruct” $\beta A$;

- Since $|u^\text{af}_\beta|$ is approx. constant (on the scale 1), vortices must be uniformly distributed (on a smaller scale) in the homogeneous case.
- The number of vortices is $\sim \beta$ and their average distance is $\sim 1/\sqrt{\beta}$.
- In presence of a trapping potential some inhomogeneity in the vortex distribution should appear.
Vortex Structure of $u^\text{af}_\beta$ (I)

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Modulus and phase of $u^a_f$ in a square trap with Dirichlet boundary conditions for $\beta = 130$. 
Modulus and phase of $u_{af}^\beta$ in a square trap with Dirichlet boundary conditions for $\beta = 230$. 
Modulus and phase of $u^a_{\beta}$ in a harmonic trap for $\beta = 25$. 
Modulus and phase of $u^a_{\beta}$ in a harmonic trap for $\beta = 140$. 
Modulus and phase of $u^a_f$ in a quartic trap for $\beta = 90$. 
Modulus and phase of $u_{\beta}^{af}$ in a quartic trap for $\beta = 195$. 
It is clear from these numerical experiments that $u^\text{af}_{\beta}$ is expected to carry a large number of vortices.

Vortices are expected to be distributed according to the density $|u^\text{af}_{\beta}|^2$ and therefore their distribution is inhomogeneous in presence of a trapping potential.

To check whether our expectations fit with the numerical data we can compare the number of vortices $N^\text{num}_v(r)$ in a disc of radius $r$ (counting the zeros of the wave function) with the expected value...

Minimizing the kinetic term $\rho |\nabla \phi + \beta A[\rho]|^2$ yields a vorticity density

$$\mu_v = \text{curl} \nabla \phi \simeq -2\pi \beta |u^\text{af}_{\beta}|^2,$$

which implies

$$N^\text{th}_v(r) = 4\pi^2 \beta \int_0^r dr \ r \ \rho^\text{TF}_\beta(r).$$
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VORTEX STRUCTURE OF $u_{\beta}^\text{af}$ (II)

Theoretical (red) and numerical (blue) vortex density for $V(x) = |x|^2$, $\beta = 140$ (left) and $V(x) = |x|^4$, $\beta = 195$ (right).
Theoretical (red) and numerical (blue) vortex density for $V(x) = |x|^2$, $\beta = 140$ (left) and $V(x) = |x|^4$, $\beta = 195$ (right).
**Aabrikosov-like problem?**

- Typically in BECs or superconductors the vortex core is much smaller than the vortex mean distance $\Rightarrow$ the density is approx. constant with isolated zeros (scale separation).

- There is a regime in BECs and superconductors where the core and mean distance are of the same order (no scale separation) $\Rightarrow$ Abrikosov problem: minimize the energy of a periodic distribution of vortices $\Rightarrow$ triangular lattice with energy $e_A$ [Aftalion, Blanc ‘07; Fournais, Kachmar ‘11; Sandier, Serfaty ‘12].

- In $\varepsilon^\text{af}_\beta$ there is only one asymptotic parameter $\Rightarrow$ no scale separation and core $\sim$ mean distance.

- It seems natural to compare $e(1, 1)$ with $e_A \approx 1.1596$, but not only $e(1, 1) > 2\pi$, apparently also $e(1, 1) > e_A$...
In Progress – Vortex Structure

Vortex Structure of $u^\text{af}_\beta$ (III)

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Perspectives

- **AF functional:**
  - Obtain more information about $e(1, 1)$;
  - Investigate the vortex structure of $u^\text{af}_\beta$, at least with numerical simulations;
  - Find an estimate of the critical value of $\beta$ for the occurrence of vortices;
  - Prove that the vorticity is uniformly distributed for $\beta$ large in the homogenous case.

- **Anyon gas:**
  - Recover the behavior $\beta \to \infty$ at the many-body level, in a limit $N \to \infty$, $\alpha = \alpha(N)$;
  - Prove the existence of the thermodynamic limit in the same setting.
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Thank you for the attention!